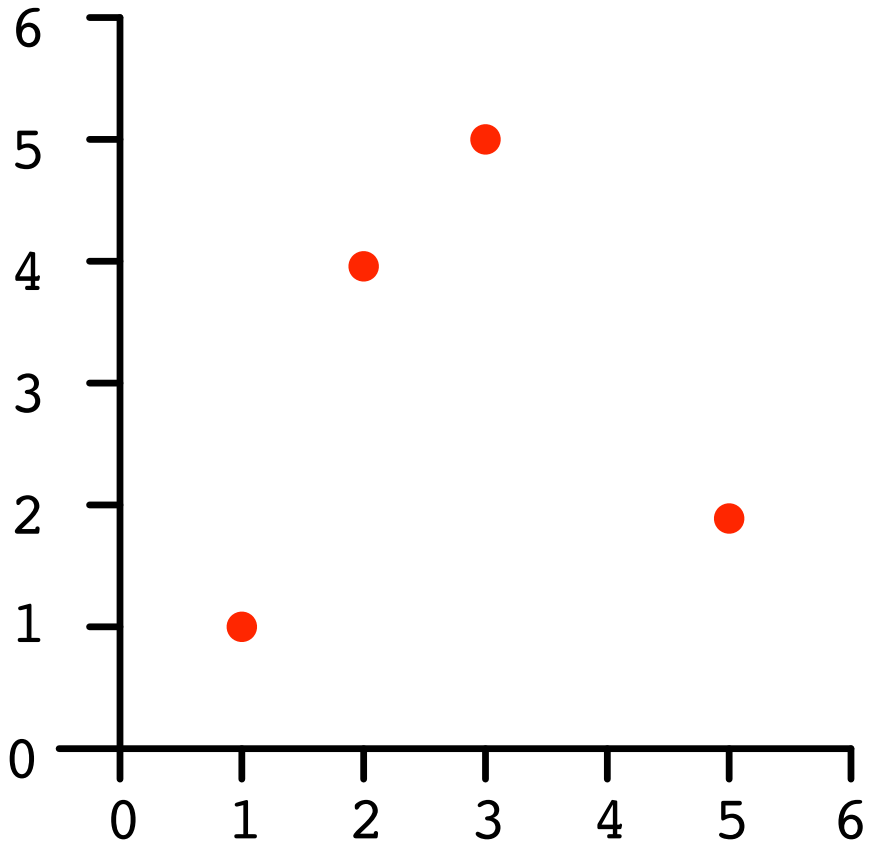
Linear algebra primer

Instructor: Taylor Berg-Kirkpatrick
Slides: Sanjoy Dasgupta

Data as vectors and matrices



Matrix-vector notation

Vector $x \in \mathbb{R}^d$:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_d \end{pmatrix}$$

Matrix-vector notation

Vector $x \in \mathbb{R}^d$:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_d \end{pmatrix}$$

Matrix $M \in \mathbb{R}^{r \times d}$:

$$M = \begin{pmatrix} M_{11} & M_{12} & \cdots & M_{1d} \\ M_{21} & M_{22} & \cdots & M_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ M_{r1} & M_{r2} & \cdots & M_{rd} \end{pmatrix}$$

$M_{ij} =$ entry at row i, column j

Transpose of vectors and matrices

$$x = \begin{pmatrix} 1 \\ 6 \\ 3 \\ 0 \end{pmatrix} \quad \text{has } \textbf{transpose} \quad x^T = \begin{pmatrix} 1, & 6, & 3, & 0 \end{pmatrix}$$

$$M = \begin{pmatrix} 1 & 2 & 0 & 4 \\ 3 & 9 & 1 & 6 \\ 8 & 7 & 0 & 2 \end{pmatrix} \quad \text{has } \textbf{transpose} \quad M^T = \begin{pmatrix} 1 & 3 & 8 \\ 2 & 9 & 7 \\ 0 & 1 & 0 \\ 4 & 6 & 2 \end{pmatrix}$$

- $(A^T)_{ij} = A_{ji}$
- $(A^T)^T = A$

Adding and subtracting vectors and matrices

$x, y \in \mathbb{R}^d$

$z = x + y$

$\forall i \in \{1, \ldots, d\}$

$z_i := x_i + y_i$

$A, B \in \mathbb{R}^{r \times d}$

$C = A + B$

$\forall i \in \{1, \ldots, r\}$

$\forall j \in \{1, \ldots, d\}$

$C_{ij} = A_{ij} + B_{ij}$

Dot product of two vectors

Dot product of vectors $x, y \in \mathbb{R}^d$:

$$x^T y = x \cdot y = x_1 y_1 + x_2 y_2 + \cdots + x_d y_d.$$

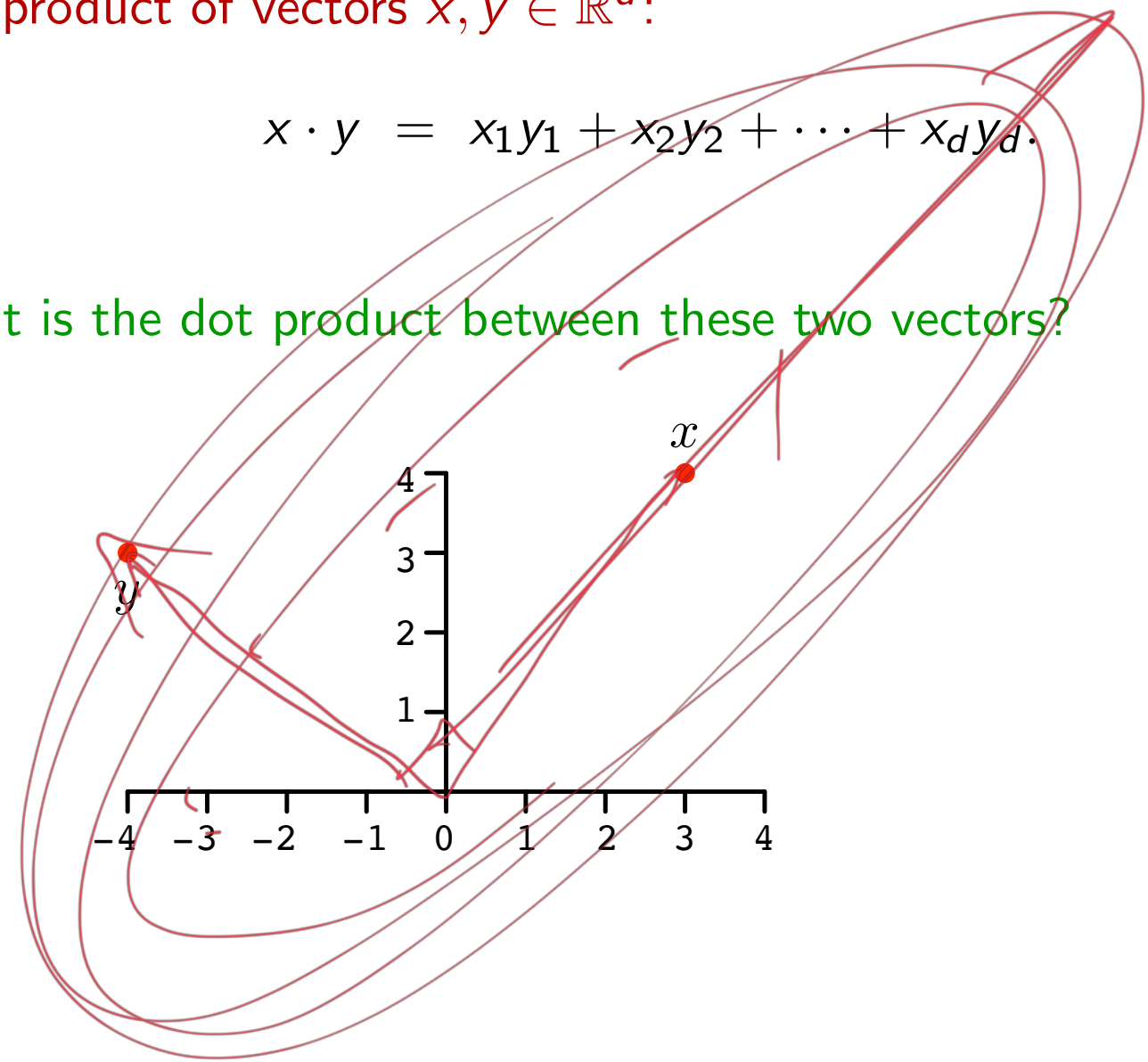
$$= \sum_{i=1}^{d} x_i y_i$$

Dot product of two vectors

Dot product of vectors $x, y \in \mathbb{R}^d$:

$$x \cdot y = x_1 y_1 + x_2 y_2 + \cdots + x_d y_d.$$

What is the dot product between these two vectors?

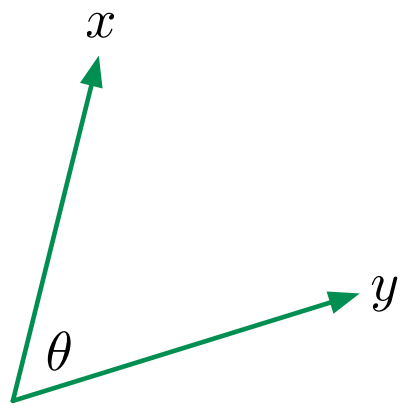


Dot products and angles

Dot product of vectors $x, y \in \mathbb{R}^d$:

$x \cdot y \;=\; x_1 y_1 + x_2 y_2 + \cdots + x_d y_d.$

Tells us the angle between x and y:

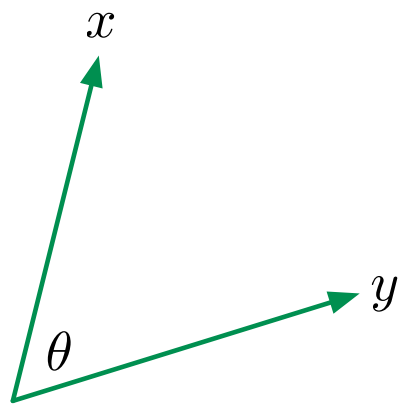


$$\cos\theta \;=\; \frac{x \cdot y}{\|x\|\,\|y\|}.$$

Dot products and angles

Dot product of vectors $x, y \in \mathbb{R}^d$:
$x \cdot y \;=\; x_1 y_1 + x_2 y_2 + \cdots + x_d y_d.$

Tells us the angle between x and y:



$$\cos\theta \;=\; \frac{x \cdot y}{\|x\|\,\|y\|}.$$

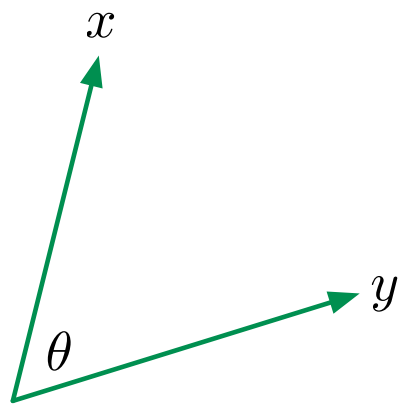
- x is **orthogonal** (at right angles) to y if and only if $x \cdot y = 0$

