Minimum Spanning Trees and Union-Find

CSE 101: Design and Analysis of Algorithms

Lecture 7

CSE 101: Design and analysis of algorithms

- Minimum spanning trees and union-find
  - Reading: Section 5.1
- Quiz 1 is today, last 40 minutes of class
- Homework 3 is due Oct 23, 11:59 PM

How to implement Kruskal's algorithm

- Sort edges by weight, go through from smallest to largest, and add if it does not create cycle with previously added edges
- How do we tell if adding an edge will create a cycle?
  - Naive: depth-first search every time
    - Need to test for every edge, m times
  - Depth-first search on a forest: only edges added to minimum spanning tree
    - As such, each depth-first search is O(n)
    - Total time O(nm)

n = |V|
m = |E|

Disjoint sets data structure (DSDS)

- Main complication: want to check if u is connected to v efficiently
- Tree T divides vertices into **disjoint sets** of connected components
- u is connected to v if they are in the same set
- Adding e to T **merges** the set containing u with the set containing v
- So we need a data structure that
  - Represents a partition of a set V into disjoint subsets
    - We will pick one element L from each subset to be the "leader" of a subset, in order to give the subsets distinct names
  - Has an operation find(u) that returns the leader of u's set
  - Has an operation union(u,v) that replaces the two sets containing u and v with their union

Kruskal's algorithm using a DSDS

```
procedure kruskal(G,w)
  Input: undirected connected graph G with edge weights w
  Output: a set of edges X that defines a minimum spanning tree of G
for all v in V
  makeset(v)
X = { }
Sort the edges in E in increasing order by weight
For all edges (u,v) in E
  if find(u) ≠ find(v):
    Add edge (u,v) to X
    union(u,v)
```

Kruskal's algorithm using a DSDS

```
procedure kruskal(G,w)
  Input: undirected connected graph G with edge weights w
  Output: a set of edges X that defines a minimum spanning tree of G
for all v in V
  makeset(v)
X = { }
Sort the edges in E in increasing order by weight
For all edges (u,v) in E until X is a connected graph
  if find(u) ≠ find(v):
    Add edge (u,v) to X
    union(u,v)
```

Trees

- Definition: A tree is an undirected connected graph with no cycles
- An undirected connected graph is a tree if and only if removing any edge results in two disconnected graphs
- An undirected connected graph with n vertices is a tree if and only if it has n -1 edges
- An undirected connected graph is a tree if and only if there is a unique path between nodes

Kruskal's algorithm using a DSDS

```
procedure kruskal(G,w)
  Input: undirected connected graph G with edge weights w
  Output: a set of edges X that defines a minimum spanning tree of G
for all v in V
  makeset(v)
X = { }
Sort the edges in E in increasing order by weight
For all edges (u,v) in E until |X| = |V| - 1
  if find(u) ≠ find(v):
    Add edge (u,v) to X
    union(u,v)
```

Kruskal's algorithm using a DSDS

procedure kruskal(G,w)
 Input: undirected connected graph G with edge weights w
 Output: a set of edges X that defines a minimum spanning tree of G
for all v in V
 makeset(v) |V| * makeset
X = { }
Sort the edges in E in increasing order by weight sort(|E|)
For all edges (u,v) in E until |X| = |V| - 1 2 * |E| * find
 if find(u) ≠ find(v):
  Add edge (u,v) to X
  union(u,v) (|V| - 1) * union

Kruskal's algorithm, DSDS subroutines

- makeset(u)
  - Creates a set with one element, u
- find(u)
  - Finds the set to which u belongs
- union(u,v)
  - Merges the sets containing u and v

- Kruskal's algorithm

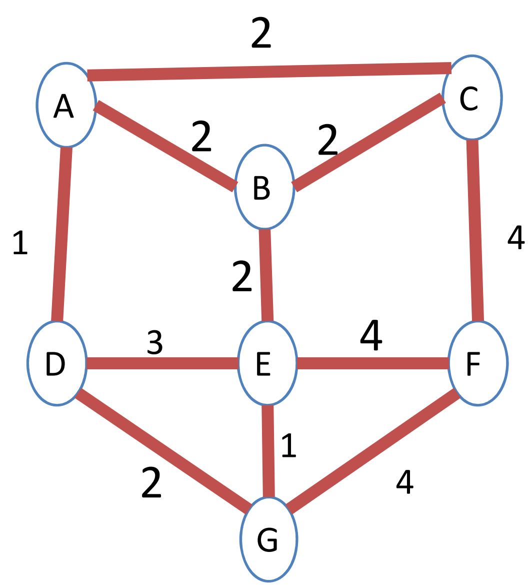
  |V| * makeset + 2 * |E| * find + (|V| - 1) * union + sort(|E|)

