On the virtues of the cumulative distribution function

matlab code

```
function [f,x] = cdist(data)

x=sort(data);
f=[0, 0:(1/length(x)):1];
x=[x(1); x(:); x(end)];
```

CDFs and sorting

- To Create a CDF you need to sort
- Can knowing the CDF help you sort?

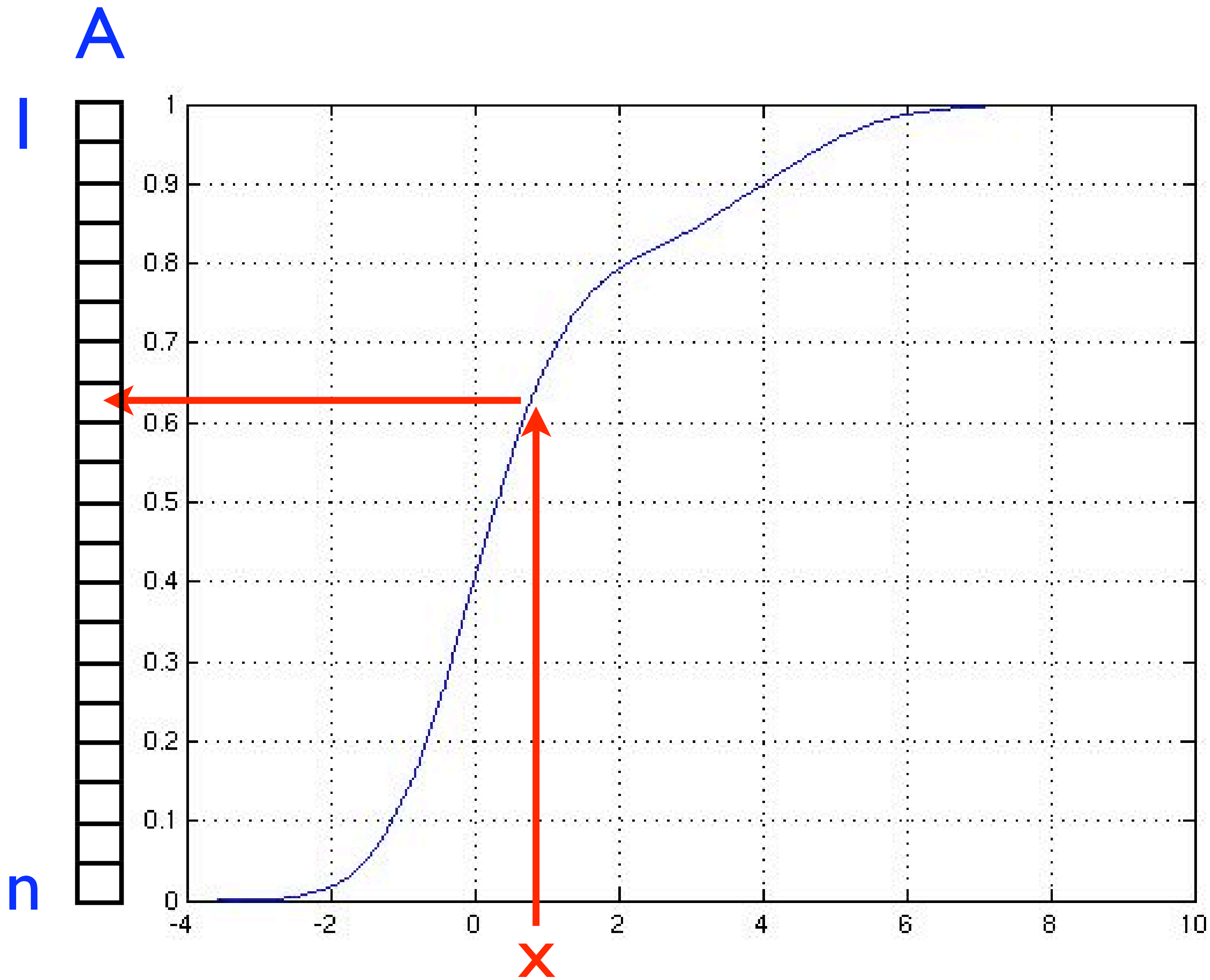
No. of comparisons needed for sorting

- There are $n!$ permutations
- Identifying the correct permutation requires $\log(n!)$ comparisons.
- $\log(n!) \sim n \log n$
- Merge-sort achieves time $O(n \log n)$

