Priority Queue Implementations and Minimum Spanning Trees

CSE 101: Design and Analysis of Algorithms

Lecture 6

CSE 101: Design and analysis of algorithms

- Priority queue implementations and minimum spanning trees
  - Reading: Sections 4.5 and 5.1
- Homework 2 is due today, 11:59 PM
- Quiz 1 is Oct 18, in class

Dijkstra's algorithm, towards low-level

```
procedure dijkstra(G,ℓ,s)
X: set of vertices we know distances to
F: set of vertices we aren't sure of the distance to
```

- Let X and F be sets
- All vertices start out in F with dist set to ∞
- Set dist(s)=0 and move s from F to X
- Repeat until F is empty or there are no edges from X to F:
  - Let w be the vertex in F with the minimum value:
    $dist(v) + \ell(v, w)$ for all v in X
  - Set $dist(w) = dist(v) + \ell(v, w)$
    (Note: only need to keep track of best current edge to each vertex in F, so only one number per node in F. Can use same array, dist to keep track of this number)
  - Move w from F to X

Dijkstra's algorithm, towards low-level

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- Let X and F be sets
- All vertices start out in F with dist set to ∞
- Set dist(s)=0 and move s from F to X

  We will maintain the invariant that for all u ∈ F, dist(u) is the minimum over all e = (v,u), v ∈ X of dist(v)+l(e)
- Repeat until F is empty or there are no edges from X to F:
  - Let w be the vertex in F with the minimum value:

    $dist(v) + \ell(v, w)$ for all v in X
  - Set $dist(w) = dist(v) + \ell(v, w)$
  - Move w from F to X

Dijkstra’s algorithm, towards low-level

procedure dijkstra(G,ℓ,s)
X: set of vertices we know distances to
F: set of vertices we aren’t sure of the distance to

- Let X and F be sets
- All vertices start out in F with dist set to ∞
- Set dist(s)=0 and move s from F to X

  For each u in F, maintain array dist:

$$dist(u) = \min_{\substack{(v,u)\in E \\ v\in X}} dist(v) + \ell(v,u)$$

- Repeat until F is empty or there are no edges from X to F:
  - Let w be the vertex in F with the minimum value:

    $dist(v) + \ell(v,w)$ for all v in X
  - Set $dist(w) = dist(v) + \ell(v,w)$
  - Move w from F to X
  - Update dist(u) for all neighbors of w

Structures in Dijkstra's algorithm

- Graph G: How does it change? Where do we access it?
- Set X: How does it change? Where do we access it?
- Set F: How does it change? Where do we access it?

Structures in Dijkstra's algorithm

- Graph G: no changes, need to list members
  - Adjacency list
- Set X: insert, check membership
  - Array of booleans
- Set F: Find and delete the element with minimum key, dist(u). Decrease the keys of some elements u'.

What do we keep track of?

- Instead of picking a theoretical minimum:
  - Let w be the vertex in F with the minimum value dist(v) + $\ell(v, w)$ for all v in X
- We only have to choose the vertex w that has the minimum dist value

Priority queue

- A priority queue is a data structure of a set of objects (vertices) along with key values for each object that can be changed (alarm settings). Additionally, it can support the following operations.
  - insert
  - deletemin
  - decreasekey

Array as a priority queue

- Array: indexed by vertex, giving key value
  [key(A),key(B),key(C),key(D),key(E),key(F)]

- insert
- deletemin
- decreasekey

Array as a priority queue

- Array: indexed by vertex, giving key value

  R = [key(A),key(B),key(C),key(D),key(E),key(F)]

Let n = |V|
Initialize the array as an array of nulls

- Insert O(1)
  - insert(v, value). R[v] = value.
- deletemin O(n)
  - Find minimum and change value to null
- decreasekey O(1)
  - decreasekey(v, value). R[v] = value.

Array as a priority queue

- Dijkstra's algorithm

  makequeue + deletemin × |V| + decreasekey × |E|

  – If we use an array, then it will take

    $O(|V|) + O(|V|)\ |V| + O(1)\ |E| = O(|V|^2 + |E|) = O(|V|^2)$

Runtime of Dijkstra's algorithm

```
procedure dijkstra(G,ℓ,s)
for all u in V
  dist(u) := infinity
  prev(u) := nil
dist(s) := 0
H := makequeue(V)
while H is not empty
  u := deletemin(H)
  for all edges (u,v) in E
    if dist(v) > dist(u) + ℓ(u,v) then
      dist(v):= dist(u) + ℓ(u,v)
      prev(v):=u
      decreasekey(H,v)
```

Runtime of Dijkstra's algorithm, array as priority queue

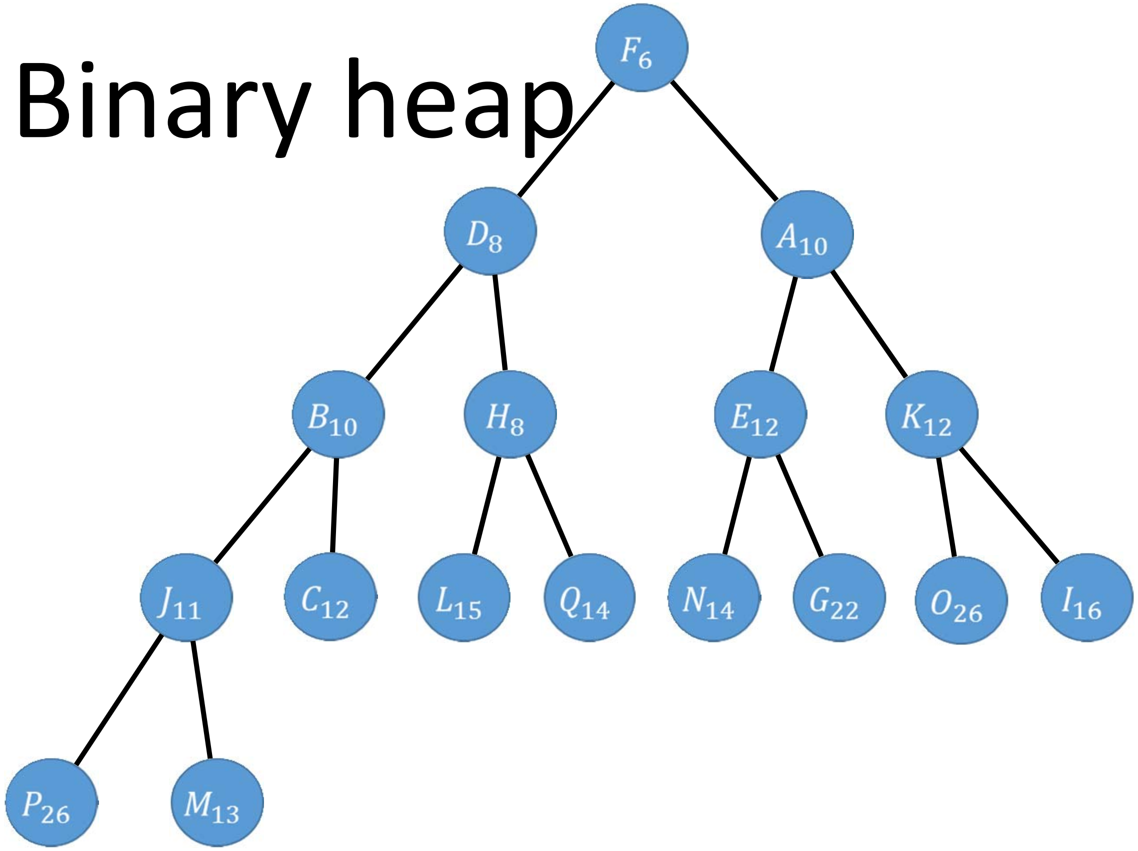
```
procedure dijkstra(G,ℓ,s)
for all u in V
  dist(u) := infinity
  prev(u) := nil
dist(s) := 0
H := makequeue(V)
while H is not empty
  u := deletemin(H) executes |V| times
  for all edges (u,v) in E deg(u)
    if dist(v) > dist(u) + ℓ(u,v) then
      dist(v):= dist(u) + ℓ(u,v)
      prev(v):=u
      decreasekey(H,v) executes |E| times
```