DSDS, leader version

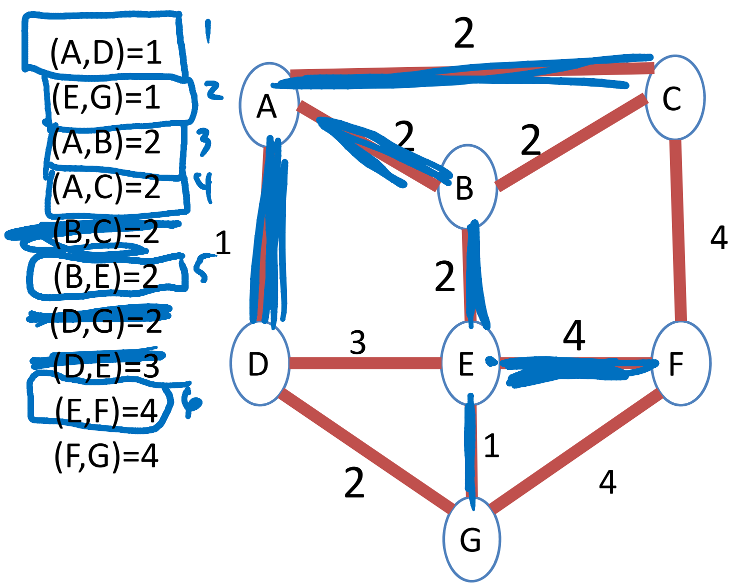
- Keep an array leader(u) indexed by element
- In each array position, keep the leader of its set
- makeset(u): leader(u) = u
- find(u): return leader(u)
- union(u,v): set leader(x) = leader(u)

Example: DSDS, leader version

(A,D)=1
(E,G)=1
(A,B)=2
(A,C)=2
(B,C)=2
(B,E)=2
(D,G)=2
(D,E)=3
(E,F)=4
(F,G)=4



Example: DSDS, leader version



A	B	C	D	E	F	G
A	B	C	D	E	F	G
A	A	A	A	E		E
				A	A	A

DSDS, leader version

- Keep an array leader(u) indexed by element
- In each array position, keep the leader of its set
- makeset(v): leader(u) = u, O(1)
- find(u): return leader(u), O(1)
- union(u,v): For each array position, if it is currently leader(v), then change it to leader(u). O(|V|)

- Kruskal's algorithm

  |V| * makeset + 2 * |E| * find + (|V| - 1) * union + sort(|E|)

  $= |V| * O(1) + 2 * |E| * O(1) + (|V| - 1) * O(|V|) + sort(|E|)$

  $= O(|V|^2)$

A more efficient implementation

- We want to optimize DSDS for other uses as well
- And it’s fun (right?)

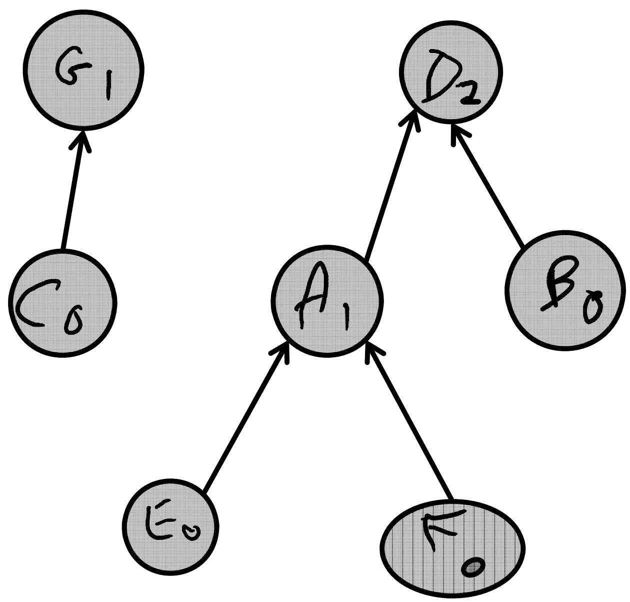
DSDS, directed trees with ranks version

- Each set is a rooted tree, with the vertices of the tree labeled with the elements of the set and the root the leader of the set
- To find, only need to go up to leader, so just need parent pointer
- To union, point one leader to other

