

# CSE 20—Discrete Math

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Day 18  
Infinite Series (cont.)  
Equivalence  
Pigeonhole Principle

Instructor: Neil Rhodes

## Harmonic Series

$1 + 1/2 + 1/3 + 1/4 + 1/5 \dots$

- Lower and upper bounds on:

$$H_n = \sum_{1 \leq k \leq n} \frac{1}{k}$$

- Put one term in group 1, two in group 2, four in group 3, etc.
- $1 + 1/2 + 1/3 + 1/4 + 1/5 + 1/6 + 1/7 + 1/8 + 1/9 + 1/10 + \dots$

- Each group has terms no bigger than first term and strictly bigger than first term of next group.
- $\frac{1}{k} < \text{sum of each group} \leq \frac{1}{k}$
- If  $n$  is in group  $k$ ,  $\frac{1}{k} < H_n \leq \frac{1}{k-1}$

$$\frac{\lfloor \lg n \rfloor}{2} < H_n \leq \lfloor \lg n \rfloor + 1$$

Gives bound to  $H_n$  within a factor of 2

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## Better Bounds for Harmonic Series

Use integration of  $f(x) = 1/x$

- Upper bound:



-  $H_n > \ln n$

- Lower bound:



-  $H_n < \ln n + 1$

Bounds to  $H_n$  within at most 1

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## Equivalence Classes

Are 1 and 101 equal?

Are there situations where we treat them as equivalent?

What other numbers are equivalent to 1 and 101 in that same situation?

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## Equivalence

If  $R$  is a relation from  $A$  to  $A$  (**binary relation**) then  $R$  is an **equivalence relation** if and only if  $R$  is:

- Symmetric: if  $x R y$ , then  $y R x$
- Transitive: if  $x R y$  and  $y R z$ , then  $x R z$
- Reflexive:  $x R x$
- The set of all elements equivalent to  $x$  is called the **equivalence class** of  $x$
- We write  $a \equiv b$  to show that  $a$  is equivalent to  $b$ .

### Example

- A partition of a set induces an equivalence relation.  $x R y$  iff  $x$  and  $y$  are in the same block of the partition
- Modulus induces an equivalence relation (all numbers that have the same remainder when divided by  $k$  are equivalent).
- Equality is an equivalence relation

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## Showing a Binary Relation Graphically

Let  $S = \{1, 2, 3, 4\}$

Let  $R = \{(1, 3), (3, 1), (1, 1), (3, 3), (2, 2), (4, 4)\}$

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## Examples

Rationals

Propositional formulas

$\leq$

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## How Many Equivalence Relations?

Given a set  $S$ , how many different equivalence relations are there?

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## Pigeonhole Principle

Version 1: If there are more than  $n$  objects distributed into  $n$  compartments, then some compartment has more than 1 object.

Version 2: If there are  $n$  compartments and each compartment has at most 1 object, then there are at most  $n$  objects.

Version 3: If more than  $n \cdot k$  objects are distributed into  $n$  compartments, then some compartment has more than  $k$  objects.

Version 4: For a non-empty, finite bag of numbers, the maximum (minimum) value is at least the average number.

Version 5: Suppose  $K$  is a partition of a set  $S$ . If  $K$  has fewer than  $|S|$  elements, then some block of  $K$  has at least two elements.

Version 6: For any function  $f: A \rightarrow B$  where  $A$  and  $B$  are finite, if  $|A| > |B|$ , then  $f$  is not injective (one-to-one)

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## Examples

What is the maximum number of people in a group where no two of them share both the same birthday (MM/DD), and the same initials (XXX)?

If there are 750 people in a room, what is the minimum number of people that share a birthday?

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## Example

Given a positive integer  $n$ , what is the smallest integer  $k$  such that, given a set of distinct integers  $\{t_1, \dots, t_k\}$ , either:

- $t_i \mid n$ , for some  $t_i$
- $(t_i - t_j) \mid n$ , for some  $t_i$  and  $t_j$ ?

- Consider  $f: \{t_1, \dots, t_k\} \rightarrow \{1, \dots, n-1\}$  where  $f(x) = x \bmod n$

- Why isn't 0 in the domain?

- What if  $f$  is *not* injective?

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## Example

If there are 48 different pairs of people who know each other at a party of 20 people, show that some person has 4 or fewer acquaintances

- Average number of acquaintances =

- Therefore, minimum is  $\leq$

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## Example

Suppose the numbers 1 through 10 are randomly positioned around a circle. Show that the sum of some set of three consecutive numbers must be at least 17.

- Average sum of three consecutive numbers =
- Maximum  $\geq$

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## Monotonic Subsequences

Let  $A = a_1, a_2, \dots, a_n$  be a sequence of distinct real numbers

- Remove some of the terms from the sequence. The remaining ones are a *subsequence*.
- If the subsequence is increasing, the subsequence is an upsequence. If the subsequence is decreasing, the subsequence is a downsequence.
- Let  $D$  be the length of the longest downsequence of  $A$ .
- Let  $U$  be the length of the longest upsequence of  $A$ .

Prove that  $n \leq D \cdot U$

- Let  $f(a_i)$  be an ordered pair of numbers  $(D_i, U_i)$  representing the maximum downsequence (upsequence) ending with  $a_i$ .
- If  $a_{i+1} > a_i$ , then:
  - $f(a_{i+1}) = (D_i, U_i + 1)$
- if  $a_{i+1} < a_i$ , then:
  - $f(a_{i+1}) = (D_i + 1, U_i)$
- Thus, each  $a_i$  has a unique pair:  $(D_i, U_i)$ . ( $f$  is an injection)
- Further  $D_i \leq D$ , and  $U_i \leq U$ .

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## Example

Show that at any party with at least six people, there exists either a set of three mutual friends, or a set of three mutual strangers.

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