Initialize $O(|V|)$

deletemin × $|V|$ + decreasekey × $|E|$
$= O(|V|) \times |V| + O(1) \times |E|$
$= O(|V|^2)$

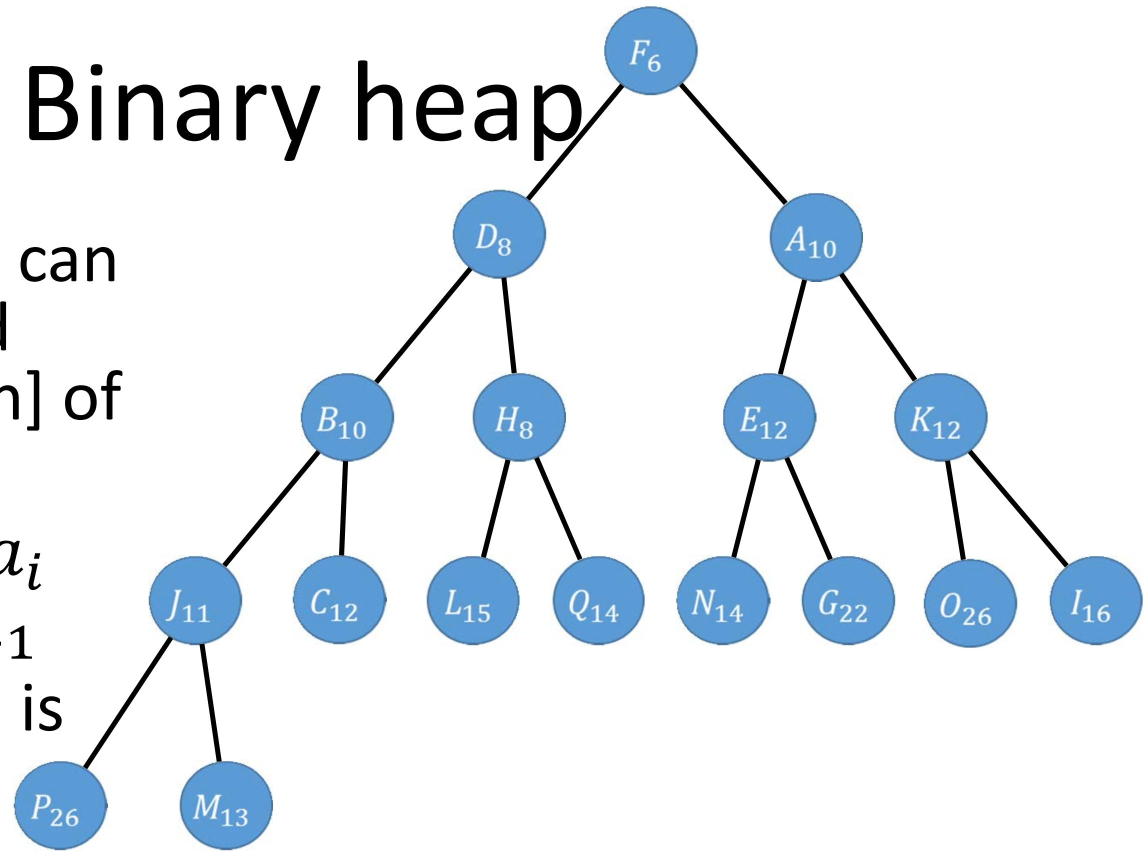
Binary heap

- A complete binary tree of objects (vertices) with the property that each key value of an object is less than the key value of its child



Binary heap

- The binary heap can be implemented with an array a[n] of vertices
- The children of a_i are a_{2i} and a_{2i+1}
- The parent of a_i is $a_{\lfloor i/2 \rfloor}$

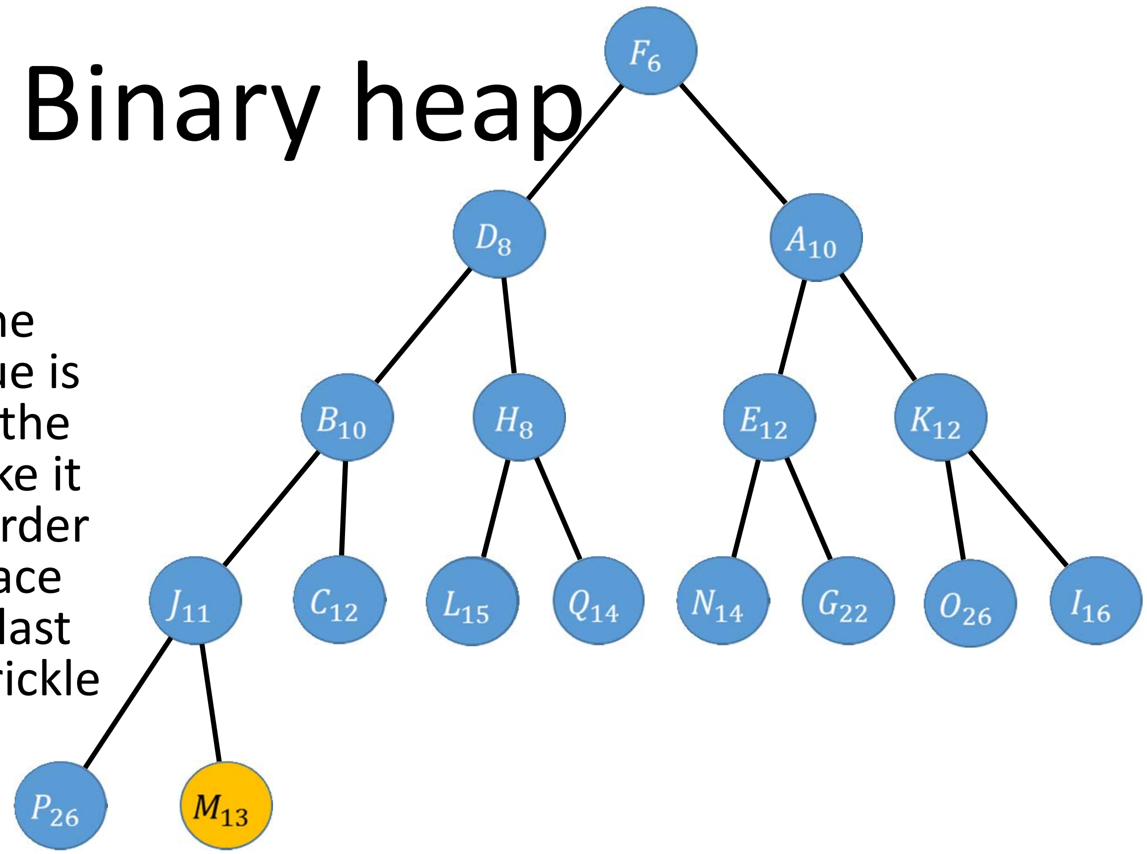


1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17

$[F_6, D_8, A_{10}, B_{10}, H_8, E_{12}, K_{12}, J_{11}, C_{12}, L_{15}, Q_{14}, N_{14}, G_{22}, O_{26}, I_{16}, P_{26}, M_{13}]$

Binary heap

- deletemin
  - The object with the minimum key value is guaranteed to be the root. Once you take it out, you must reorder the tree. You replace the root with the last object and let it trickle down.