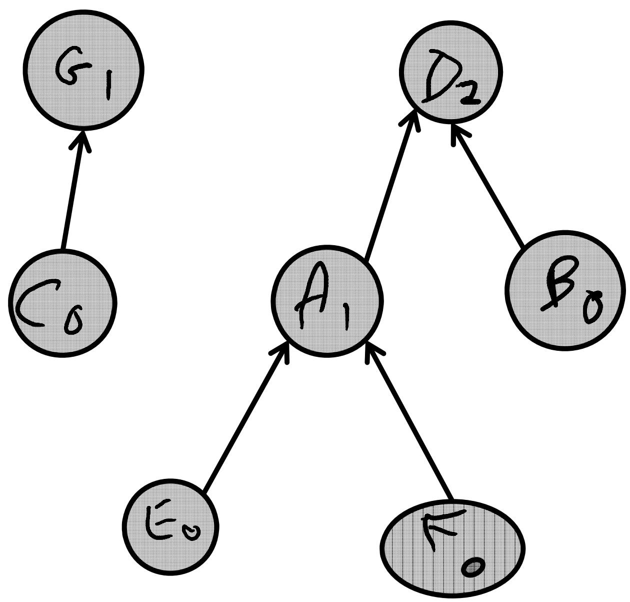
DSDS, directed trees with ranks version

- Vertices of the trees are elements of a set and each vertex points to its parent that eventually points to the root
- The root points to itself
- The root is a convenient representation or name of the set containing it and all of its children
- In addition to the parent pointer of x, $\pi(x)$, each vertex also has a rank that tells you the height of the subtree hanging from that vertex

Directed trees with ranks



Directed trees with ranks



rank(A)=1
rank(B)=0
rank(C)=0
rank(D)=2
rank(E)=0
rank(F)=0
rank(G)=1

$\pi(A) = D$
$\pi(B) = D$
$\pi(C) = G$
$\pi(D) = D$
$\pi(E) = A$
$\pi(F) = A$
$\pi(G) = G$

DSDS, directed trees with ranks version

procedure makeset(x)

$\pi(x) := x$

rank(x):=0

DSDS, directed trees with ranks version

procedure find(x)

while $\left(x \neq \pi(x)\right)$

Goes up parent pointers until root is found

$x := \pi(x)$

return x

DSDS, directed trees with ranks version

```
procedure union(x,y)
  rx:=find(x)
  ry:=find(y)
  if rx=ry then return
  if rank(rx)>rank(ry) then
    π(ry):= rx
  else
    π(rx):= ry
    if rank(rx)=rank(ry) then
      rank(ry):=rank(rx)+1
```