Dot products and angles

Dot product of vectors $x, y \in \mathbb{R}^d$:

$x \cdot y \;=\; x_1 y_1 + x_2 y_2 + \cdots + x_d y_d.$

Tells us the angle between x and y:



$$\cos\theta \;=\; \frac{x \cdot y}{\|x\|\,\|y\|}.$$

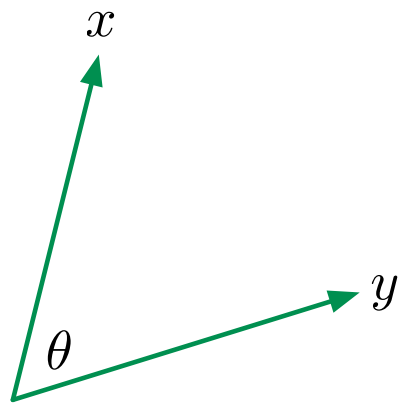
- x is **orthogonal** (at right angles) to y if and only if $x \cdot y = 0$
- When x, y are **unit vectors** (length 1): $\cos\theta = x \cdot y$

Dot products and angles

Dot product of vectors $x, y \in \mathbb{R}^d$:

$x \cdot y \;=\; x_1 y_1 + x_2 y_2 + \cdots + x_d y_d.$

Tells us the angle between x and y:

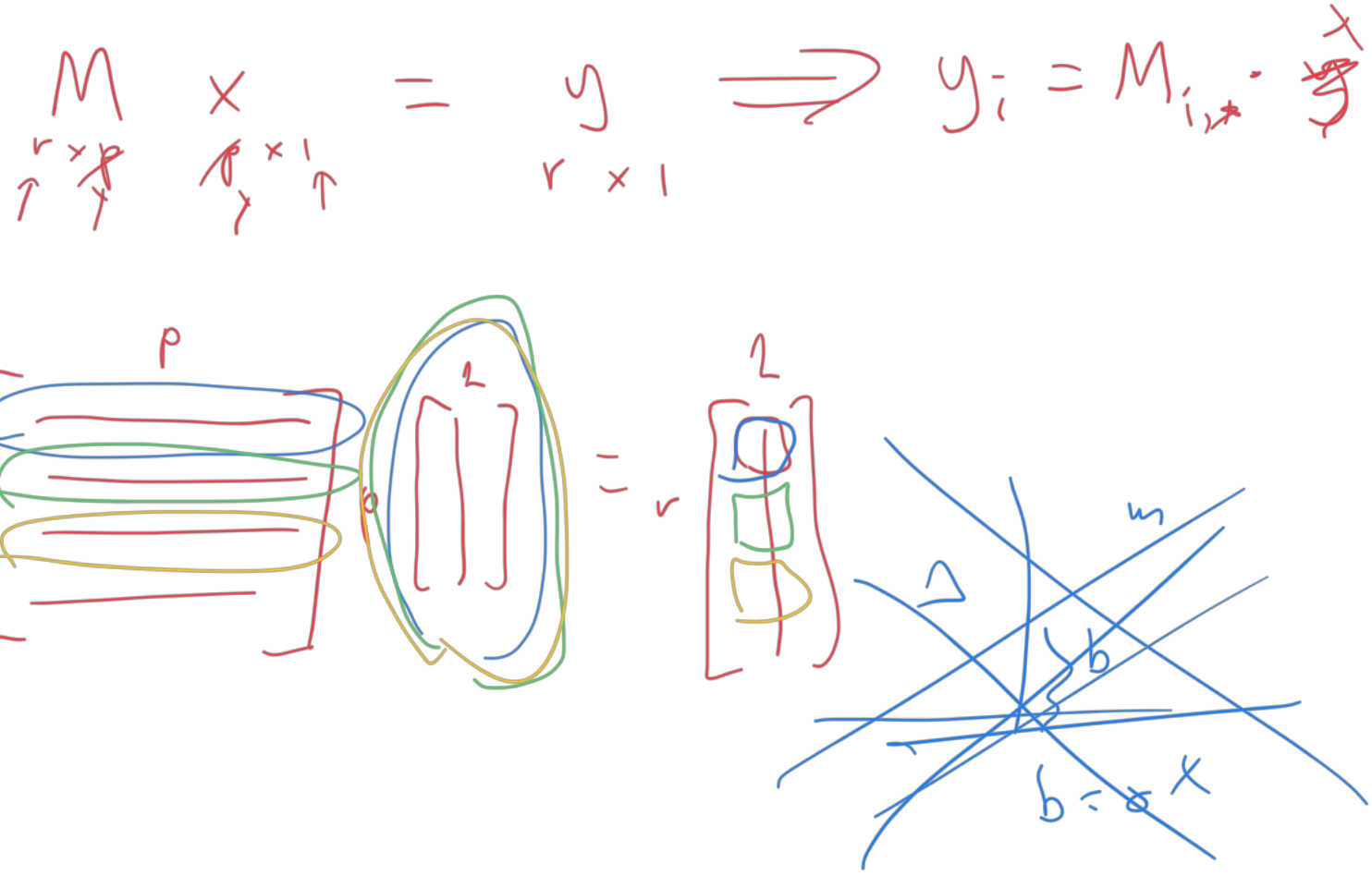


$$\cos\theta \;=\; \frac{x \cdot y}{\|x\|\,\|y\|}.$$

- x is **orthogonal** (at right angles) to y if and only if $x \cdot y = 0$
- When x, y are **unit vectors** (length 1): $\cos\theta = x \cdot y$
- What is $x \cdot x$?

