Divide and Conquer Algorithms

CSE 101: Design and Analysis of Algorithms

Lecture 12

CSE 101: Design and analysis of algorithms

- Divide and conquer algorithms
  - Reading: Sections 2.1 and 2.2
- Homework 5 is due today, 11:59 PM
- Quiz 2 is Nov 8, in class
  - Graph search and minimum spanning trees
- Homework 6 will be assigned Nov 13
  - No homework this week

Divide and conquer

- Break a problem into similar subproblems
- Solve each subproblem recursively
- Combine

Multiplying binary numbers, divide and conquer

- Suppose we want to multiply two n-bit numbers together where n is a power of 2
- One way we can do this is by splitting each number into their left and right halves which are each n/2 bits long
- x= | xL | xR |
- y= | yL | yR |

Multiplying binary numbers, divide and conquer

- Suppose we want to multiply two n-bit numbers together where n is a power of 2
- One way we can do this is by splitting each number into their left and right halves which are each n/2 bits long

$$x = 2^{n/2}xL + xR$$
$$y = 2^{n/2}yL + yR$$

Multiplying binary numbers, divide and conquer

$$x = 2^{n/2}x_L + x_R$$
$$y = 2^{n/2}y_L + y_R$$

$$xy = \left(2^{\frac{n}{2}}x_L + x_R\right)\left(2^{\frac{n}{2}}y_L + y_R\right)$$
$$xy = 2^n x_L y_L + 2^{\frac{n}{2}}(x_L y_R + x_R y_L) + x_R y_R$$

4 multiplications

Algorithm multiply

function **multiply**(x,y) T(n)
Input: n-bit integers x and y
Output: the product xy
If n=1: return xy
x_L, x_R and y_L, y_R are the left-most and right-most n/2 bits of x and y, respectively.
$P_1 = \textbf{multiply}(x_L, y_L)$ T(n/2)
$P_2 = \textbf{multiply}(x_L, y_R)$ T(n/2)
$P_3 = \textbf{multiply}(x_R, y_L)$ T(n/2)
$P_4 = \textbf{multiply}(x_R, y_R)$ T(n/2)
return $P_1 * 2^n + (P_2 + P_3) * 2^{\frac{n}{2}} + P_4$

Algorithm multiply runtime

- Let $T(n)$ be the runtime of the multiply algorithm
- Then, $T(n) = 4T\left(\frac{n}{2}\right) + O(n)$

Non-recursive part

Algorithm multiplyKS

function multiplyKS(x,y) T(n)

Input: n-bit integers x and y

Output: the product xy

If n=1: return xy

x_L, x_R and y_L, y_R are the left-most and right-most n/2 bits of x and y, respectively.

$R_1 = \mathbf{multiplyKS}(x_L, y_L)$ T(n/2)

$R_2 = \mathbf{multiplyKS}(x_R, y_R)$ T(n/2)

$R_3 = \mathbf{multiplyKS}((x_L + x_R)(y_L + y_R))$ T(n/2)

3 multiplies instead of 4

return $R_1 * 2^n + (R_3 - R_1 - R_2) * 2^{\frac{n}{2}} + R_2$

Algorithm multiplyKS runtime

- Let $T(n)$ be the runtime of the multiplyKS algorithm
- Then, $T(n) = 3T\left(\frac{n}{2}\right) + O(n)$

Non-recursive part

Recurrence of the same form

- multiply runtime $T(n) = 4T\left(\frac{n}{2}\right) + O(n)$
- multiplyKS runtime $T(n) = 3T\left(\frac{n}{2}\right) + O(n)$

Recurrence of the same form

- multiply runtime $T(n) = 4T\left(\frac{n}{2}\right) + O(n)$
- multiplyKS runtime $T(n) = 3T\left(\frac{n}{2}\right) + O(n)$
- How do you solve a recurrence of the form

$$T(n) = aT\left(\frac{n}{b}\right) + O\left(n^d\right)$$

- We will use the master theorem

Summation lemma

- Consider the summation

$$\sum_{k=0}^{n} r^k$$

Geometric series

- It behaves differently for different values of r

Summation lemma

- Consider the summation

$$\sum_{k=0}^{n} r^k$$

- It behaves differently for different values of r
- What happens if $r < 1$?