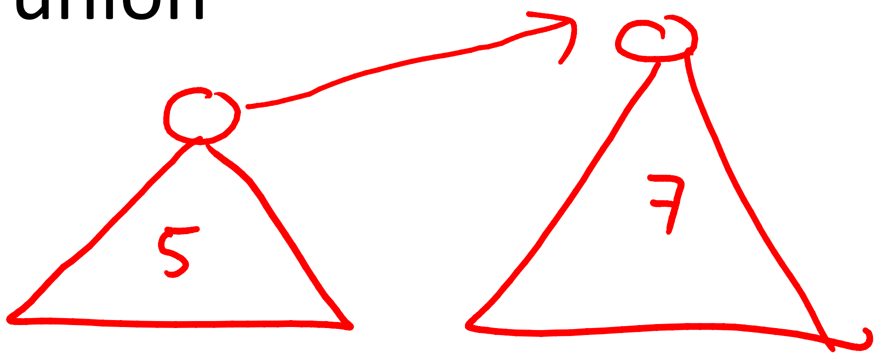
DSDS, directed trees with ranks version

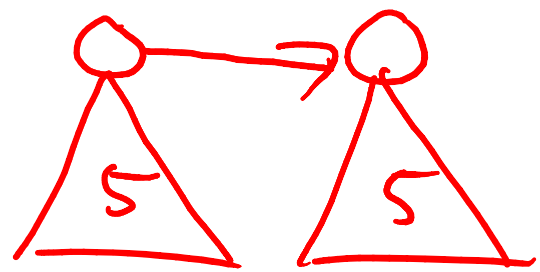
```
procedure union(x,y)
  rx:=find(x)
  ry:=find(y)
  if rx=ry then return
  if rank(rx)>rank(ry) then
    π(ry):= rx
  else
    π(rx):= ry
    if rank(rx)=rank(ry) then
      rank(ry):=rank(rx)+1
```

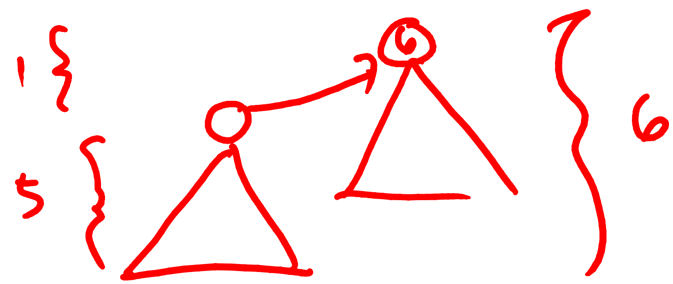
- To save on runtime, we must keep the heights of the trees short
- As such, union of two ranks points the smaller rank to the bigger rank, that way, the tree will stay the same height
- If the ranks are equal, then it increments one rank and points the smaller to the bigger
  - This is the only way a rank can increase

DSDS, directed trees with ranks version

- union







DSDS, directed trees with ranks version

procedure makeset(x)
 $\pi(x) := x$
 rank(x):=0

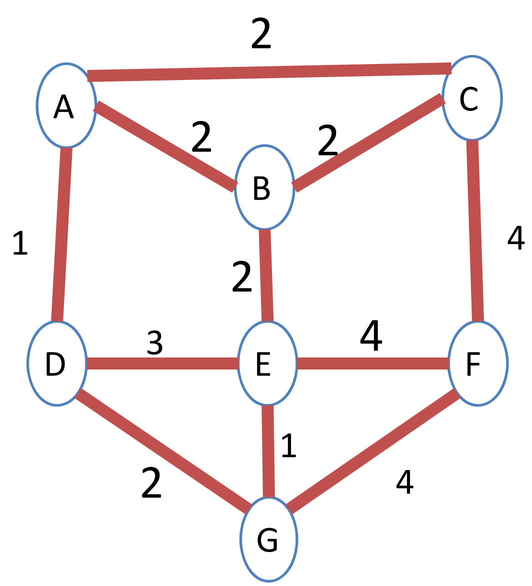
procedure find(x)
 while $(x \neq \pi(x))$
  $x := \pi(x)$
 return x

makeset O(1)
find O(height of tree containing x)
union O(find)

procedure union(x,y)
 rx:=find(x)
 ry:=find(y)
 if rx=ry then return
 if rank(rx)>rank(ry) then
  $\pi(ry) := rx$
 else
  $\pi(rx) := ry$
  if rank(rx)=rank(ry) then
   rank(ry):=rank(rx)+1

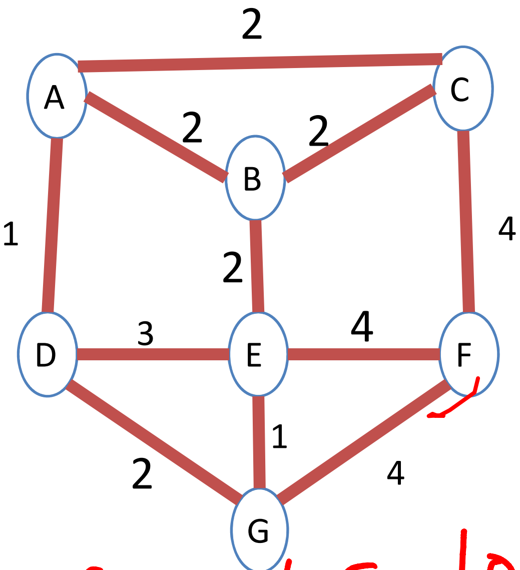
Example: DSDS, directed trees with ranks version