Matrix-vector product

Product of matrix $M \in \mathbb{R}^{r \times p}$ and vector $x \in \mathbb{R}^p$:



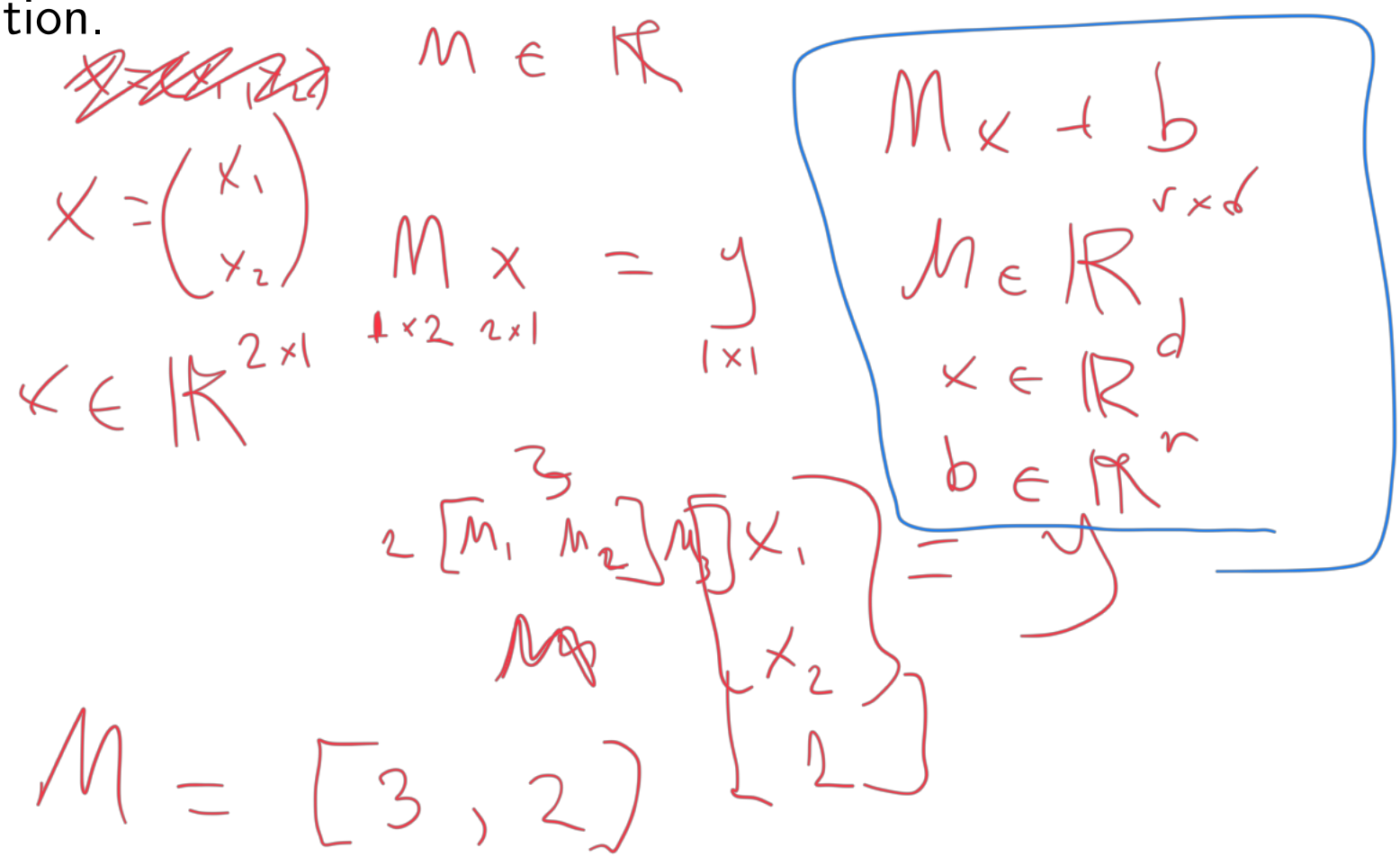
Linear functions

Let M be any $r \times d$ matrix.
The function $x \mapsto Mx$ sends $\mathbb{R}^d$ to $\mathbb{R}^r$.

It is a **linear function**: $M(x + x') = Mx + Mx'$.