Summation lemma

- Consider the summation

$$\sum_{k=0}^{n} r^k$$

- It behaves differently for different values of r
- If $r < 1$, then this sum converges. This means that the sum is bounded above by some constant c. Therefore,

if $r < 1$, then $\sum_{k=0}^{n} r^k < c$ for all n, so $\sum_{k=0}^{n} r^k \in \boxed{O(1)}$

Summation lemma

- Consider the summation

$$\sum_{k=0}^{n} r^k$$

- It behaves differently for different values of r
- What happens if $r = 1$?

Summation lemma

- Consider the summation

$$\sum_{k=0}^{n} r^k$$

- It behaves differently for different values of r
- If $r = 1$, then this sum is just summing 1 over and over $n + 1$ times. Therefore,

$$\text{if } r = 1, \text{ then } \sum_{k=0}^{n} r^k = \sum_{k=0}^{n} 1 = n + 1 \in \boxed{O(n)}$$

Summation lemma

- Consider the summation

$$\sum_{k=0}^{n} r^k$$

- It behaves differently for different values of r
- What happens if $r > 1$?

Summation lemma

- Consider the summation

$$\sum_{k=0}^{n} r^k$$

- It behaves differently for different values of r
- If $r > 1$, then this sum is exponential with base r. Therefore,

  if $r > 1$, then $\sum_{k=0}^{n} r^k < cr^n$ for all n, so $\sum_{k=0}^{n} r^k \in \boxed{O(r^n)}$

  (note that $c > \frac{r}{r-1}$)

Summation lemma

- Consider the summation

$$\sum_{k=0}^{n} r^k$$

- It behaves differently for different values of r

$$\sum_{k=0}^{n} r^k \in \begin{cases} O(1) & \text{if } r < 1 \\ O(n) & \text{if } r = 1 \\ O(r^n) & \text{if } r > 1 \end{cases}$$

Master theorem

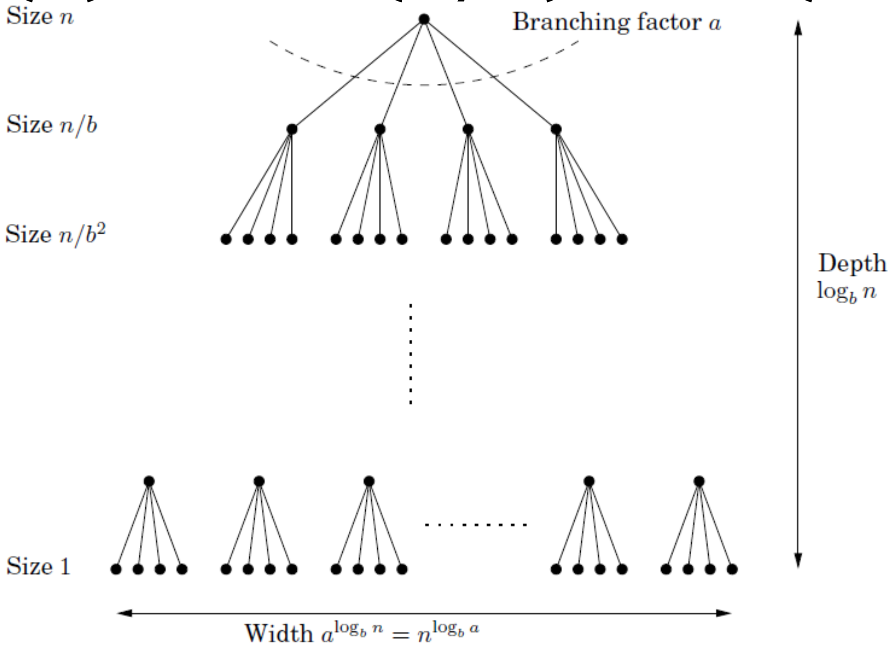
- If $T(n) = aT(n/b) + O(n^d)$ for some constants $a > 0, b > 1, d \geq 0$, then

$$T(n) \in \begin{cases} O\left(n^d\right) & \text{if } a < b^d \\ O\left(n^d \log n\right) & \text{if } a = b^d \\ O\left(n^{\log_b a}\right) & \text{if } a > b^d \end{cases}$$

- The master theorem tell us us the running times of most of the divide and conquer procedures we are likely to use