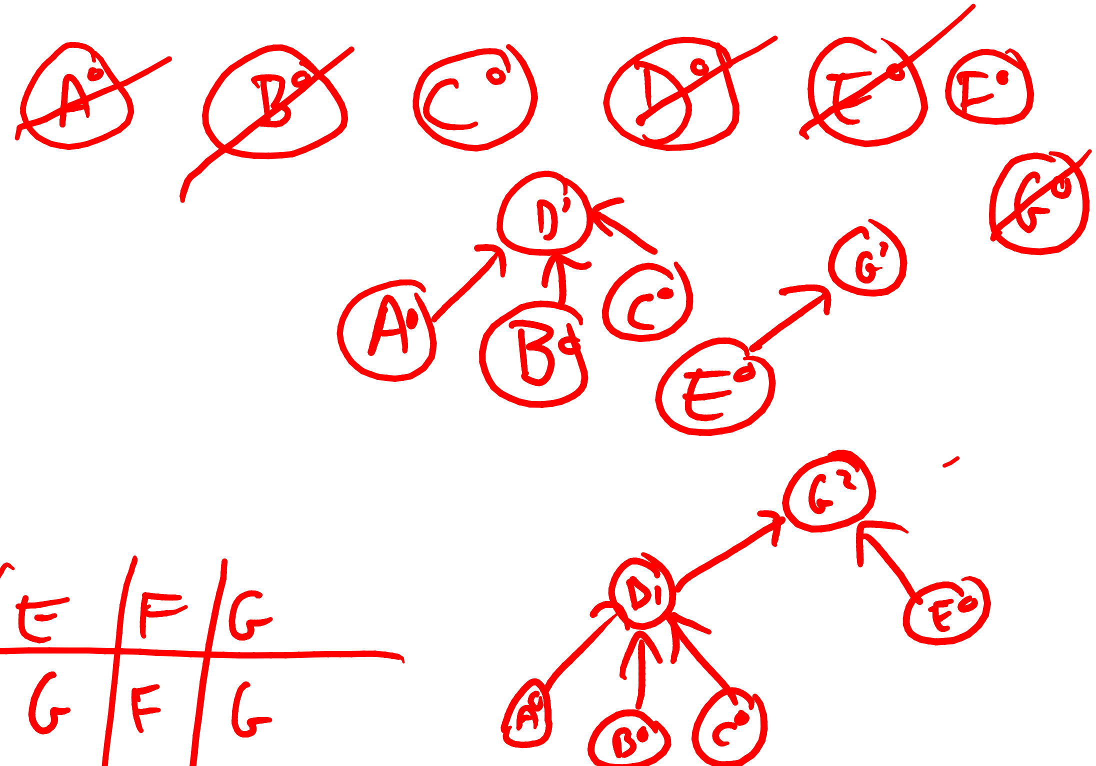
(A,D)=1
(E,G)=1
(A,B)=2
(A,C)=2
(B,C)=2
(B,E)=2
(D,G)=2
(D,E)=3
(E,F)=4
(F,G)=4



Example: DSDS, directed trees with ranks version

(A,D)=1
(E,G)=1
(A,B)=2
(A,C)=2
~~(B,C)=2~~
(B,E)=2
(D,G)=2
(D,E)=3
(E,F)=4
(F,G)=4





	A	B	C	D	E	F	G
π	D	D	D	G	G	F	G
r	0	0	0	1	0	0	2

Height of tree

- Any root node of rank k has at least 2^k vertices in its tree
- Proof
  - Base Case: a root of rank 0 has 1 vertex
  - Suppose a root of rank k has at least 2^k vertices in its tree. Then, a root of rank k+1 can only be made by unioning 2 roots each of rank k. So, a root of rank k+1 must have at least $2^k + 2^k = 2^{k+1}$vertices in its tree.

Next lecture

- Greedy algorithms
  - Reading: Kleinberg and Tardos, sections 4.1, 4.2, and 4.3