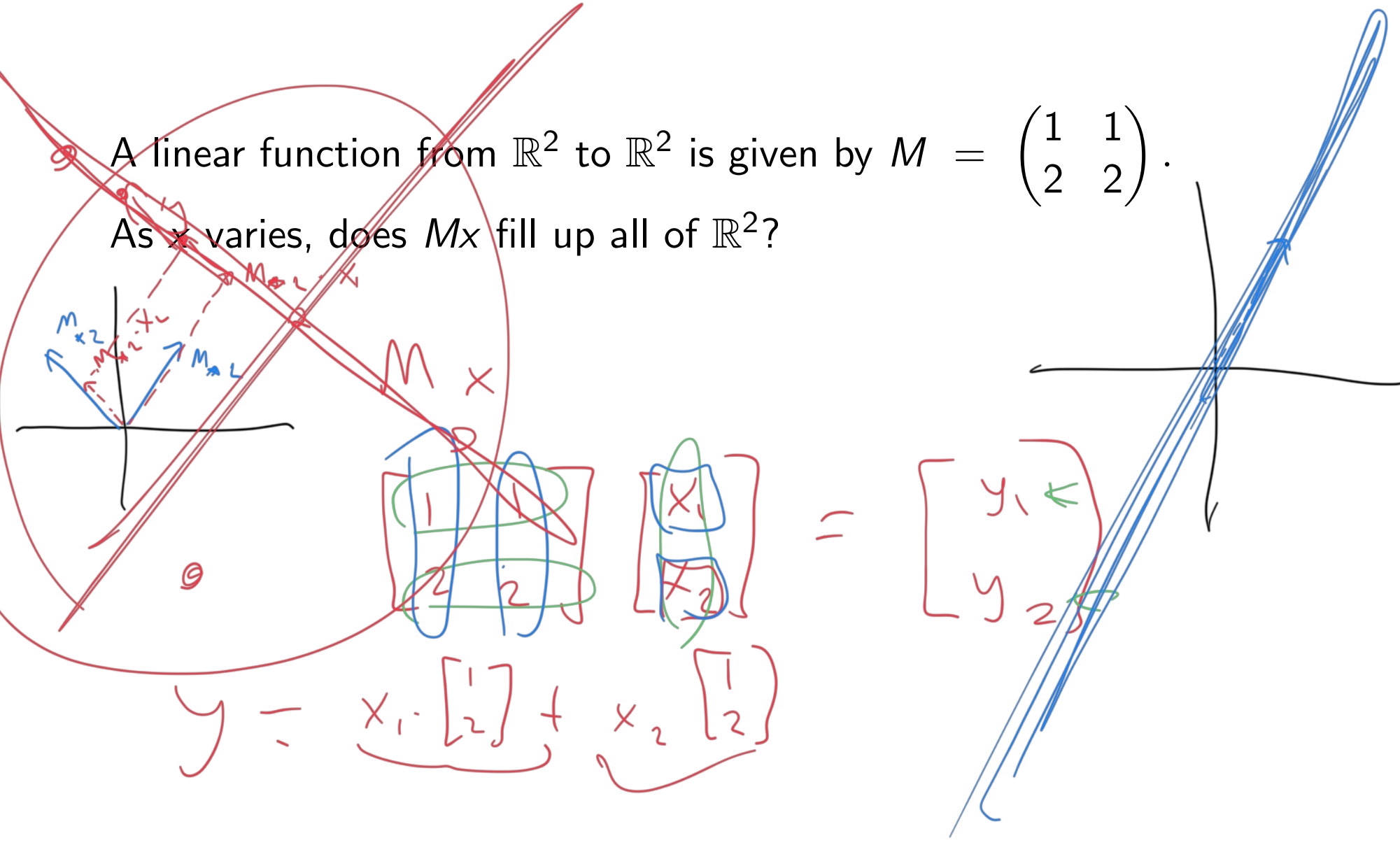
E.g. $M = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$

Write the linear function $f(x_1, x_2) = 3x_1 + 2x_2$ using vector notation.



A linear function from $\mathbb{R}^2$ to $\mathbb{R}^2$ is given by $M = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$.

As x varies, does Mx fill up all of $\mathbb{R}^2$?



The identity matrix

The $d \times d$ **identity matrix** I_d sends each $x \in \mathbb{R}^d$ to itself.