Efficient sorting for uniform distribution

- n Elements drawn IID from uniform distribution over $[0,1]$
- A = array[1:n] of lists
- given x insert it into list at $A[\text{floor}(x\ n)]$
- Lists would rarely have > 1 element
- time: $O(n)$

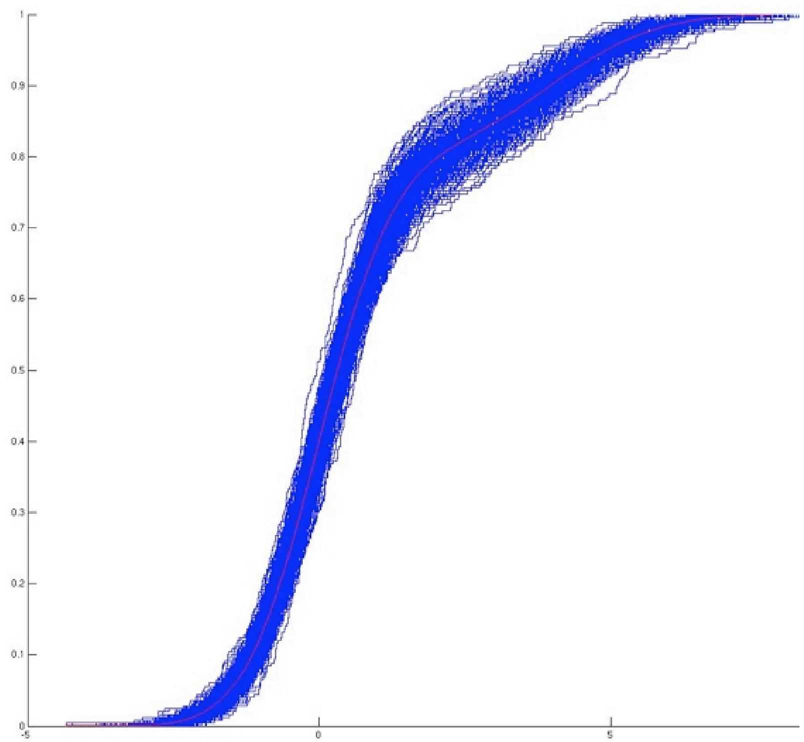
Efficient sorting using the CDF



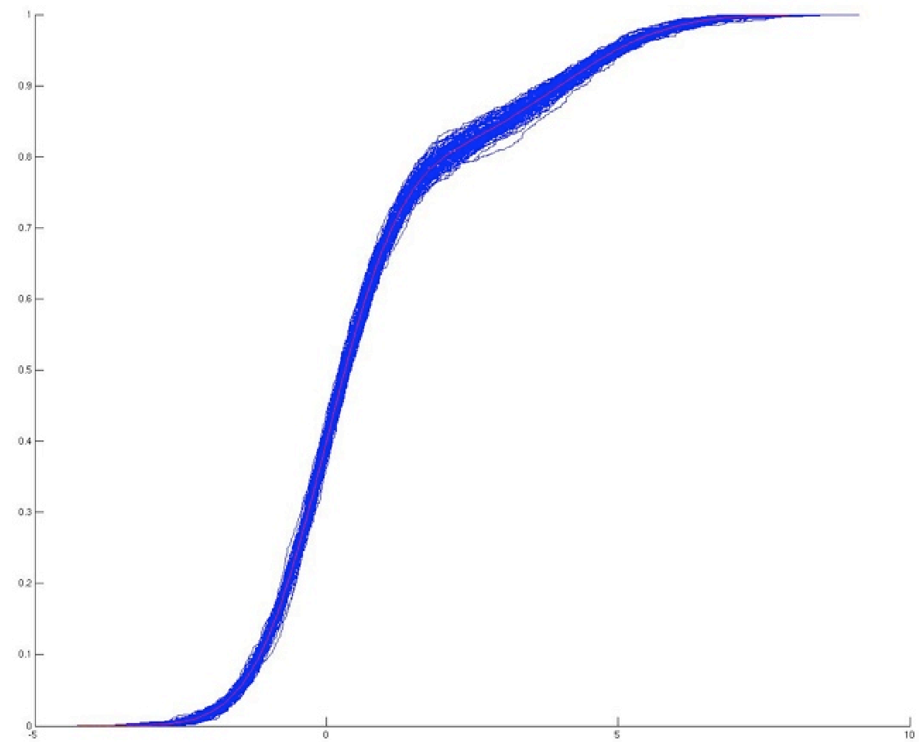
Efficient sorting for an IID source

- Source is IID but distribution is unknown
- Sort first m instances to estimate CDF
- Use CDF to sort rest of data.
- How large should m be?