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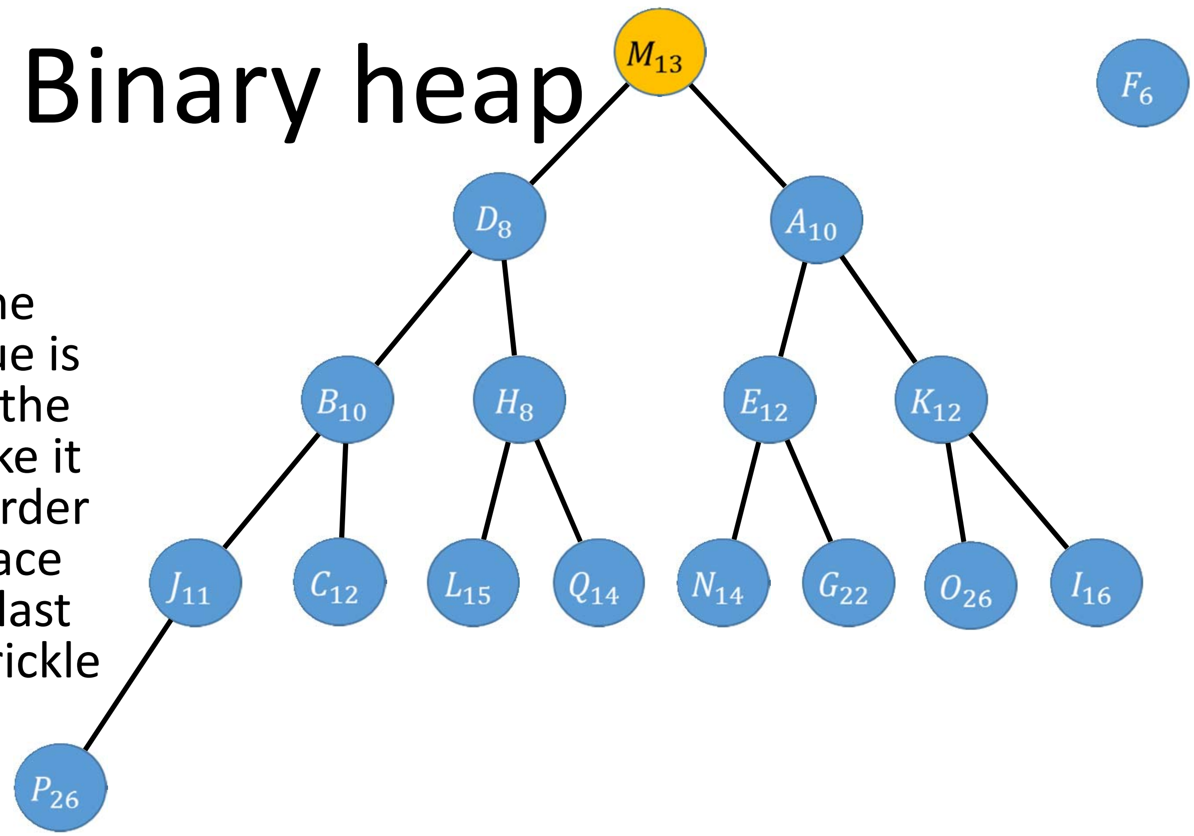

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16

$[M_{13}, D_8, A_{10}, B_{10}, H_8, E_{12}, K_{12}, J_{11}, C_{12}, L_{15}, Q_{14}, N_{14}, G_{22}, O_{26}, I_{16}, P_{26}]$

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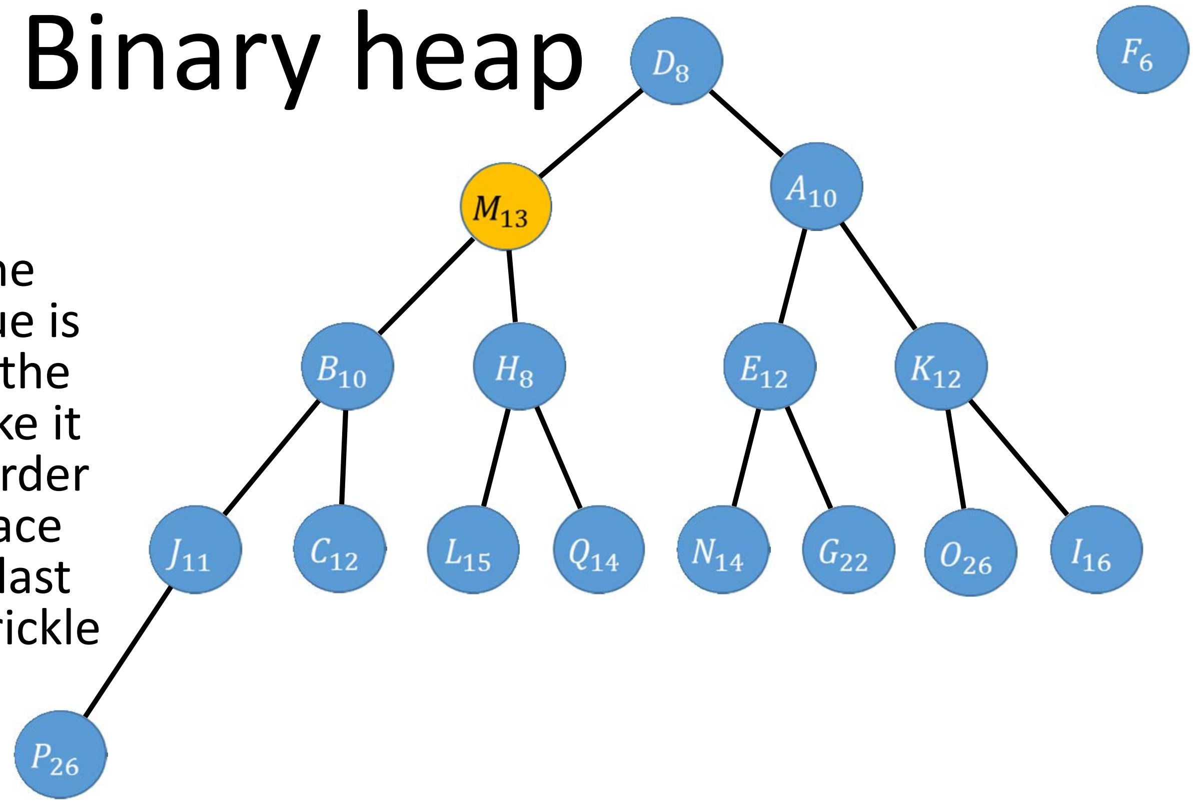

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$[D_8, M_{13}, A_{10}, B_{10}, H_8, E_{12}, K_{12}, J_{11}, C_{12}, L_{15}, Q_{14}, N_{14}, G_{22}, O_{26}, I_{16}, P_{26}]$