$$I_d = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

$U = c \cdot I_d$

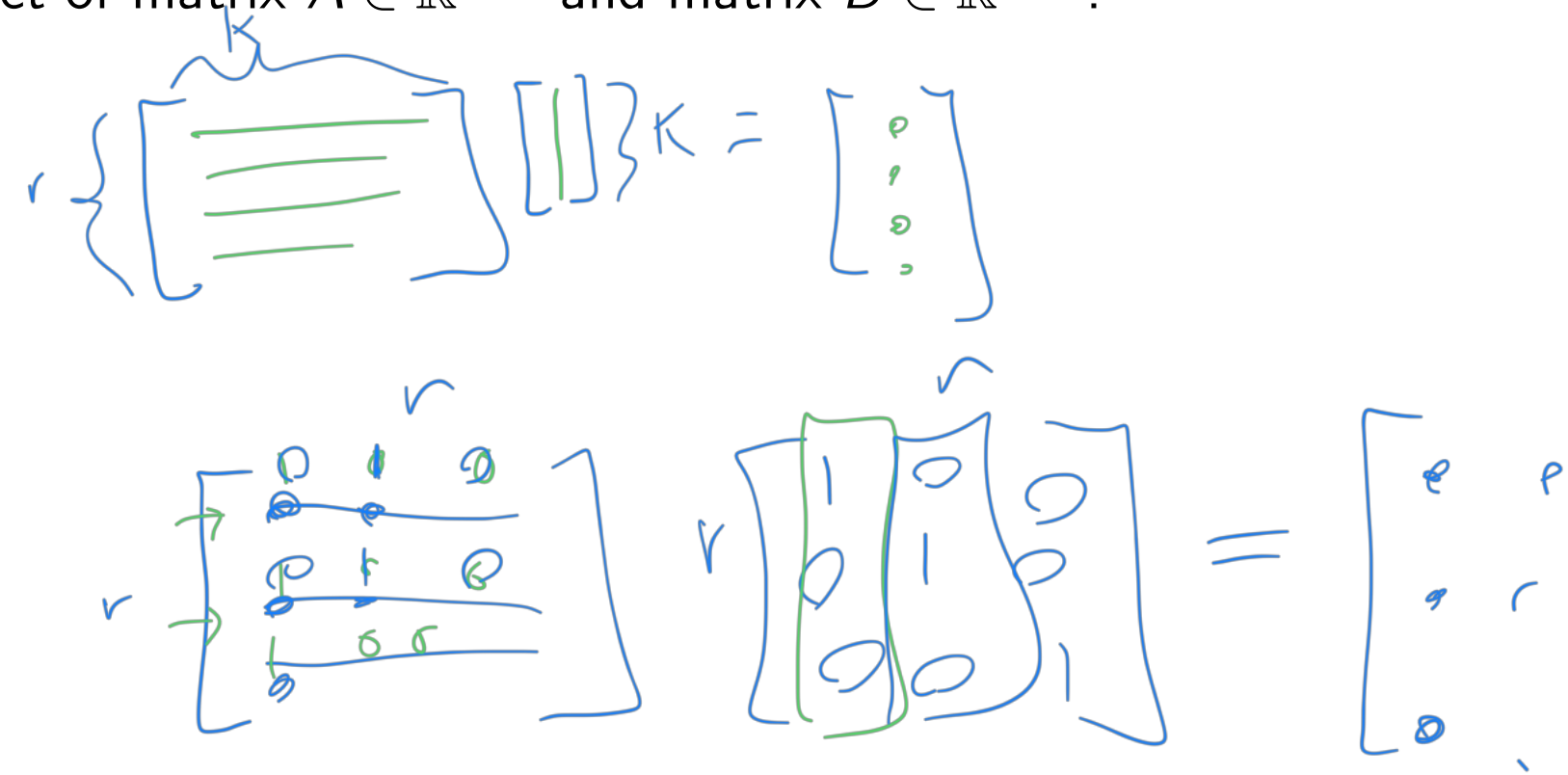
$M \in \mathbb{R}^{d \times d}$

$M \cdot U = cM$

$M \cdot I_d = M$

Matrix-matrix product

Product of matrix $A \in \mathbb{R}^{r \times k}$ and matrix $B \in \mathbb{R}^{k \times p}$:



Matrix products

If $A \in \mathbb{R}^{r \times k}$ and $B \in \mathbb{R}^{k \times p}$, then AB is an $r \times p$ matrix with

$$(AB)_{ij} = (\text{dot product of } i\text{th row of } A \text{ and } j\text{th column of } B)$$

$$= \sum_{\ell=1}^{k} A_{i\ell} B_{\ell j}$$

Matrix products

If $A \in \mathbb{R}^{r \times k}$ and $B \in \mathbb{R}^{k \times p}$, then AB is an $r \times p$ matrix with

$$(AB)_{ij} = (\text{dot product of } i\text{th row of } A \text{ and } j\text{th column of } B)$$

$$= \sum_{\ell=1}^{k} A_{i\ell} B_{\ell j}$$

- $I_k B = B$ and $A I_k = A$

Matrix products

If $A \in \mathbb{R}^{r \times k}$ and $B \in \mathbb{R}^{k \times p}$, then AB is an $r \times p$ matrix with

$$(AB)_{ij} = (\text{dot product of } i\text{th row of } A \text{ and } j\text{th column of } B)$$

$$= \sum_{\ell=1}^{k} A_{i\ell} B_{\ell j}$$

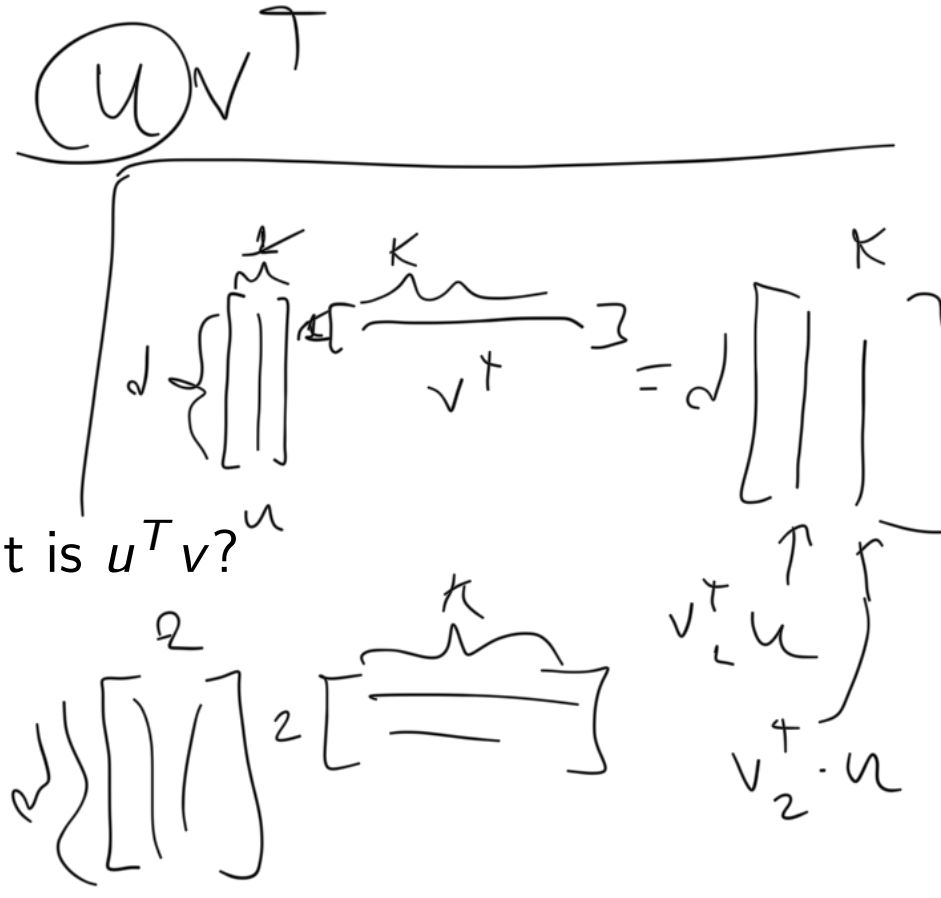
- $I_k B = B$ and $A I_k = A$
- Can check: $(AB)^T = B^T A^T$