Master theorem, solving the recurrence

$$T(n) = aT(n/b) + O(n^d)$$



1 subproblem

a subproblems

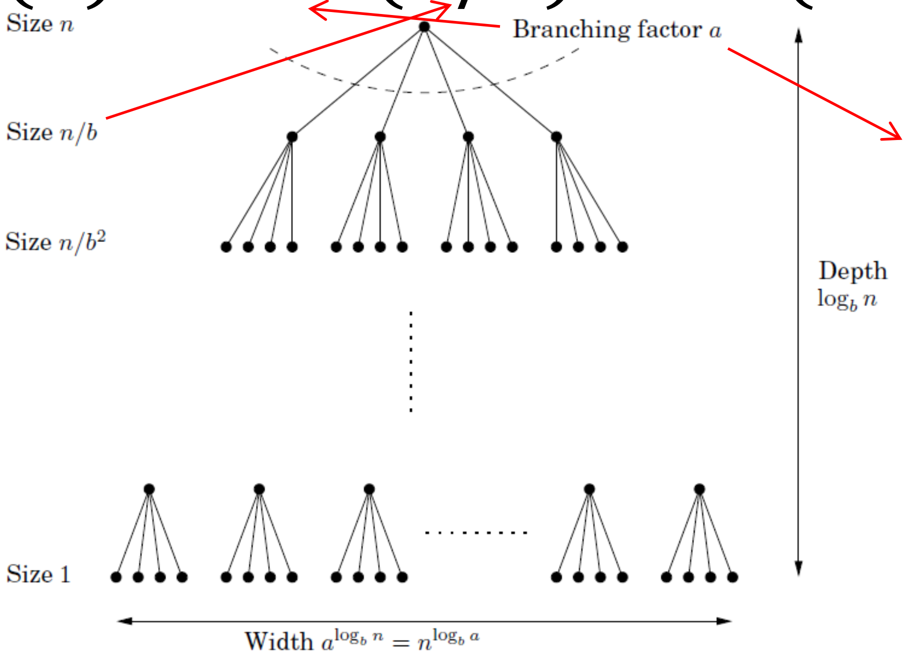
a^2 subproblems

⋮

$a^{\log_b n}$ subproblems

Master theorem, solving the recurrence

$$T(n) = aT(n/b) + O(n^d)$$



1 subproblem

a subproblems

a^2 subproblems

⋮

$a^{\log_b n}$ subproblems

Time analysis, multiply

Adding n bit numbers = cn time

n/2 bit adds=cn/2

n/2 bit adds=cn/2

n/2 bit adds=cn/2

n/2 bit adds=cn/2

Total 1^{st} level down : 4 * cn/2 = 2cn

cn/4 cn/4 cn/4 cn/4 cn/4 cn/4 cn/4 cn/4 cn/4 cn/4 cn/4 cn/4 cn/4 cn/4 cn/4 cn/4

Total 2^{nd} level down: 16 * cn/4 = 4cn

Time analysis, multiplyKS

Adding n bit numbers = cn time

n/2 bit adds=cn/2

n/2 bit adds=cn/2

n/2 bit adds=cn/2

Total 1st level down : 3 * cn/2 = (1.5) cn

cn /4 cn /4 cn /4

cn /4 cn /4 cn /4

cn /4 cn /4 cn /4

Total 2nd level down: 9 * cn/4 = (1.5)^2 cn

Master theorem, solving the recurrence

- After k levels, there are a^k subproblems, each of size n/b^k
- So, during the kth level of recursion, the time complexity is

$$O\left(\left(\frac{n}{b^k}\right)^d\right)a^k = O\left(a^k\left(\frac{n}{b^k}\right)^d\right)$$

$$= O\left(n^d\left(\frac{a}{b^d}\right)^k\right)$$

Master theorem, solving the recurrence

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$$= O\left(n^d\left(\frac{a}{b^d}\right)^k\right)$$

- After $\log_b n$ levels, the subproblem size is reduced to 1, which usually is the size of the base case
- So, the entire algorithm is a sum of each level

$$T(n) = O\left(n^d \sum_{k=0}^{\log_b n}\left(\frac{a}{b^d}\right)^k\right)$$

Master theorem, solving the recurrence

- After k levels, there are a^k subproblems, each of size n/b^k
- So, during the kth level of recursion, the time complexity is

$$O\left(\left(\frac{n}{b^k}\right)^d\right)a^k = O\left(a^k\left(\frac{n}{b^k}\right)^d\right)$$