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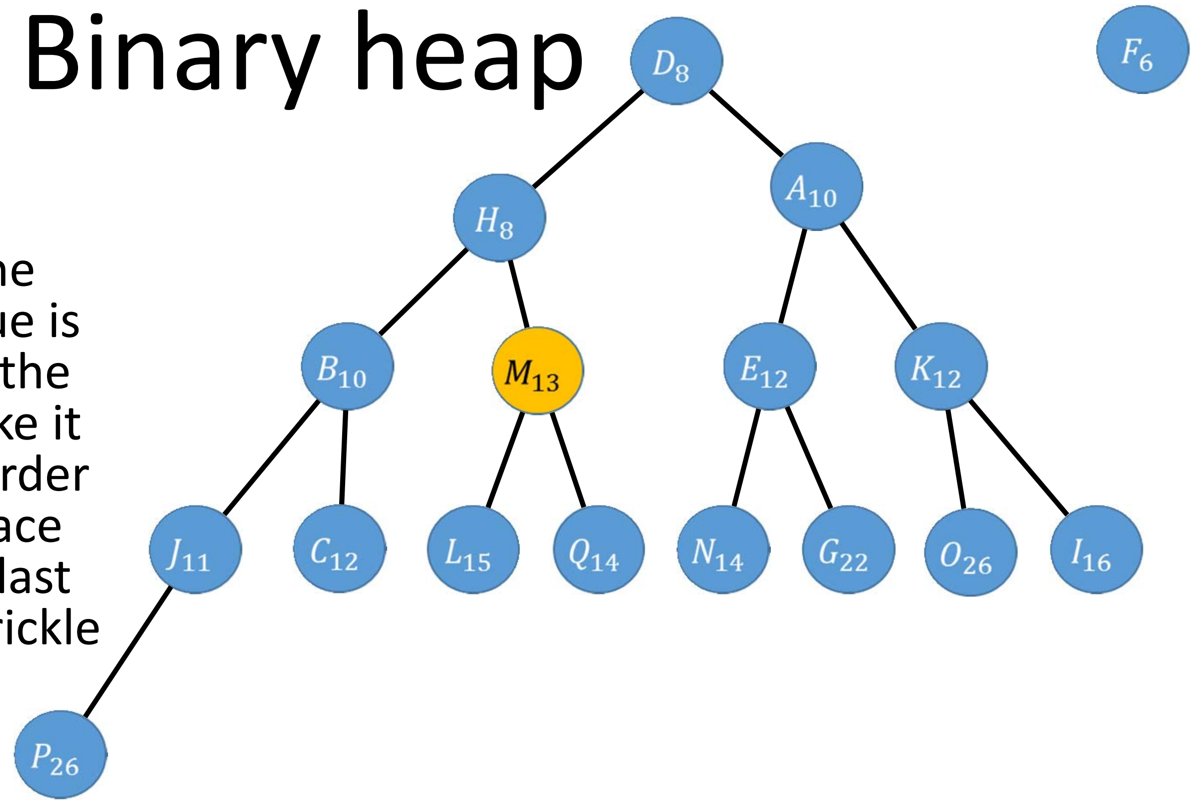


1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16

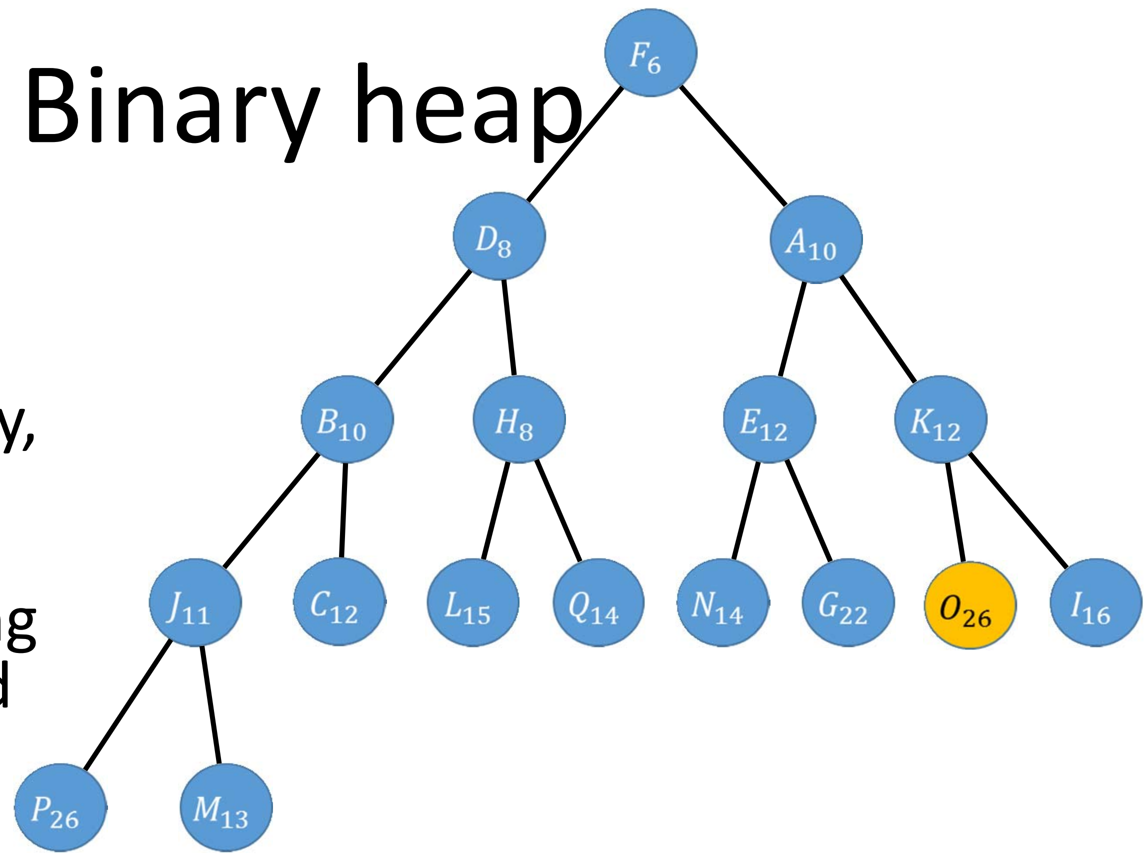
[D_8, H_8, A_{10}, B_{10}, M_{13}, E_{12}, K_{12}, J_{11}, C_{12}, L_{15}, Q_{14}, N_{14}, G_{22}, O_{26}, I_{16}, P_{26}]

Binary heap

- deletemin
  - When the last object is put in as the root, it may trickle down the entire length of the heap, the time taken is O(log(n)) where n is the number of objects in the heap
  - When performing Dijkstra's algorithm, the number of objects in the heap is |V| so the time taken is O(log(|V|))

Binary heap

- decreasekey
  - When you decrease a key, you may have to adjust the heap by having the decreased key object bubble up



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$[F_6, D_8, A_{10}, B_{10}, H_8, E_{12}, K_{12}, J_{11}, C_{12}, L_{15}, Q_{14}, N_{14}, G_{22}, O_{26}, I_{16}, P_{26}, M_{13}]$

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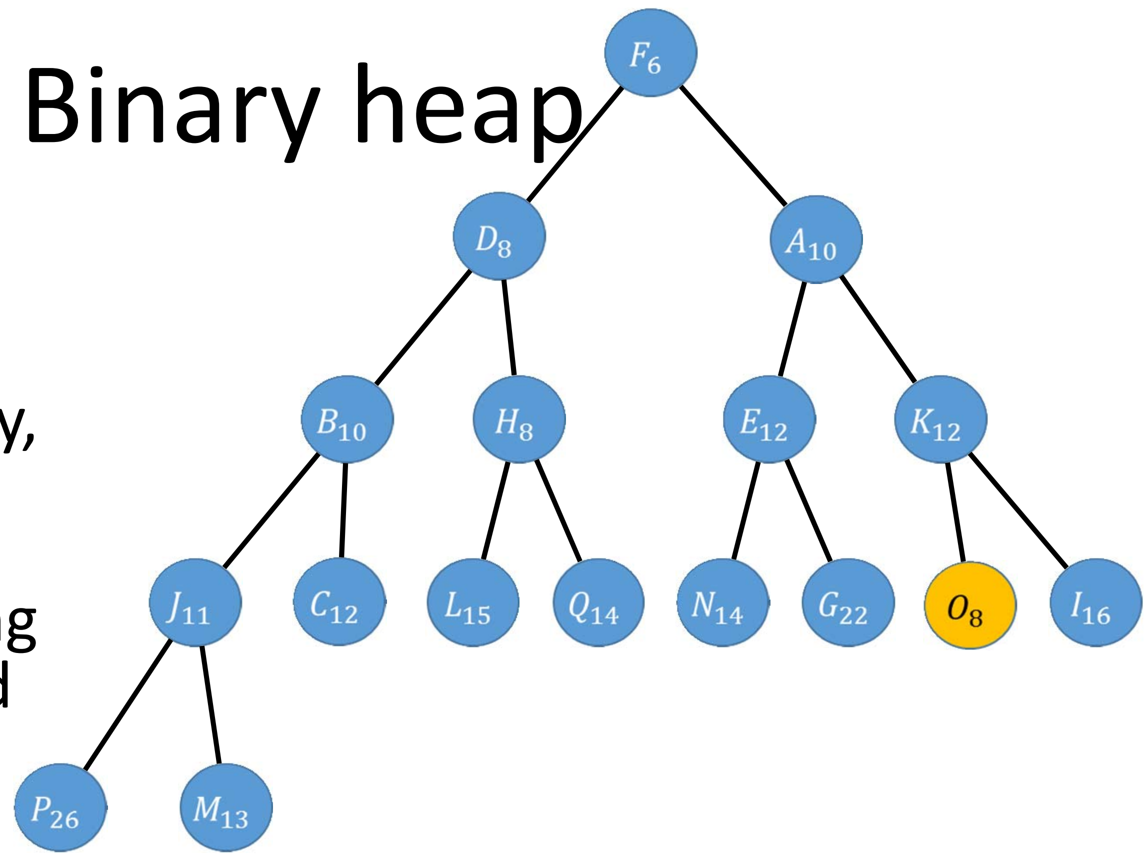