Stability of empirical CDF



250 instances

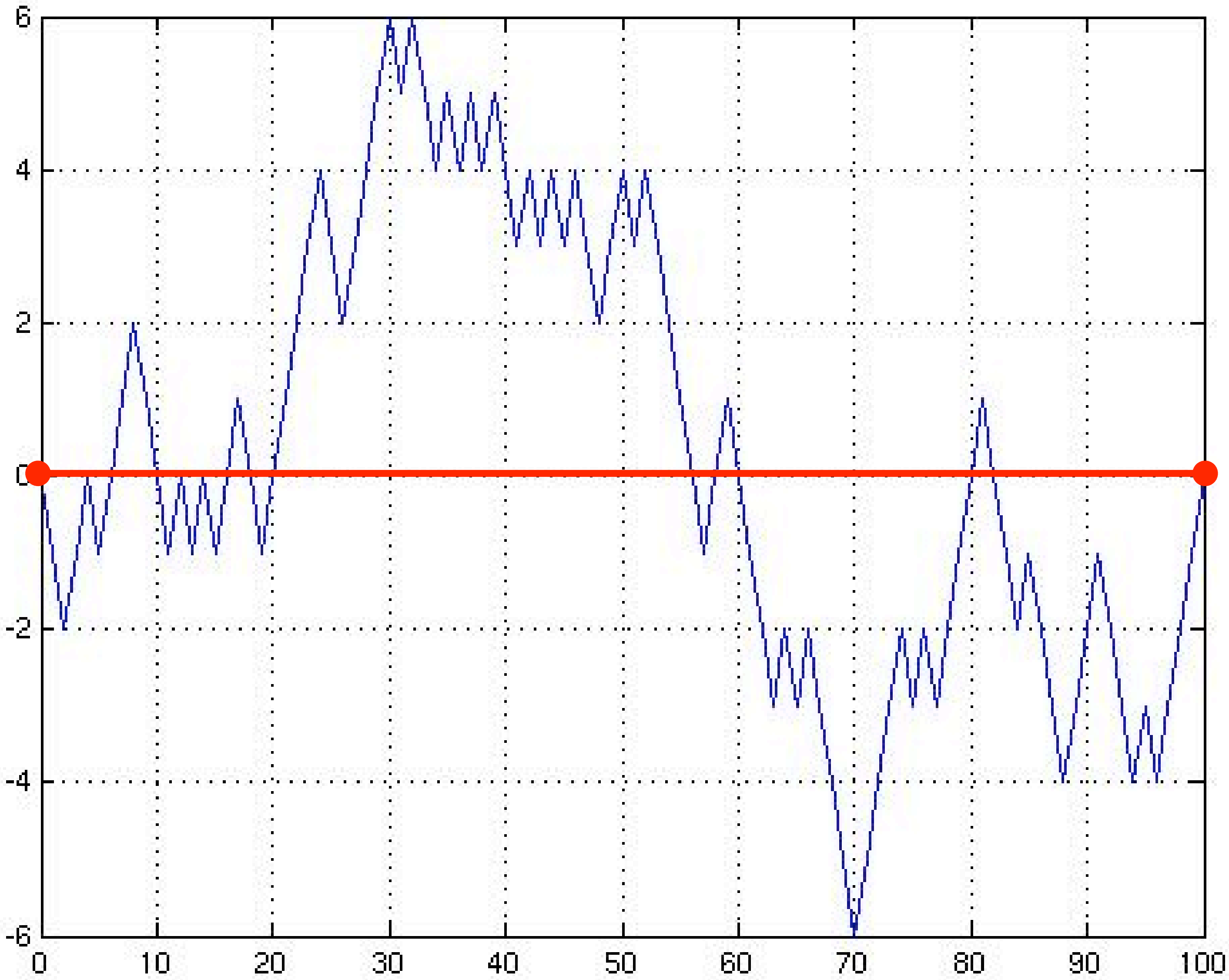


1000 instances

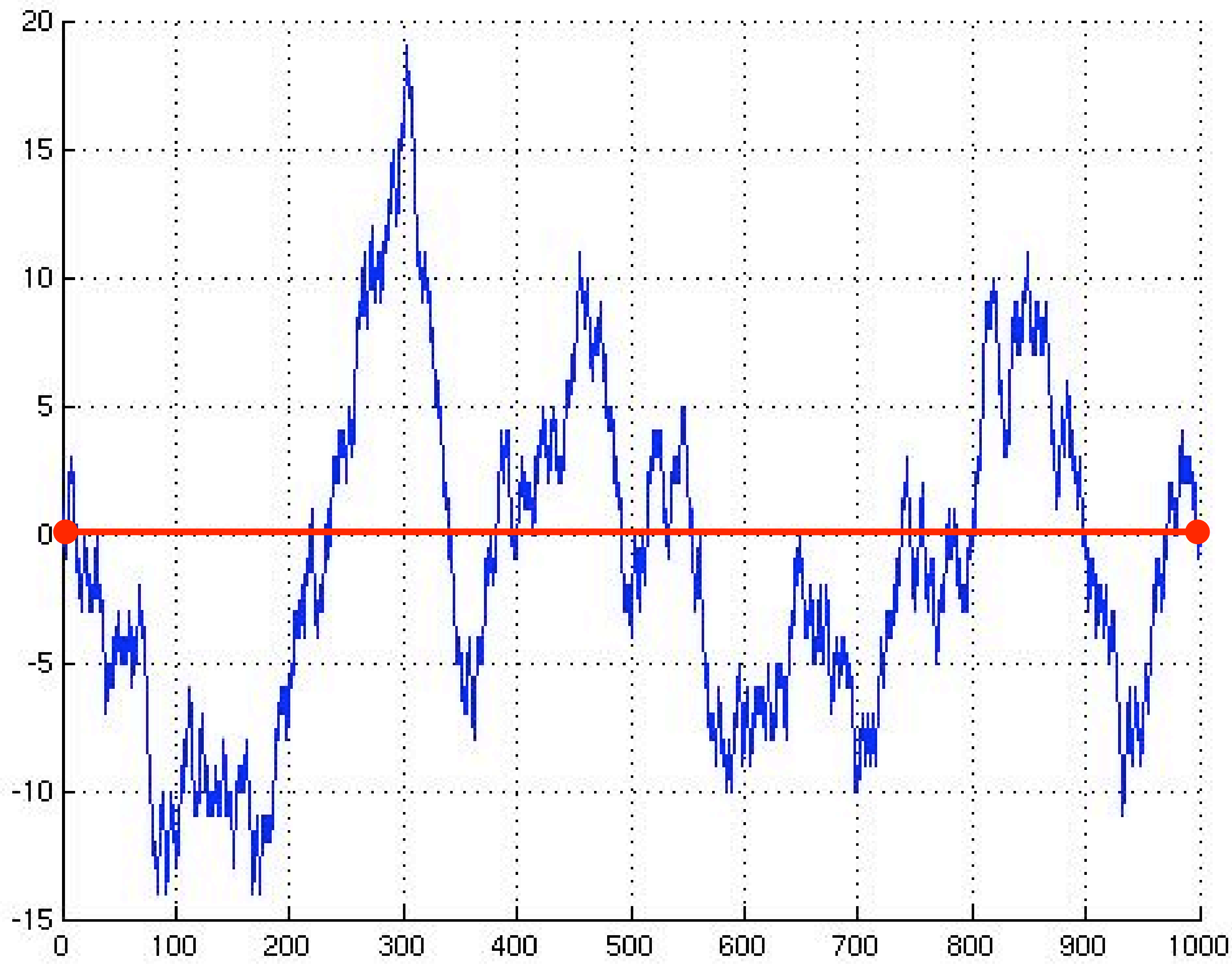
The difference between two empirical CDFs

```
% input: samples d1,d2
n = length(d1) % assuming length(d1)==length(d2)
vals = [d1; d2];
labels = [repmat(1,n,1); repmat(-1,n,1)];

[s,i] = sort(vals);    % sort labels by
s_labels = labels(i); % increasing vals

d = cumsum(s_labels);
plot(0:length(d),[0;d]);
```