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- So, the entire algorithm is a sum of each level

$$T(n) = O\left(n^d \sum_{k=0}^{\log_b n}\left(\frac{a}{b^d}\right)^k\right)$$

Master theorem, proof

$$T(n) = O\left(n^d \sum_{k=0}^{\log_b n} \left(\frac{a}{b^d}\right)^k \right)$$

- Case 1: $a < b^d$
- Then, $\frac{a}{b^d} < 1$ and the series converges to a constant so

$$T(n) = O\left(n^d\right)$$

Master theorem, proof

$$T(n) = O\left(n^d \underbrace{\sum_{k=0}^{\log_b n} \left(\frac{a}{b^d}\right)^k}_{O(1)}\right)$$

- Case 1: $a < b^d$
- Then, $\frac{a}{b^d} < 1$ and the series converges to a constant so

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Master theorem, proof

$$T(n) = O\left(n^d \sum_{k=0}^{\log_b n} \left(\frac{a}{b^d}\right)^k\right)$$

- Case 2: $a = b^d$
- Then, $\frac{a}{b^d} = 1$ and so each term is equal to 1 so

$$T(n) = O\left(n^d \log_b n\right)$$

Master theorem, proof

$$T(n) = O\left(n^d \sum_{k=0}^{\log_b n} \underbrace{\left(\frac{a}{b^d}\right)^k}_{1} \right)$$

- Case 2: $a = b^d$
- Then, $\frac{a}{b^d} = 1$ and so each term is equal to 1 so

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Master theorem, proof

$$T(n) = O\left(n^d \sum_{k=0}^{\log_b n} \left(\frac{a}{b^d}\right)^k\right)$$

- Case 3: $a > b^d$
- Then, the summation is exponential and grows proportional to its last term $\left(\frac{a}{b^d}\right)^{\log_b n}$ so

$$T(n) = O\left(n^d \left(\frac{a}{b^d}\right)^{\log_b n}\right) = O\left(n^{\log_b a}\right)$$

Master theorem, proof

$$T(n) = O\left(n^d \sum_{k=0}^{\log_b n} \left(\frac{a}{b^d}\right)^k\right)$$

- Case 3: $a > b^d$
- Then, the summation is exponential and grows proportional to its last term $\left(\frac{a}{b^d}\right)^{\log_b n}$ so

$$T(n) = O\left(n^d \left(\frac{a}{b^d}\right)^{\log_b n}\right) = O\left(n^{\log_b a}\right)$$ Exercise

Master theorem

- If $T(n) = aT(n/b) + O(n^d)$ for some constants $a > 0, b > 1, d \geq 0$, then

$$T(n) \in \begin{cases} O(n^d) & \text{if } a < b^d \\ O(n^d \log n) & \text{if } a = b^d \\ O(n^{\log_b a}) & \text{if } a > b^d \end{cases}$$

Top-heavy

Steady-state

Bottom-heavy

Master theorem applied to multiply

$$T(n) = aT(n/b) + O(n^d)$$

$$T(n) \epsilon \begin{cases} O\left(n^d\right) & \text{if } a < b^d \\ O\left(n^d \log n\right) & \text{if } a = b^d \\ O\left(n^{\log_b a}\right) & \text{if } a > b^d \end{cases}$$

- The recursion for the runtime of multiply is

$$T(n) = 4T\left(\frac{n}{2}\right) + O(n)$$

- So, we have that $a = 4$, $b = 2$, and $d = 1$. In this case, $a > b^d$ so

$$T(n) \epsilon O\left(n^{\log_2 4}\right) = O(n^2)$$

- Not any improvement on grade-school method

Master theorem applied to multiply

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Master theorem applied to multiplyKS

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- The recursion for the runtime of multiplyKS is

$$T(n) = 3T\left(\frac{n}{2}\right) + O(n)$$

- So, we have that $a = 3$, $b = 2$, and $d = 1$. In this case, $a > b^d$ so

$$T(n) \epsilon O\left(n^{\log_2 3}\right) = O(n^{1.58})$$

- An *improvement* on grade-school method

Master theorem applied to multiplyKS

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- An *improvement* on grade-school method

Can we do better than $n^{1.58}$?

- Could any multiplication algorithm have a faster asymptotic runtime than $\Theta(n^{1.58})$?
- Ideas

Can we do better than $n^{1.58}$?

- Instead of splitting the numbers in half, we split them into thirds
- x= | xL | xM | xR |
- y= | yL | yM | yR |

Can we do better than $n^{1.58}$?

