

CSE 20—Discrete Math

Fall, 2005

September 29 (Day 3)
Propositional Logic

Instructor: Neil Rhodes

Simplifying a Boolean Function

Example

- $\sim (\sim p \wedge q) \wedge (p \vee q)$

2

Logic

Propositional Logic

- Deals with individual statements that have true/false values
 - no in-between
- Corresponds 1-1 with Boolean Functions and Combinational Circuits

(First-order) Predicate Logic

- Can actually talk about groups: For all, There exists,
- Covered next lecture

3

Propositional Logic Formulas

Well-formed formulas (wff)

- Any lower-case letter is a wff (atomic formula)
 - p
 - q
 - r
 - ...
- If Φ is a wff, then $\sim \Phi$ is a wff
- If Φ and Π are wffs, then:
 - $(\Phi \vee \Pi)$ is a wff
 - $(\Phi \wedge \Pi)$ is a wff
 - $(\Phi \rightarrow \Pi)$ is a wff
 - $(\Phi \leftrightarrow \Pi)$ is a wff

As a shortcut, we sometimes omit unneeded parentheses

4

Formula

An atomic formula (p) represents some statement that has a truth value

- Either p is true, or it's negation is true
 - *tertium non datur*
- It's either true or false (bivalent)

Example English propositions

- It will rain tomorrow
- The sky is blue
- The sky is fluorescent orange
- The program will finish in less than 3 minutes

The truth value of a complex formula can be determined from the truth values of the atomic formulas within

5

Equivalent Formulas

Formulas that differ syntactically are said to have different forms

- Formulas with different forms are *equivalent* if their truth tables are the same
- Example:

6

Formulas Equivalent to a Constant

A formula whose truth table values are all true is called a *tautology*

- Linguistic origin:
- True regardless of the values of the atomic formulas
- Example:

A formula whose truth table values are all false is called a *contradiction*

- Linguistic origin:
- False regardless of the values of the atomic formulas
- Example:

7

Implication

Implication (also called if-then)

- $p \rightarrow q$
- For the formula to be true, if p is true, then q can't be false.
- If p is false, the formula is true
- Example:
 - If it is raining tomorrow, the driveway will get wet

p	q	$p \rightarrow q$
f	f	
f	t	
t	f	
t	t	

8

Implication

Common English forms

- *If p then q*
 - If it is raining, then the driveway will get wet
- *p only if q*
 - The team will play in the playoffs *only if* the team reaches the semifinals
- *p if and only if q*
 $p \text{ iff } q$
 - The team will be the champion *if and only if* the team wins the championship game
- *p is necessary for q*
A necessary condition for q is p
 - Reaching the semifinals *is necessary for* playing in the playoffs

9

Implication

Common English forms

- *p is sufficient for q*
A sufficient condition for q is p
 - "John is eligible to vote" *is sufficient (to ensure the truth of)* "John is at least 18 years old"
- *p is necessary and sufficient for q*
 - "X is even" *is necessary and sufficient (to ensure the truth of)* "X is evenly divisible by 2"

10

Relatives of Implication

Given $p \rightarrow q$

- Inverse: $\sim p \rightarrow \sim q$
- Converse: $q \rightarrow p$
- Contrapositive: $\sim q \rightarrow \sim p$

p	q	$p \rightarrow q$	$\sim p \rightarrow \sim q$	$q \rightarrow p$	$\sim q \rightarrow \sim p$
f	f	t			
f	t	t			
t	f	f			
t	t	t			

11

Anything can be Proved from a Contradiction

If we can establish:

- A contradiction implies anything because:
 - contradiction $\rightarrow q$
 - contradiction
 - Therefore, q

12

Proof by Contradiction

To prove q by contradiction

- Assume $\sim q$
- Show this leads to a contradiction
- Example
 - Prove that there is no largest integer (p)
 - Assume, to the contrary that _____ ($\sim p$)
 - Then, _____ (formula that follows from p)
 - Now, _____ ($p \wedge \sim p$)

13

Argument forms

Classical logic uses syllogisms

- Two premises
- One conclusion
- *Modus ponens*
 - If it rained, the driveway got wet
 - It rained
 - Therefore, the driveway got wet
- *Modus tollens*
 - If it rained, the driveway got wet
 - The driveway didn't get wet
 - Therefore, it didn't rain

14

Argument Forms

Generalization

- p
- Therefore, $p \vee q$

Specialization

- $p \wedge q$
- Therefore, p

Conjunction

- p
- q
- Therefore, $p \wedge q$

Elimination

- $p \vee q$
- $\sim q$
- Therefore, p

Transitivity

- $p \rightarrow q$
- $q \rightarrow r$
- Therefore, $p \rightarrow r$

15

Argument Forms

Proof by Division into Cases

- $p \vee q$
- $p \rightarrow r$
- $q \rightarrow r$
- Therefore, r

Contradiction rule

- $\sim p \rightarrow \text{false}$
- Therefore, p

16

Algebraic Rules for Propositional Formulas

Equivalences between propositional formulas (similar to algebraic equivalences):

- Associative
- Distributive
- Idempotent
- Double negation
- DeMorgan's
- Commutative
- Absorption
- Bound
- Negation