CSE 20: Week 7 Howework Solutions

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SF: Solutions to 2.15, 2.16, 2.17, 2.18 are found in the back of your textbook.

SF 2.17(a): False, suppose $X = \{a, b\}$ and $Y = \{c\}$ and $f(a) = f(b) = c$. If $A = \{a\}$, $f(A) = \{c\}$, and $f^{-1}(f(A)) = \{a, b\}$. Note that $\{a, b\} \neq \{a\}$.

IS: Solutions to 1.7, 1.16, 1.4 are found in the back of your textbook.

IS 1.4 (using the perturbation method): Let

$$S_n = \sum_{k=1}^{n} k^3$$

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Then S_{n+1} can be written as

$$\begin{aligned} S_n + (n+1)^3 &= 1^3 + \sum_{k=2}^{n+1} k^3 \\ &= 1 + \sum_{k=1}^{n} (k+1)^3 \\ &= 1 + \sum_{k=1}^{n} (k^3 + 3k^2 + 3k + 1) \\ &= 1 + S_n + \sum_{k=1}^{n} 3k^2 + \sum_{k=1}^{n} 3k + \sum_{k=1}^{n} 1 \end{aligned}$$

The S_n's cancel out, leaving us with:

$$\begin{aligned} 3\sum_{k=1}^{n} k^2 &= (n+1)^3 - 1 - 3\sum_{k=1}^{n} k - n \\ &= (n+1)^3 - \frac{3(n)(n+1)}{2} - (n+1) \\ &= (n+1)((n+1)^2 - 3n/2 - 1) \\ &= (n+1)n(n+1/2) \end{aligned}$$

Then we have

$$\sum_{k=1}^{n} k^2 = \frac{(n)(n+1)(2n+1)}{6}$$

Pumpkins: The problem with the proof lies in using the incorrect base case. If we could prove a base case for sets of size 2, the proof would hold. That is, if all pumpkins are pair-wise equal, then all pumpkins are equal.

However, if we apply the same argument we used to get from sets of size $n-1$ to n on sets of size 1, we find a problem. Since the intersection of two sets of size 1 is empty, we cannot use transitivity to get sets of size 2. The proof by induction depends on using sets of size 2 to get to sets of size 3, and so on. This is why the proof breaks down.

Precondition and Loop Invariant: **Precondition:** $i \in \mathbb{N}, j \in \mathbb{R}$. That is, i is a nonnegative integer and j is a real number. If you try running the algorithm with $i \in \mathbb{R} - \mathbb{N}$, you will find that the loop will never terminate. For example, see what happens when i is -1, or 2.1.

Loop Invariant: $0 \leq$ `i2` $\leq$ `i` and `total` = (`i`$-$`i2`)$*$`j`. A good way to discover the loop invariant is to try an example. In class we set $i = 3$ and $j = 2.1$. The values for `total` and `i2` will be $\{(0, 3), (2.1, 2), (4.2, 1), (6.3, 0)\}$.

Inductions with Predicates: The following predicates are true:

- P(2)
- P(n) $\rightarrow$ P(n-1) for $n \geq 1$
- (P(2) AND P(n)) $\rightarrow$ P(2n)

Prove that P(i) is true for all non-negative integers i.

Proof by induction:

Base case (P(1) is true): P(2) $\rightarrow$ P(1), since P(2) and P(n) $\rightarrow$ P(n-1).

Base case (P(0) is true): P(1) $\rightarrow$ P(0), since P(1) and P(n) $\rightarrow$ P(n-1).

Inductive Hypothesis: P(n) is true.

We will show that P(n+1) is true.

- P(n) is true by the inductive hypothesis
- P(2n) is true, since (P(2) AND P(n)) $\rightarrow$ P(2n)
- P(2n - 1) is true, since P(2n) $\rightarrow$ P(2n-1)
- By the same reasoning, P(2n), P(2n-1),...,P(n+1), P(n), ..., P(1) are all true.
- Therefore P(n+1) is true.

We have shown, using a proof by induction, that P(i) is true for all non-negative integers i.