- Instead of splitting the numbers in half, we split them into thirds
- $x = 2^{2n/3}xL + 2^{n/3}xM + xR$
- $y = 2^{2n/3}yL + 2^{n/3}yM + yR$

Multiplying trinomials

$$(ax^2 + bx + c)(dx^2 + ex + f)$$
$$= adx^4 + (ae + bd)x^3 + (af + be + cd)x^2 + (bf + ce)x + cf$$

- How many multiplications?

Multiplying trinomials

$$(ax^2 + bx + c)(dx^2 + ex + f)$$
$$= adx^4 + (ae + bd)x^3 + (af + be + cd)x^2 + (bf + ce)x + cf$$

- 9 multiplications means 9 recursive calls
- Each multiplication is 1/3 the size of the original

$$T(n) = 9T\left(\frac{n}{3}\right) + O(n)$$

Multiplying trinomials

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Multiplying trinomials

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$$= adx^4 + (ae + bd)x^3 + (af + be + cd)x^2 + (bf + ce)x + cf$$

$$T(n) = 9T\left(\frac{n}{3}\right) + O(n)$$

- Master theorem

Same form

$$T(n) = aT\left(\frac{n}{b}\right) + O\left(n^d\right)$$

Master theorem applied to multiplying trinomials

$$T(n) = aT(n/b) + O(n^d)$$

$$T(n) \in \begin{cases} O\left(n^d\right) & \text{if } a < b^d \\ O\left(n^d \log n\right) & \text{if } a = b^d \\ O\left(n^{\log_b a}\right) & \text{if } a > b^d \end{cases}$$

- The recursion for the runtime of multiplying trinomials is

$$T(n) = 9T\left(\frac{n}{3}\right) + O(n)$$

- So, we have that $a = 9$, $b = 3$, and $d = 1$. In this case, $a > b^d$ so

$$T(n) \in O\left(n^{\log_3 9}\right) = O(n^2)$$

Master theorem applied to multiplying trinomials

$$T(n) = aT(n/b) + O(n^d)$$

$$T(n) \epsilon \begin{cases} O(n^d) & \text{if } a < b^d \\ O(n^d \log n) & \text{if } a = b^d \\ O(n^{\log_b a}) & \text{if } a > b^d \end{cases}$$

- The recursion for the runtime of multiplying trinomials is

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$$T(n) \epsilon O\left(n^{\log_3 9}\right) = O(n^2)$$

Multiplying trinomials

$$(ax^2 + bx + c)(dx^2 + ex + f)$$
$$= adx^4 + (ae + bd)x^3 + (af + be + cd)x^2 + (bf + ce)x + cf$$

- When multiplying binomials, we can reduce the number of multiplications from 4 to 3
- Similarly, when multiplying trinomials, we can reduce the number of multiplications from 9 to 5
- Then, the recursion becomes

$$T(n) = 5T\left(\frac{n}{3}\right) + O(n)$$

Multiplying trinomials

$$(ax^2 + bx + c)(dx^2 + ex + f)$$
$$= adx^4 + (ae + bd)x^3 + (af + be + cd)x^2 + (bf + ce)x + cf$$

- When multiplying binomials, we can reduce the number of multiplications from 4 to 3
- Similarly, when multiplying trinomials, we can reduce the number of multiplications from 9 to 5
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Master theorem applied to multiplying trinomials

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$$T(n) \epsilon \begin{cases} O\left(n^d\right) & \text{if } a < b^d \\ O\left(n^d \log n\right) & \text{if } a = b^d \\ O\left(n^{\log_b a}\right) & \text{if } a > b^d \end{cases}$$

- The recursion for the runtime of multiplying trinomials is

$$T(n) = 5T\left(\frac{n}{3}\right) + O(n)$$

- So, we have that $a = 5$, $b = 3$, and $d = 1$. In this case, $a > b^d$ so

$$T(n) \epsilon O\left(n^{\log_3 5}\right) = O(n^{1.43})$$

Master theorem applied to multiplying trinomials

$$T(n) = aT(n/b) + O(n^d)$$

$$T(n) \epsilon \begin{cases} O(n^d) & \text{if } a < b^d \\ O(n^d \log n) & \text{if } a = b^d \\ O(n^{\log_b a}) & \text{if } a > b^d \end{cases}$$

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$$T(n) \epsilon O\left(n^{\log_3 5}\right) = O(n^{1.43})$$

Dividing into k subproblems

- What happens if we divide into k subproblems each of size n/k?

$$(a_{k-1}x^{k-1} + a_{k-2}x^{k-2} + \cdots + a_1x + a_0)(b_{k-1}x^{k-1} + b_{k-2}x^{k-2} + \cdots + b_1x + b_0)$$

- How many terms (multiplications)?