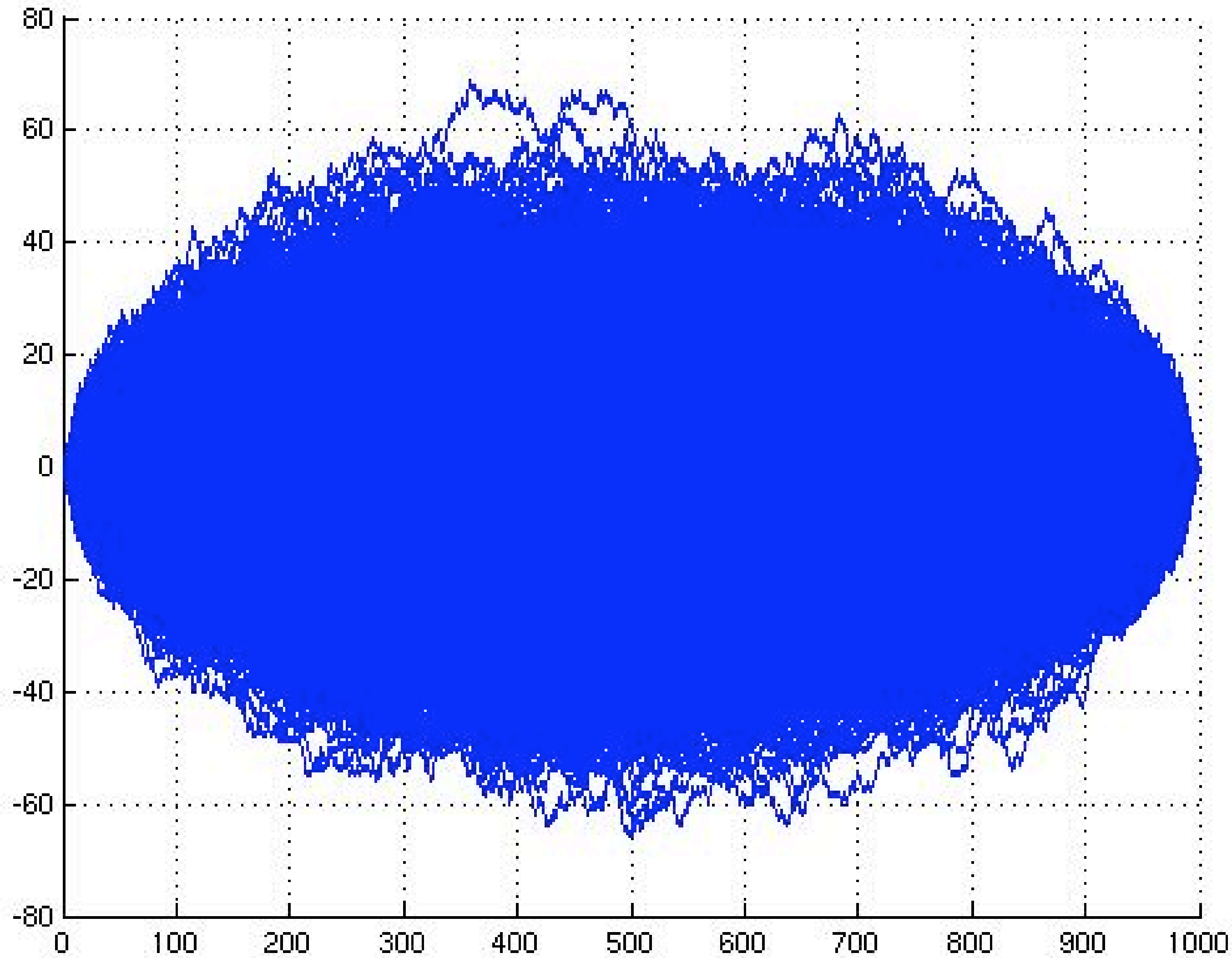
Typical cumulative difference



Typical cumulative difference



Maximal divergence of a returning random walk



The Glivenko-Cantelli Theorem

$F(x)$ = CDF

$F_n(x)$ = empirical CDF using n random instances

ϵ = error tolerance

$\mathbf{x}_1^n = \langle x_1, x_2, \ldots, x_n \rangle$ = sample

$$P_{\mathbf{x}_1^n}\left(\max_x |F_n(x) - F(x)| \geq \epsilon\right) \leq ae^{-bn\epsilon^2}$$