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17

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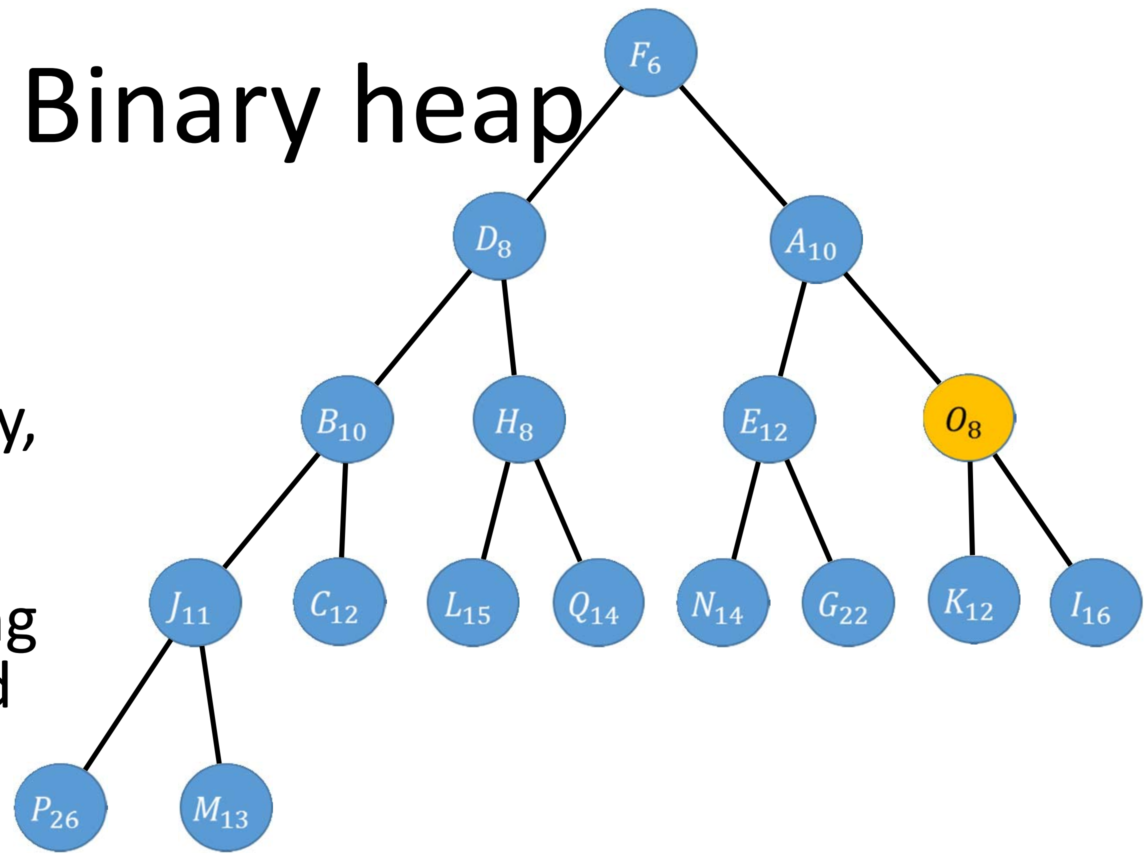

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
F_6	D_8	A_{10}	B_{10}	H_8	E_{12}	O_8	J_{11}	C_{12}	L_{15}	Q_{14}	N_{14}	G_{22}	K_{12}	I_{16}	P_{26}	M_{13}

Binary heap

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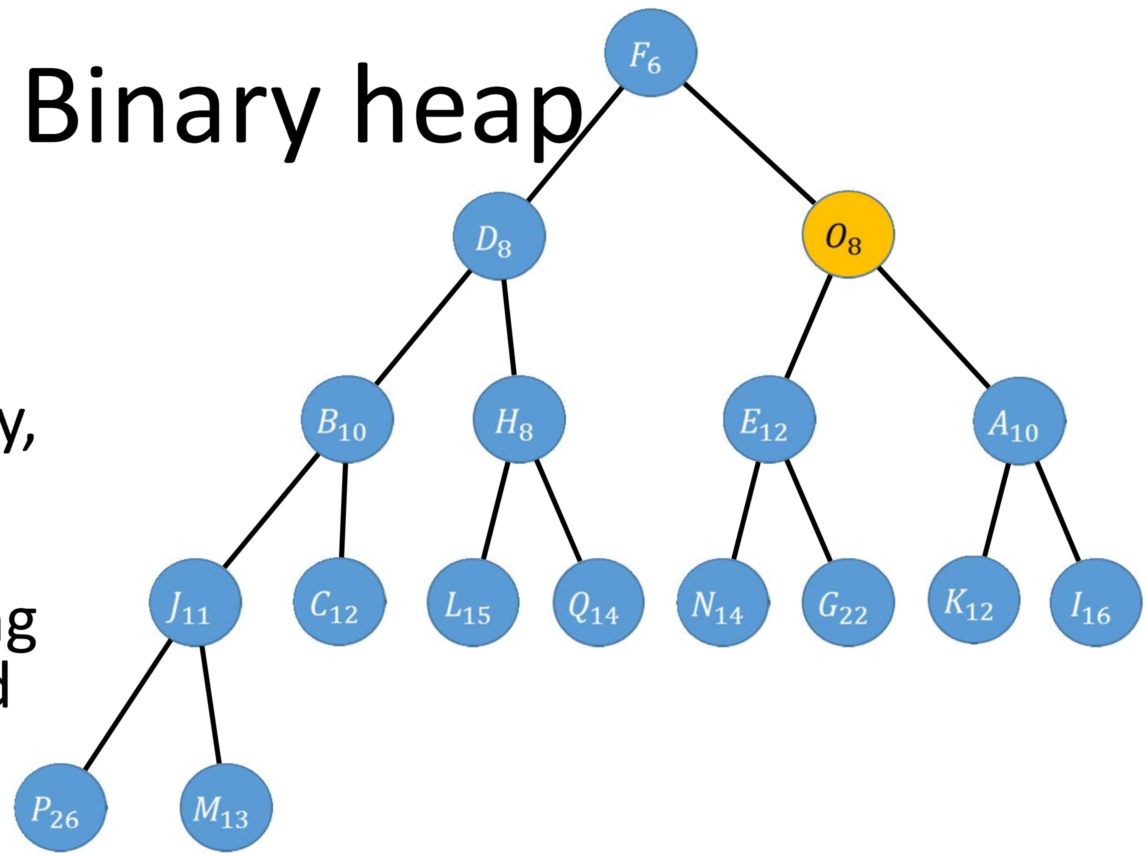