Matrix products

If $A \in \mathbb{R}^{r \times k}$ and $B \in \mathbb{R}^{k \times p}$, then AB is an $r \times p$ matrix with

$$(AB)_{ij} = (\text{dot product of } i\text{th row of } A \text{ and } j\text{th column of } B)$$

$$= \sum_{\ell=1}^{k} A_{i\ell} B_{\ell j}$$



- $I_k B = B$ and $A I_k = A$
- Can check: $(AB)^T = B^T A^T$
- For two vectors $u, v \in \mathbb{R}^d$, what is $u^T v$?

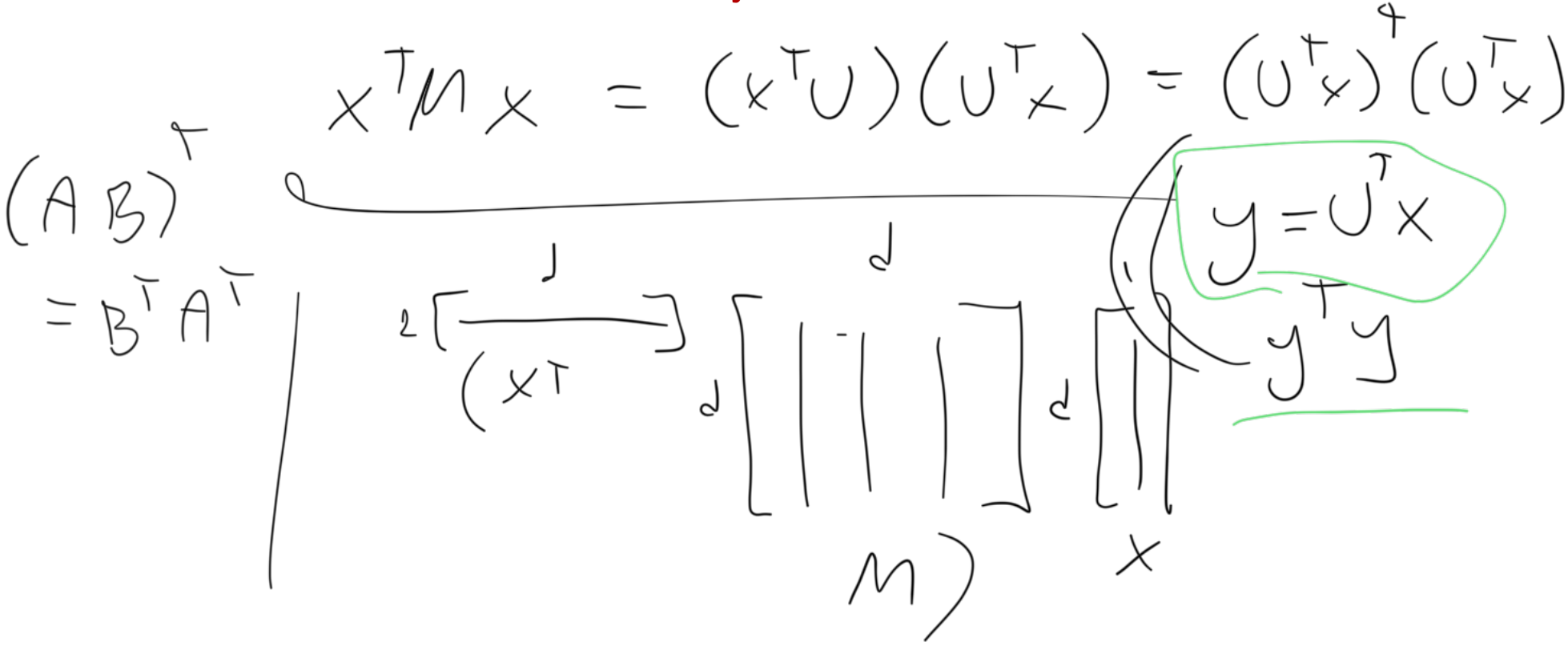
Some special cases

For vector $x \in \mathbb{R}^d$, what are $x^T x$ and xx^T?

Special case, generalized

For vector $x \in \mathbb{R}^d$, we have $x^T x = \|x\|^2$.

What about $x^T Mx$, for arbitrary $d \times d$ matrix M?