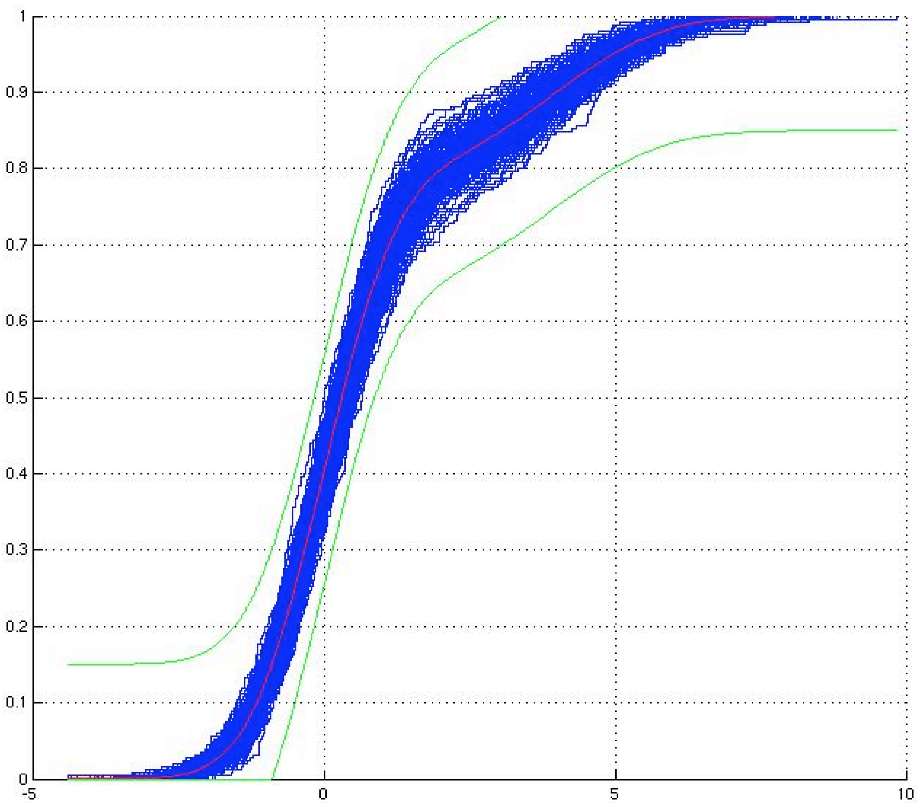
Elegant proof.

Devroye, Gyorfi, Lugosi / A probabilistic Theory of Pattern Recognition, page 192

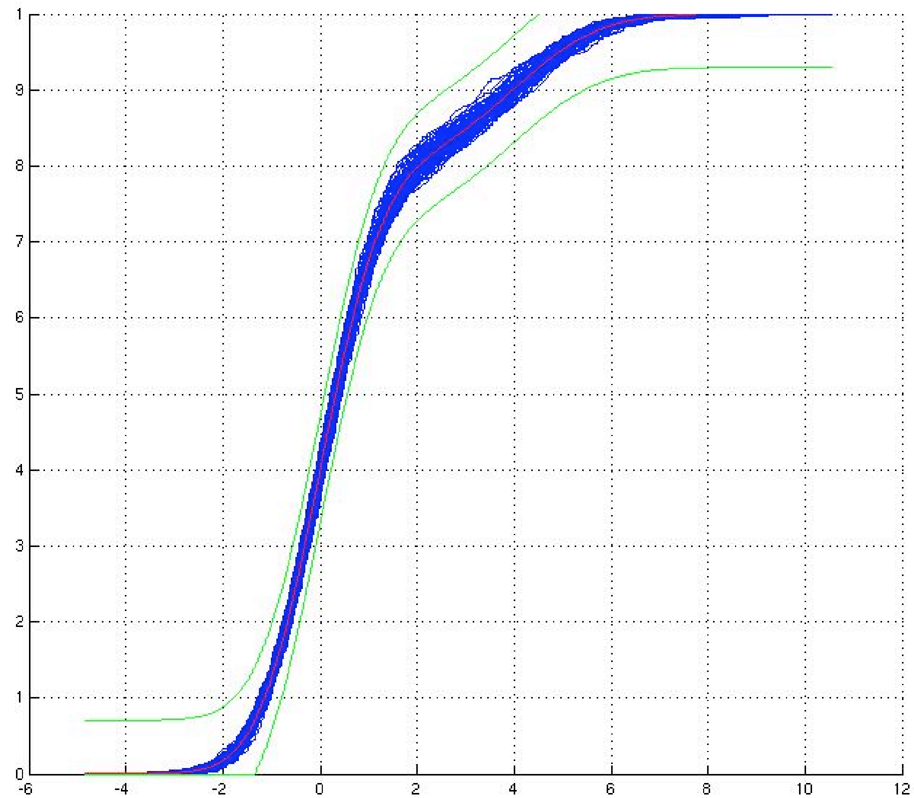
Best Constants: a=b=2, Complex proof.