Dividing into k subproblems

- What happens if we divide into k subproblems each of size n/k?

$$(a_{k-1}x^{k-1} + a_{k-2}x^{k-2} + \cdots + a_1 x + a_0)(b_{k-1}x^{k-1} + b_{k-2}x^{k-2} + \cdots + b_1 x + b_0)$$

- How many multiplications (terms)?
- There are k^2 multiplications. The recursion is

$$T(n) = k^2 T\left(\frac{n}{k}\right) + O(n)$$

Master theorem applied

$$T(n) = aT(n/b) + O(n^d)$$
$$T(n) \in \begin{cases} O\left(n^d\right) & \text{if } a < b^d \\ O\left(n^d \log n\right) & \text{if } a = b^d \\ O\left(n^{\log_b a}\right) & \text{if } a > b^d \end{cases}$$

- The recursion for the runtime is

$$T(n) = k^2 T\left(\frac{n}{k}\right) + O(n)$$

- So, we have that $a = k^2$, $b = \mathrm{k}$, and $d = 1$. In this case, $a > b^d$ so

$$T(n) \in O\left(n^{\log_{\mathrm{k}} k^2}\right) = O(n^2)$$

Master theorem applied

$$T(n) = aT(n/b) + O(n^d)$$

$$T(n) \in \begin{cases} O(n^d) & \text{if } a < b^d \\ O(n^d \log n) & \text{if } a = b^d \\ O(n^{\log_b a}) & \text{if } a > b^d \end{cases}$$

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- So, we have that $a = k^2$, $b = \mathrm{k}$, and $d = 1$. In this case, $a > b^d$ so

$$T(n) \in O\left(n^{\log_{\mathrm{k}} k^2}\right) = O(n^2)$$

Cook-Toom algorithm

- In fact, if you split up your number into k equally sized parts, then you can combine them with 2k-1 multiplications instead of the k^2 individual multiplications

3 instead of 4, k = 2, binomials
5 instead of 9, k = 3, trinomials

- This means that you can get an algorithm that runs in

$$T(n) = (2k-1)T\left(\frac{n}{k}\right) + O(n)$$

Master theorem applied

$$T(n) = aT(n/b) + O(n^d)$$

$$T(n) \in \begin{cases} O(n^d) & \text{if } a < b^d \\ O(n^d \log n) & \text{if } a = b^d \\ O(n^{\log_b a}) & \text{if } a > b^d \end{cases}$$

- The recursion for the runtime is

$$T(n) = (2k-1)T\left(\frac{n}{k}\right) + O(n)$$

- So, we have that $a = 2k - 1$, $b = \mathrm{k}$, and $d = 1$. In this case, $a > b^d$ so

$$T(n) \in O\left(n^{\log_{\mathrm{k}}(2k-1)}\right) = O\left(n^{\frac{\log(2k-1)}{\log k}}\right)$$

Master theorem applied

$$T(n) = aT(n/b) + O(n^d)$$

$$T(n) \epsilon \begin{cases} O(n^d) & \text{if } a < b^d \\ O(n^d \log n) & \text{if } a = b^d \\ O(n^{\log_b a}) & \text{if } a > b^d \end{cases}$$

- The recursion for the runtime is

$$T(n) = (2k - 1)T\left(\frac{n}{k}\right) + O(n)$$

- So, we have that $a = 2k - 1$, $b =$ k, and $d = 1$. In this case, $a > b^d$ so

$$T(n) \epsilon O\left(n^{\log_k(2k-1)}\right) = O\left(n^{\frac{\log(2k-1)}{\log k}}\right)$$

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$\log_2 3 = 1.58$ binomials

$\log_3 5 = 1.43$ trinomials

Cook-Toom algorithm

- $T(n) = (2k-1)T\left(\frac{n}{k}\right) + O(n)$
- $T(n) = O\left(n^{\frac{\log(2k-1)}{\log k}}\right)$
- We can have a near-linear time algorithm if we take k to be sufficiently large. The O(n) term in the recursion takes a lot of time the bigger k gets. So is it worth it to make k very large?

Next lecture

- Divide and conquer algorithms
  - Reading: Section 2.6