

CSE 20—Discrete Math

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Predicate Logic

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Predicates

Grammatical origin

Alternate definitions of a predicate:

- A sentence with a finite number of variables. Replacing variables with specific values yields a *statement*. The domain is the set of all possible values to substitute.
- A function whose codomain is (true/false) statements.

Example:

- " $x^2 > 1$ "
 - Domain?
 - Example statements?
- "Boy x likes girl y " (alternate syntax:)
 - Domain?
 - Example statements?

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Truth Set of a Predicate, $P(x)$

The set of all elements in the domain that make $P(x)$ true

Denoted:

- $\{x \in D: P(x)\}$

Example:

- What is the truth set of "x is a factor of 8"?
 - If D is $\{0, 1, 2\}$?
 - if D is in $\{0, 1, 2, 3, \dots\}$ ($\mathbf{N}$)?
 - If D is in $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ ($\mathbf{Z}$)?

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Quantifiers

Universal

- $\forall x \in D, S(x)$

Existential

- $\exists x \in D: S(x)$

Common to drop $\in D$ if it is clear from context

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Examples

Let $L(x, y)$ be the predicate “Boy x likes girl y ”

- $\exists y: \exists x: L(x, y)$
- $\exists y: \forall x: L(x, y)$
- $\forall x \exists y L(x, y)$
- $\forall y L(\text{“Jose”}, y)$

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Negation and Quantifiers

$$\sim(\exists x: P(x)) \leftrightarrow \forall x \sim P(x)$$

$$\sim(\forall x P(x)) \leftrightarrow \exists x: \sim P(x)$$

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Examples

$L(x, y)$ = “ x likes y ” Domain: all people in my son's 1st grade class

- Everyone likes Jill
- Nobody likes Jill
- Not everybody likes Jill
- There exists a person that nobody likes
- Nobody dislikes everybody
- Everybody likes him/herself
- $\forall x, \forall y, \forall z, L(x, y) \wedge L(y, z) \rightarrow L(x, z)$
- $\forall x, \forall y, L(x, y) \rightarrow L(y, x)$

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Converting to Predicates

If a number is a natural number, it is an integer

The square of any real number greater than 2 is greater than 4

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Quantifiers and Adversaries

Statements involving quantifiers can be viewed as a game

- Two players
 - You, trying to prove the statement is true
 - An adversary, trying to prove the statement is false
- Work from left to right through the quantifiers
 - $\forall y$: the adversary selects an y (think of the adversary as choosing the worst possible y). That y remains *bound* the rest of the game
 - $\exists x$: you select an x that will satisfy the statement. That x remains *bound* the rest of the game

Example:

- $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}: 2x = y$
- $\exists x \in \mathbb{Z}: \forall y \in \mathbb{Z}, xy = 0$
- $\forall y \in \mathbb{Z}, \exists x \in \mathbb{Q}: xy = 1$

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Limit

The limit of a sequence a is L means that a gets arbitrarily close to L

$$\lim_{n \rightarrow \infty} a_n = L$$

$$\forall \varepsilon \in \mathbb{R}^+, \exists n_0 \in \mathbb{N} : \forall n \in \mathbb{N}, n \geq n_0 \rightarrow L - \varepsilon \leq a_n \leq L + \varepsilon$$

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Order of Growth of a Function

We say that $T(n)$ has an asymptotic upper bound of $f(n)$ if:

$$\exists n_0 \in \mathbb{N} : \exists c \in \mathbb{R}^+ : \forall n \in \mathbb{Z}, n \geq n_0 \rightarrow 0 \leq T(n) \leq cf(n)$$

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Proving Quantified Statements

Prove universal statement: $\forall x \in D, P(x) \rightarrow Q(x)$

- Exhaustive enumeration
- Generalizing from the generic particular
 - "Suppose x is in D and $P(x)$ "
 - ...
 - Therefore $Q(x)$
- Example: The difference of two odd numbers is even

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Proving Quantified Statements

Prove existential statement: $\exists x \in D: P(x)$

- Constructive proof
 - Display an x
 - Give a set of directions for finding x
- Nonconstructive proof
 - Proof by contradiction (assume non-existence and show a contradiction)
 - Show x must exist
- Example:
 - In a group of 367 people, at least two share a birthday

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Disproving Quantified Statements

Disprove universal statement: $\forall x \in D, P(x) \rightarrow Q(x)$

- Counterexample
 - Show an x in D where $P(x)$, but not $Q(x)$
- Example: All primes are of the form $2^n - 1$

Disprove existential statement: $\exists x \in D: P(x)$

- Equivalent to:
 - Prove
 - Or, alternatively,
 - Therefore, best bet is generalizing from the generic particular.
- Example: There exists a prime which can be written as the square of an integer > 1

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