Homework #2: Numbers, Propositional Logic, and Proof by Contradiction
Due Date: October 6, 2005
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Book Problems

Solutions to BF $\{2.9, 2.10, 2.13, 2.15, 2.16\}$ and LO $\{1.4, 1.7, 1.10, 1.15, 1.17, 1.23\}$ are also in the book.

Notes:

1.4 - If a flag is set to 0, then there is no error. So if the computer is out of memory, you should say p.

1.7 - The trick is to use the associativity property and remove some of the parenthesis

1.15 - The book answer is not quite the contrapositive. The contrapositive is "If Dennis is unsure of victory, then he will not enter the America's Cup". Of course, the statement and its contrapositive are equivalent.

1.23 - Watch out for the way they use "necessary" and "sufficient". If you say "A necessary cond for Y is X", that's equivalent to "X necessary for Y".

Webpage Problems

1

Write the contrapositive, inverse, and converse of the statement "If n is an integer for which $n \cdot n$ is even, then n is even." For all the statements below, assume n is an integer.

Contrapositive: "If n is not even, then $n \cdot n$ is not even".

Inverse: "If $n \cdot n$ is not even, then n is not even".

Converse: "If n is even, then $n \cdot n$ is even".

2

Prove, by contradiction, that there are an infinite number of primes.

Assume that n is the largest prime.

Consider $x = n! + 1$. Now, break the problem into two cases:

1. x is prime
2. x is not prime

Case 1: x is a prime number, but $x = n! + 1 > n$. This contradicts the assumption that n is the largest prime.

Case 2: x is not a prime number (and it's not 1). It is a composite number. By definition, a composite number can be broken into prime factors. This means that there must be at least one prime number that divides x.

Pick *any* prime number d.

By the original assumption, all prime numbers in the universe including d are less than or equal to n.

Consider x/d

$$\begin{aligned}\frac{x}{d} &= \frac{n!+1}{d} \\ &= \frac{n!}{d} + \frac{1}{d} \\ &= \frac{1 \cdot 2 \cdot 3 \cdot \ldots \cdot d \cdot \ldots \cdot n}{d} + \frac{1}{d} \\ &= (1 \cdot 2 \cdot 3 \cdot \ldots \cdot d - 1 \cdot d + 1 \cdot \ldots \cdot n) + \frac{1}{d}\end{aligned}$$

This means x/d will have a remainder of 1 for any prime d. This contradicts the definition of a composite number.

Since all cases lead to a contradiction, our only assumption must be false. Therefore, there is no largest prime. There are an infinite number of primes.

Comments: A good example of when $n! + 1$ is not prime is if $n = 5$. $5! + 1 = 121$. Because we assume 5 is the largest prime, the only primes we know about are $2, 3, 5$. It requires knowledge of a larger prime 11 to divide 121 into its prime factors!

Another comment: We never showed that $x = n! + 1$ is prime or not. There are times when it is, and there are times when it is not. As long as we consider all cases, and all cases lead to a contradiction, the proof works.

3

Prove, by contradiction, that there are at least two people on the planet who were born on the same second of the same hour of the same day of the same year in the twentieth centry. (You can assume that there were at least 4 billion–2**32–people born in that century)

First, calculate the number of seconds in the century: :$100 \cdot 365 \cdot 24 \cdot 60 \cdot 60 = 3,153,600,000$.

Let $S = \{s_1, s_2, \ldots, s_{3,153,600,000}\}$, the set of all seconds in the century.

Let $P = \{p_1, p_2, \ldots, p_{2^{32}}\}$, the set of all people born in the century.

Now, assume for the sake of contradiction, that no two people are born on the same second of the century. That is, at most one p_i is born on any s_i. Since people must be born on some second, we can conclude that:

$$|P| \leq |S|$$
$$4 \text{ billion} \leq 3 \text{ billion}$$

This is a contradiction. Therefore, our only assumption is false, and there is a second where at least two different people are born.

Comment: Problems of this form can be proved by invoking the, “pigeonhole principle” as well.