1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

$[F_6, D_8, O_8, B_{10}, H_8, E_{12}, A_{10}, J_{11}, C_{12}, L_{15}, Q_{14}, N_{14}, G_{22}, K_{12}, I_{16}, P_{26}, M_{13}]$

Binary heap

- decreasekey
  - But, how do we know where v is in the binary heap?
  - Keep a supplemental array indexed by v, and keep pointers in both directions between this array and binary heap elements

Binary heap

- decreasekey
  - When the object decreases key, it may bubble up the entire heap, the time taken is O(log(n)) where n is the number of objects in the heap
  - When performing Dijkstra's algorithm, the number of objects in the heap is |V| so the time taken is O(log(|V|))

Binary heap as a priority queue

- Dijkstra's algorithm

  makequeue + deletemin × |V| + decreasekey × |E|

  – If we use a binary heap, then it will take

    $O(|V|) + O(\log(|V|))\ |V| + O(\log(|V|))\ |E|$

    $= O((|V| + |E|) \log(|V|))$

Dijkstra's algorithm with different priority queues

- Runtime of array

  $O(|V|^2)$

- Runtime of binary heap

  $O((|V| + |E|) \log(|V|))$

- Runtime of Fibonacci heap

  $O(|V| \log(|V|) + |E|)$

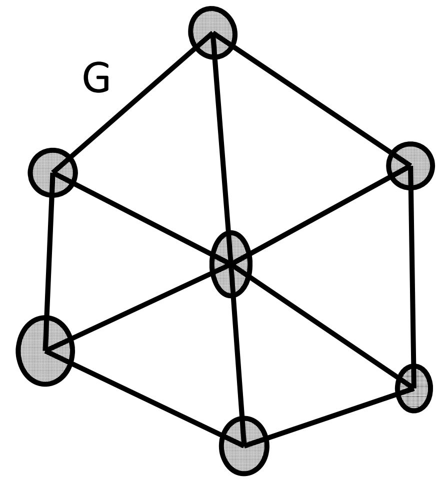
Advantages of flexibility

- By working with a higher level version of the algorithm, we can understand what is going on without getting caught up in details
- Flexibility allows us to fit data structures to be efficient for the given circumstance

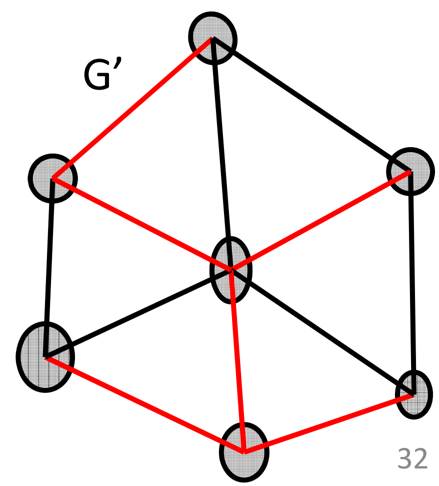
	Array	Binary heap
Dense graphs: $\|E\| = O(\|V\|^2)$	$O(\|V\|^2)$	$O(\|V\|^2 \log \|V\|)$
Sparse graphs	$O(\|V\|^2)$	$O((\|V\| + \|E\|) \log \|V\|)$

MINIMUM SPANNING TREES

Spanning trees



- A spanning tree of an undirected graph G=(V,E) is a subgraph G'=(V,E') such that G' is a tree and all vertices in V are connected
- An output tree of depth-first search or breadth-first search is a spanning tree



Example

- Suppose you have a network of computers that were linked pairwise
- Suppose each link has a positive maintenance cost
- Your job is to cut some links so that the cost of the network is minimized and the network stays connected

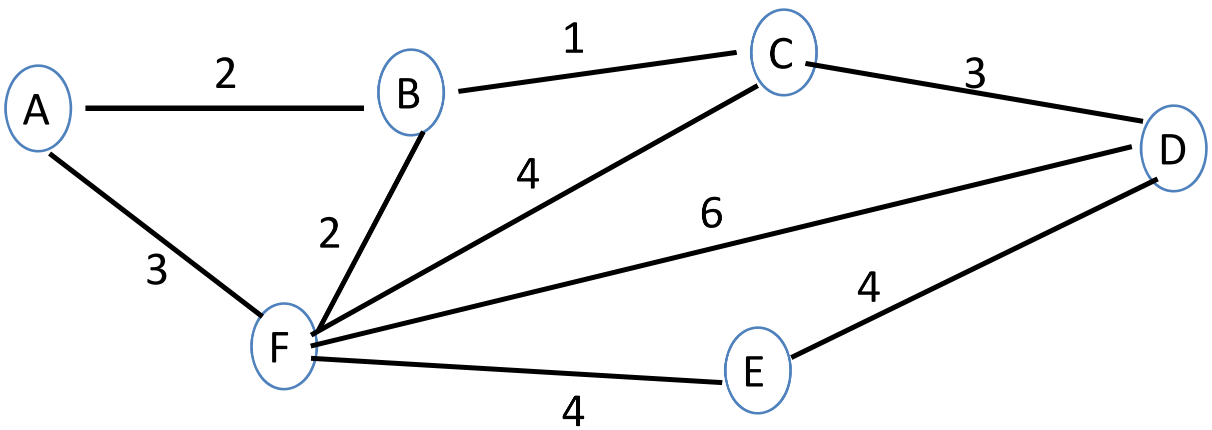
The minimized graph is a tree

- When is a connected undirected graph not a tree?

The minimized graph is a tree

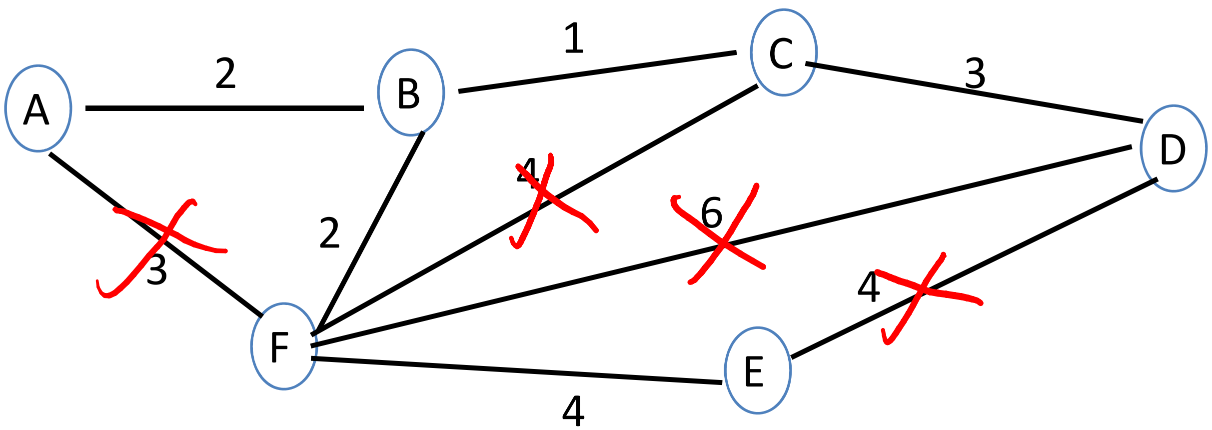
- When is a connected undirected graph not a tree?
  - Suppose it was not a tree -> there is a cycle
  - Removing any cycle edge does not disconnect the graph
  - Removing any cycle edge will decrease cost without disconnection

Example



Greedy rule: which edge should we pick first?