Quadratic functions

Let M be any $d \times d$ (square) matrix.
For $x \in \mathbb{R}^d$, the mapping $x \mapsto x^T M x$ is a **quadratic function** from $\mathbb{R}^d$ to $\mathbb{R}$:

$$x^T M x = \sum_{i,j=1}^{d} M_{ij} x_i x_j.$$

What is the quadratic function associated with $M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 3 & 4 & 5 \end{pmatrix}$?

$$x^T M x + U x + b$$

$$x \in \mathbb{R}^d$$

Write the quadratic function $f(x_1, x_2) = x_1^2 + 2x_1x_2 + 3x_2^2$ using matrices and vectors.

$$x^T M x = \sum_{ij} M_{ij} x_i x_j$$

$$M = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$$

$M^T = M$



$i = j = 1 : M_{11} = 1$

$i = 1, j = 2 : [M_{12} + M_{21}] = 2$

$i = j = 2 \quad M_{22} = 3$

Special cases of square matrices

- **Symmetric**: $M = M^T$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 3 & 4 & 6 \end{pmatrix}$$

- **Diagonal**: $M = \mathrm{diag}(m_1, m_2, \ldots, m_d)$

$$\mathrm{diag}(1, 4, 7) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

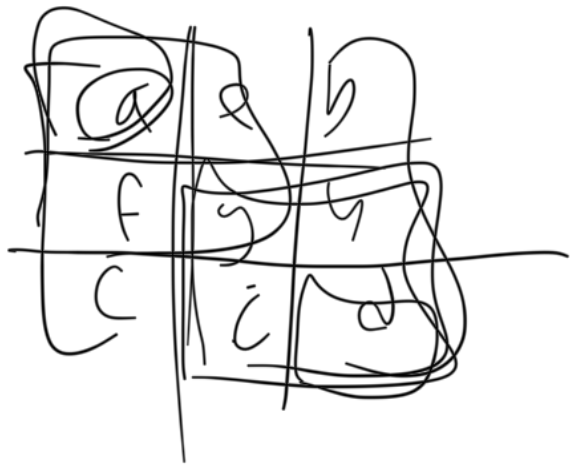
$$\begin{bmatrix} m_1 & m_2 \end{bmatrix} \quad \mathrm{diag}(1, 2)$$

$$= \begin{bmatrix} m_1 & 2m_2 \end{bmatrix}$$

Determinant of a square matrix

Determinant of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $|A| = ad - bc$.

Example: $A = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$



Inverse of a square matrix

The **inverse** of a $d \times d$ matrix A is a $d \times d$ matrix B for which $AB = BA = I_d$.

Notation: A^{-1}.

$$A = \begin{bmatrix} e & e \\ e & e \end{bmatrix}$$

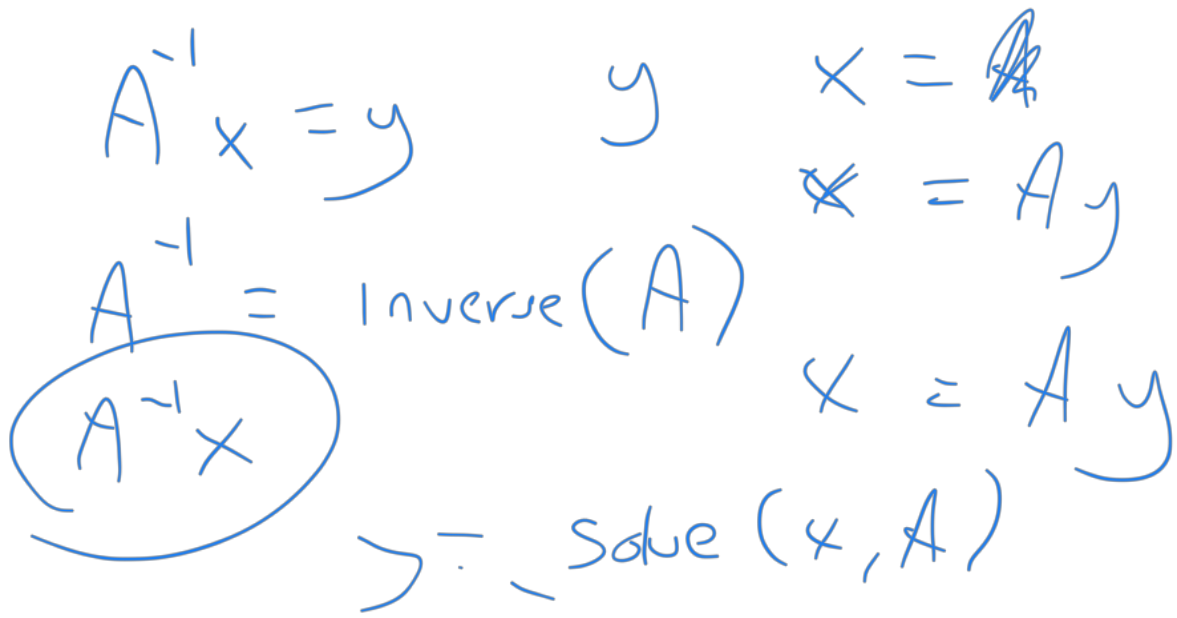
$$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

$$A + \varepsilon I$$

Inverse of a square matrix

The **inverse** of a $d \times d$ matrix A is a $d \times d$ matrix B for which $AB = BA = I_d$.
Notation: A^{-1}.

Example: if $A = \begin{pmatrix} 1 & 2 \\ -2 & 0 \end{pmatrix}$ then $A^{-1} = \begin{pmatrix} 0 & -1/2 \\ 1/2 & 1/4 \end{pmatrix}$. Check!



Inverse of a square matrix, cont'd

The **inverse** of a $d \times d$ matrix A is a $d \times d$ matrix B for which $AB = BA = I_d$.
Notation: A^{-1}.

$\text{diag}(a_1, \ldots, a_d)^{-1} = \text{diag}(\frac{1}{a_1}, \ldots, \frac{1}{a_d})$

- Not all square matrices have an inverse
- Square matrix A is invertible if and only if $|A| \neq 0$
- What is the inverse of $A = \text{diag}(a_1, \ldots, a_d)$?

