

CSE 20—Discrete Math

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Day 19
Pigeonhole Principle (cont.)
Order

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Monotonic Subsequences

Let $A = a_1, a_2, \dots, a_n$ be a sequence of distinct real numbers

- Remove some of the terms from the sequence. The remaining ones are a *subsequence*.
- If the subsequence is increasing, the subsequence is an upsequence. If the subsequence is decreasing, the subsequence is a downsequence.
- Let D be the length of the longest downsequence of A .
- Let U be the length of the longest upsequence of A .

Prove that that $n \leq D \cdot U$

- Let $f(a_i)$ be an ordered pair of numbers (D_i, U_i) representing the maximum downsequence (upsequence) ending with a_i .
- If $a_{i+1} > a_i$, then:
 - $f(a_{i+1}) = (?, U_i + 1)$
- if $a_{i+1} < a_i$, then:
 - $f(a_{i+1}) = (D_i + 1, ?)$
- If $i > j$, either $D_i > D_j$ or $U_i > U_j$
- Thus, each a_i has a unique pair: (D_i, U_i) . (f is an injection)
- Further $D_i \leq D$, and $U_i \leq U$.

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Example

Show that at any party with at least six people, there exists either a set of three mutual friends, or a set of three mutual strangers.

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Partially Ordered Set (Poset)

Binary relation R on S is an **order relation** if it is:

- Reflexive
 - $aRb \rightarrow bRa$
- Transitive
 - $aRb \wedge bRc \rightarrow aRc$
- Anti-symmetric
 - $aRb \wedge \sim bRa \rightarrow a=b$

Order relations are commonly written $a \leq b$ (precedes)

- Rather than aRb
- $a < b$ (a strictly precedes)

Combination of a set and an order relation on the set is called a **partially ordered set (poset)**

Example

- $(\mathbf{N}, \leq)$
- S , a set of sets, with $\subseteq$ as an order relation

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Diagramming Posets

Hasse diagram of a poset $(S, \preceq)$: graph with:

- Each element of S as a node of the graph
- Line from a to b if a *immediately precedes* b
- Lines always point up
- Used for project management, for instance
 - **Critical chain**: longest line
 - PERT (Program Evaluation and Review Technique)
 - CPM (Critical Path Method)

Example: poset $(P(\{1,2,3\}), \subseteq)$

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Poset Example

Let the set be positive integers

Let the order relation be $a \mid b$

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Poset Examples

Set is $\mathbf{Z}$, order relation is $a \preceq b$ if $b = a^r$ for some positive integer r

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Poset Examples

$(\{\}, R)$ (empty set)

$(S, \{\})$ (empty relation)

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Poset Examples

Prerequisites for CSE classes

- 8A
- 8B: requires 8A
- 12: requires 8B,
- 20: requires 12
- 21: requires 20
- 30: requires 12, 20
- 100: requires 12, 21
- 101: requires 12, 21, 100
- 105: requires 12, 21
- 120: requires 100, 101, 141
- 131A: requires 30, 100, 105
- 131B: requires 30, 100, 105, 131A
- 140: requires 20, 30
- 141: requires 140

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Ordered Subset

If $(S, \leq)$ is a poset, then,

- Take $X \subseteq S$
- Take $\leq_X = \{a \leq b: a, b \in X\}$

$(X, \leq_X)$ is a poset

We would normally write: X is an **ordered subset** of S

- Assume the order relation of X is the relation induced by the order relation of S

Example: given $(\mathbb{Z}, |)$ take $\{1, 2, 3, 4, 5, 6, 7, 8\} \subseteq \mathbb{Z}$

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Product Set

Given two posets $(S, \leq_S)$, and $(T, \leq_T)$, we define the product set $(R, \leq_R)$

- $R = S \times T$
- $(s, t) \leq_R (s', t')$ iff $s \leq_S s'$ and $t \leq_T t'$

The product set is a poset

Example

- $(\mathbb{N} \times \mathbb{N}, \leq)$

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Linearly Ordered Sets

If $(S, \leq)$ is a poset, then a and b in S are **comparable** iff

- $a \leq b$ or $b \leq a$

If $(S, \leq)$ is a poset, and each pair of elements of S are comparable, then S is a **linearly ordered set** (or **totally ordered set**)

- $\leq$ is a **total ordering** of S
- S is called a **chain**

Examples

- $(\mathbb{N}, \leq)$
- $(\mathbb{N}, |)$
- $(\{1, 2, 32, 8, 4\}, |)$

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Maximal, Minimal, Least, Greatest

Given a poset $(S, \preceq)$, and an element e in S

- e is **minimal** if there is no preceding element of S (other than e)
- e is **maximal** if e precedes no element of S (other than e)
- e is **least** if it precedes all elements of S
- e is **greatest** if it is preceded by all elements of S

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Topological Sort of a Poset

Given a poset $(S, \preceq)$, it's always possible to enumerate the elements of S in a linear ordering $s_1, s_2, s_3, \dots, s_{|S|}$. That linear ordering is called a **topological sort** of the poset

- $s_i \preceq s_j \rightarrow i < j$

Is the topological sort unique?

Constructing a topological sort from a Hasse diagram

```
Topological-sort(S)
  if S is empty then
    return S
  else
    e = a minimal element of S
    return Concat(e, Topological-sort(S - e))
```

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Example Topological Sort

Topological sort of prerequisites

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Lexicographic Relation

Given $(S, \preceq)$ is a poset

- If two items have different lengths, the shorter one sorts before the longer one if the shorter one is a prefix of the longer one
- Words in dictionary
 - $A < AA < AAA < B < BAT < \dots < Z < ZY < ZZ < ZZZ < ZZZZZZZZ$
- Tuples of varying lengths
 - $S^0 = \epsilon$ (empty string)
 - $S^1 = S$
 - $S^n = SxSx\dots xS$ (n times)
 - $S^* = \text{union of } S^0, S^1, S^2, \dots$
- How to order S^* ?
 - $\epsilon \preceq x$, for all $x \in S^*$
 - $(a_0, \dots, a_m) \preceq (b_0, \dots, b_n)$ if:
 - $m \leq n$ and $\forall i, a_i = b_i$
 - or, $\exists j \leq \min(m, n): a_j \neq b_j, a_i = b_i, \forall i < j$

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Lexicographic Relation is Lexicographic Ordering

Reflexive

Transitive

Antisymmetric