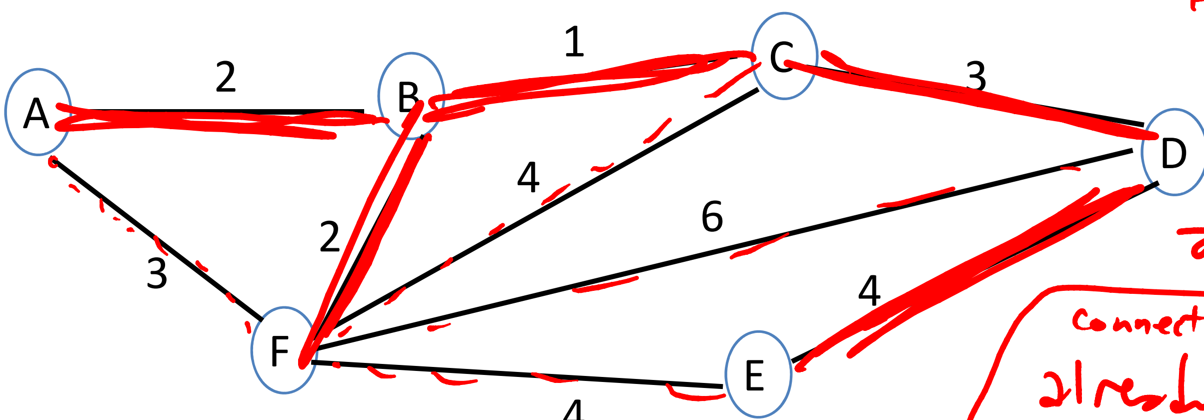
Example



delete the maximum edge if it doesn't disconnect the graph.

Greedy rule: which edge should we pick first?

Example



Keep adding in the smallest edge as long as it doesn't

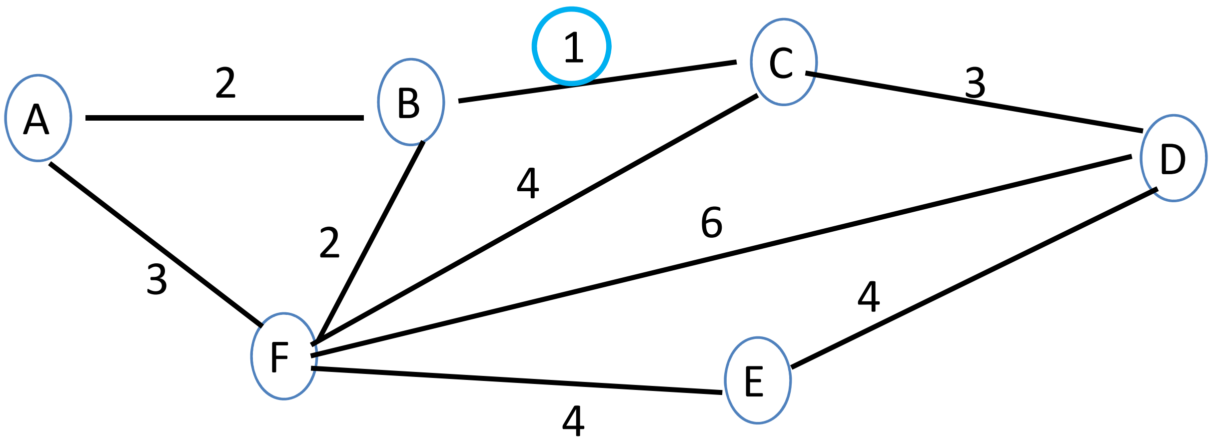
connect two vertices already connected

create a cycle

until graph is connected.

Greedy rule: which edge should we pick first?

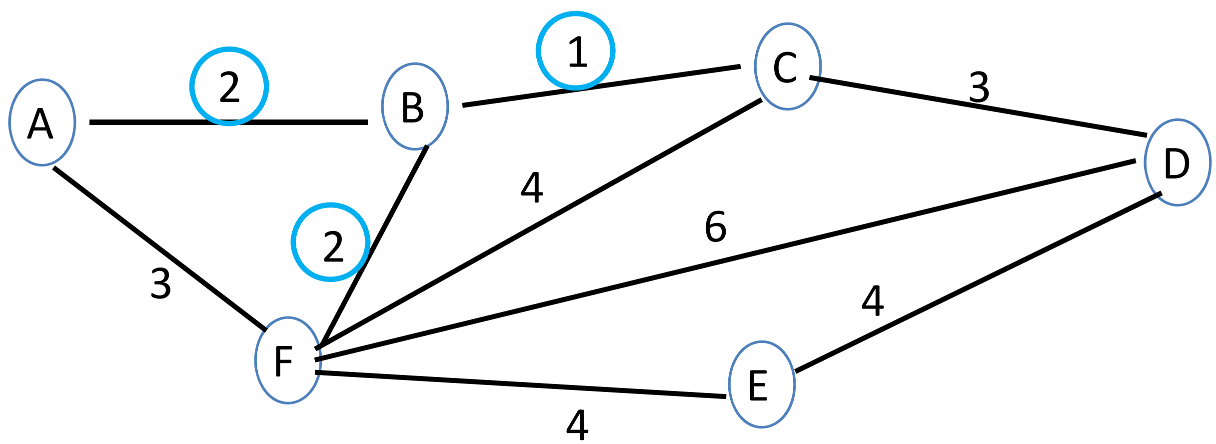
Example



Greedy rule: which edge should we pick first?

Pick the smallest weight edge

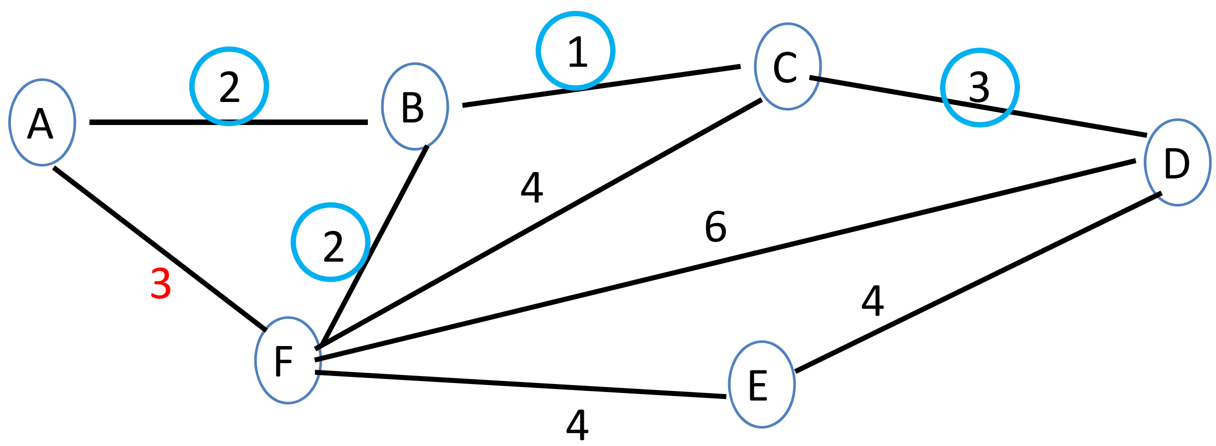
Example



Greedy rule: which edge should we pick first?