MASSART, P. (1990). The tight constant in the Dvoretzky-Kiefer-Wolfowitz inequality. Ann. of Probability 18, 1269-1293.

Empirical test of the Glivenko-Cantelli Theorem



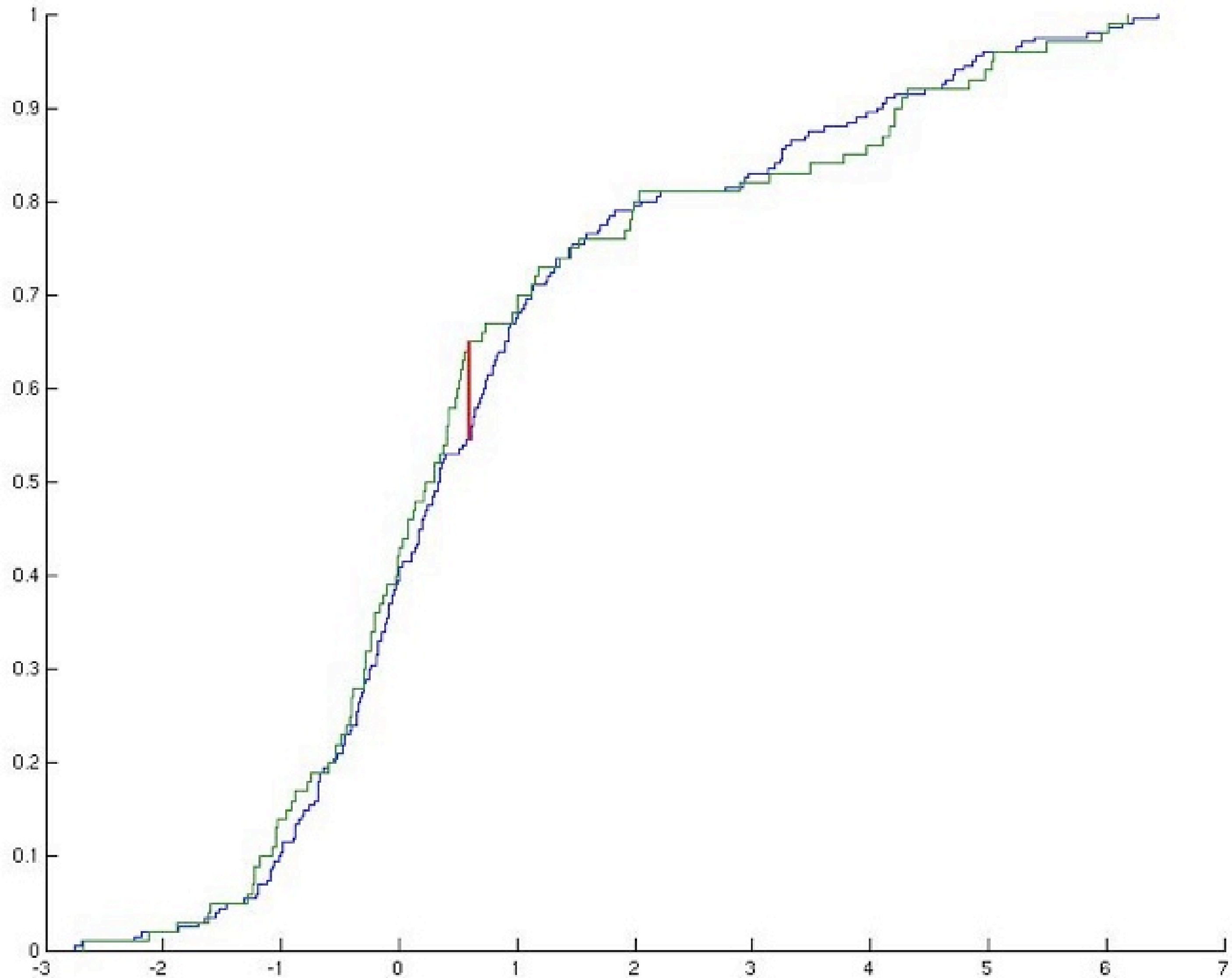
250 instances



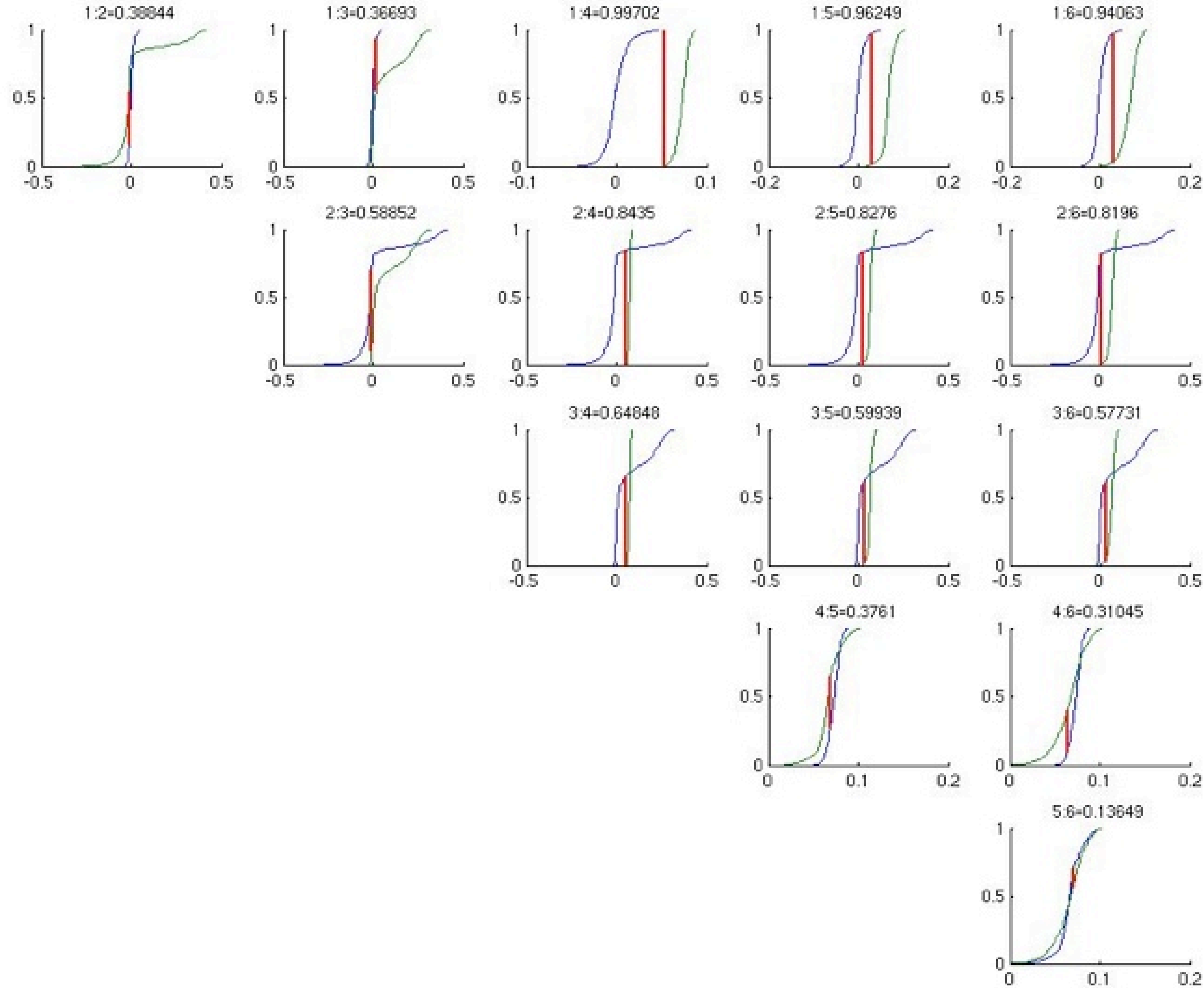
1000 instances

The Kolmogorov-Smirnoff Test

the maximal difference between the two empirical CDFs



KS for hue distributions



Conclusions

- Computing CDFs ~ Sorting.
- CDFs are very stable for all distributions
- Proof: Glivenko Cantelli - related to properties of random walks.
- KS test is a powerful test for n>1000
- When there is systematic variation - KS might be too sensitive.
- For colors of fruit - we need to estimate parameters of distribution - Use distribution model.