Pick the smallest weight edge

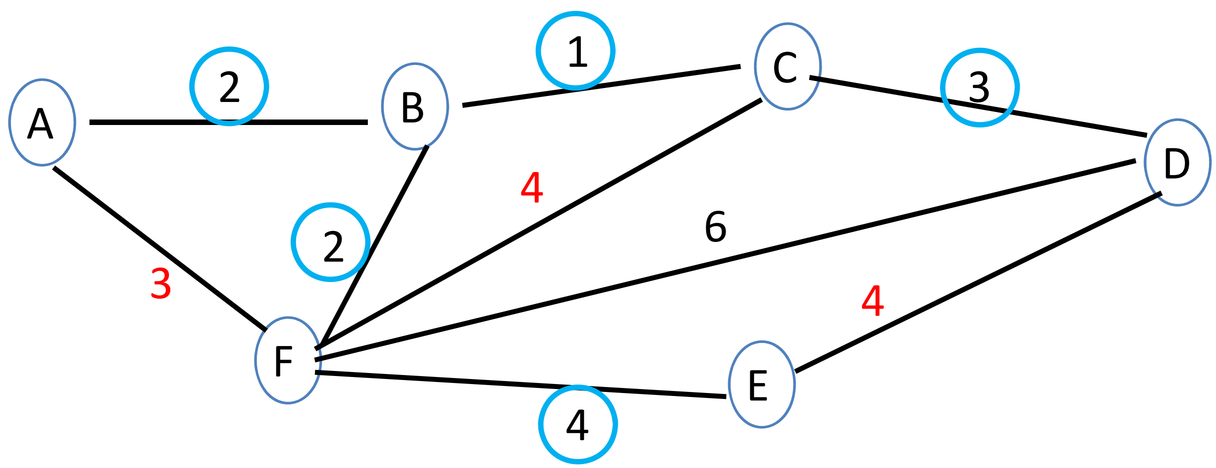
Example



Greedy rule: which edge should we pick first?

Pick the smallest weight edge unless its vertices are already connected

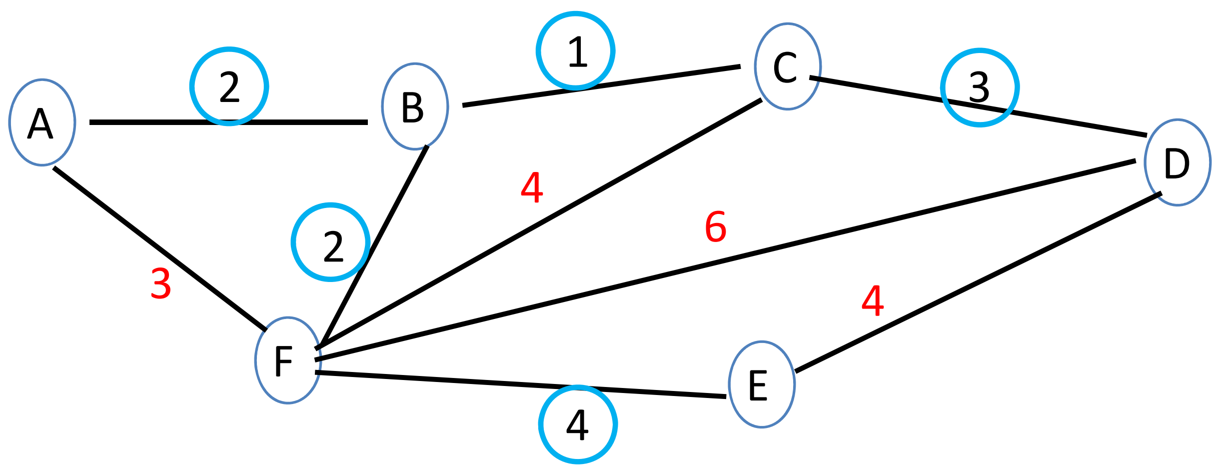
Example



Greedy rule: which edge should we pick first?

Pick the smallest weight edge unless its vertices are already connected

Example

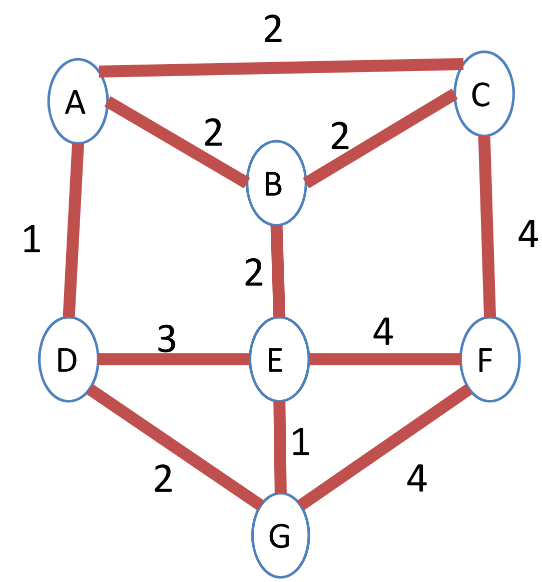


Greedy rule: which edge should we pick first?

Pick the smallest weight edge unless its vertices are already connected

Kruskal's algorithm for finding the minimum spanning tree

- Start with a graph with only the vertices (no edges)
- Repeatedly add the next lightest edge that does not form a cycle



High-level to mid-level

- High level Kruskal's algorithm
  - Given an undirected, connected graph with positive edge weights
  - Start with only the vertices
  - Repeat until graph is connected:
    - Add the lightest edge that does not create a cycle
      - Run depth-first search

- With your neighbors, discuss how to implement Kruskal's algorithm
  - At this level, just describe what you want the data structures to do, not actually how to implement them. We will leave those details until later.

High-level to mid-level

- High level Kruskal's algorithm
  - Given an undirected, connected graph with positive edge weights
  - Start with only the vertices
  - Repeat until graph is connected:
    - Add the lightest edge that does not create a cycle
      - Run depth-first search

- How do we know which edge is the lightest edge?

- How do we know that particular edge does not create a cycle?

High-level to mid-level

- High level Kruskal's algorithm
  - Given an undirected, connected graph with positive edge weights
  - Start with only the vertices
  - Repeat until graph is connected:
    - Add the lightest edge that does not create a cycle
      - Run depth-first search

- How do we know which edge is the lightest edge?
  - Sort
- How do we know that particular edge does not create a cycle?
  - Remove *e* from the graph to get a new graph G'. Run explore on G' from one of the end-points of *e* and if the other endpoint is visited, then return True, else return False.

How to implement Kruskal’s algorithm

Start with an empty graph R (only vertices, no edges)
Sort edges by weight from smallest to largest O(|E| log|E|)
For each edge e in sorted order:
If e does not create a cycle in R **then** O(|V| + |E|), |E| times
Add e to R
otherwise
do not add e to R

Total time: O(|E| log|E|) + O(|E| (|V| + |E|))

- How do we tell if adding an edge will create a cycle?
  - Remove e from the graph to get a new graph G’. Run explore on G’ from one of the end-points of e and if the other endpoint is visited, then return True, else return False. O(|V| + |E|)

Telling if an edge is in a cycle

- How do we tell if adding an edge will create a cycle?
- Homework 2, problem 2. $O(|\mathrm{V}| + |\mathrm{E}|)$
- Need to test for every edge, $|\mathrm{E}|$ times

How to implement Kruskal's algorithm

Start with an empty graph R (only vertices, no edges)
Sort edges by weight from smallest to largest O(|E| log|E|)
  For each edge e in sorted order:
    If e does not create a cycle in R **then** O(|V| + |E|) = O(|E|) see below
      Add e to R
    otherwise
      do not add e to R

Total time: O(|E| log|E| + |E|^2) = O(|E|^2)

- Every graph we want to check is a forest so there are O(|V|) edges
- Since input is connected, |V| = O(|E|)
- In the worst case, we have to check every edge, so |E| times

Disjoint sets data structure (DSDS)

- What can it do?
- Given a set of objects, DSDS manage partitioning the set into disjoint subsets
- It does the following operations:
  - makeset(S): puts each element of S into a set by itself
  - find(u): returns the name of the subset containing u
  - union(u,v): unions the set containing u with the set containing v

Kruskal's algorithm using a DSDS

```
procedure kruskal(G,w)
  Input: undirected connected graph G with edge weights w
  Output a set of edges X that defines a minimum spanning tree of G
makeset(V)
X = { }
Sort the edges in E in increasing order by weight
For all edges (u,v) in E
  if find(u) ≠ find(v):     Separate connected components
    Add edge (u,v) to X
    union(u,v)
```

Next lecture

- Minimum spanning trees and union-find
  - Reading: Section 5.1