

CSE 20—Discrete Math

November 16, 2005

Day 16
Infinite Sequences

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Midterm 2

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Midterm 2

1. 8 pts. Given any function, f , which of the following are true about the domain, codomain, image, and coimage of f ? Circle all that are true.

- (a) The size of the coimage is equal to the size of the image.
- (b) The size of the codomain is equal to the size of the domain.
- (c) The image is a partition of the codomain.
- (d) The coimage is a partition of the domain.
- (e) The coimage is a subset of the domain.
- (f) The image is a subset of the codomain.
- (g) The size of the coimage is no bigger than the size of the domain.
- (h) The size of the image is no bigger than the size of the codomain.

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2. 8 pts. Following are some statements about the greatest common divisor (gcd) and least common multiple (lcm) functions and integers m and n . Circle all that are true

- (a) $\gcd(m, n) \times \text{lcm}(m, n) = m \times n$,
- (b) $\gcd(m, n) = \gcd(m - cn, n)$ for all integers c ,
- (c) $\gcd(m, n) = km + jn$ for some integers k and j ,
- (d) $\gcd(m, n) = 1$ iff m and n are relatively prime.

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3. 10 pts. Give a partition of $\{a, b, c, d, e, f, g, h\}$ of cardinality 3.

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4. 12 pts. What is the value of the following sum and why?

$$\sum_{i=0}^n \binom{n}{i}$$

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5. 15 pts. Prove using induction that for all integers $n \geq 1$:

$$\sum_{k=1}^n k(k!) = (n+1)! - 1$$

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6. 15 pts. Let $A = \{a, b, c, d\}$ and $B = \{1, 2, 3\}$. For this exercise, answer each part with an actual number as well as showing your work. [Hint: recall that the Stirling numbers are defined as $S(n, k) = S(n-1, k-1) + kS(n-1, k)$.]

(a) How many surjective functions are there with domain A and codomain B ?

(b) How many *non*-surjective functions are there with domain A and codomain B ?

(c) How many injective functions are there with domain B and codomain A ? [Note: the domain and codomain are reversed from parts a) and b).]

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7. 10 pts. Here is a mistaken "proof" that for all sets A and B , $A^C \cup B^C \subseteq (A \cup B)^C$.

Proof: Suppose A and B are sets, and $x \in A^C \cup B^C$. Then $x \in A^C$ or $x \in B^C$. It follows that $x \notin A$ or $x \notin B$ by definition of complement, and so $x \notin A \cup B$ by definition of union. Thus, $x \in (A \cup B)^C$ by definition of complement, and hence $A^C \cup B^C \subseteq (A \cup B)^C$.

Circle the mistake in this proof and explain why it is wrong.

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8. 12 pts.

Define $A \times B \times B$ as $\{(r, s, t) : r \in A, s, t \in B\}$. Given sets $A = \{a, b, c\}$ (with $a < b < c$), $B = \{0, 1, 2, 3, 4\}$ (with $0 < 1 < 2 < 3 < 4$), consider $C = (A \times B) \cup (A \times B \times B)$.

- What is the first member of C in lexicographic order?
- What is the second member of C in lexicographic order?
- What is the second-to-last member of C in lexicographic order?

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9. 15 pts. Extra Credit. Evaluate the following sum to come up with a closed form (simplify your result as much as possible). [Note: you can consider using the perturbation method, or guessing the solution and then proving it with induction.]

$$S_n = \sum_{0 \leq k \leq n} k(-1)^k$$

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Monotonicity

A sequence a_n is

- strictly increasing** if each term is bigger than the previous one
- strictly decreasing** if each term is smaller than the previous one
- nondecreasing** (or weakly increasing) if each term is no smaller than the previous one
- nonincreasing** (or weakly decreasing) if each term is no bigger than the previous one
- monotonic** (or monotone) if it is either nondecreasing or nonincreasing

A sequence is **eventually [...]** if some tail of the sequence is [...]

Examples

- 1, 3, 5, ...
- $x^2 - 8x + 1$:

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Bounded and Monotone

If a sequence is bounded and eventually monotone, then it converges.

Example:

- limit of $2 + 1/n$
- limit of $2 - 1/n$

Sandwich theorem:

- If a_n has a limit of A and c_n has the same limit, and each term of b_n has the property that:
 - $a_n \leq b_n \leq c_n$
- Then, the limit of b_n is A

Example:

- limit of $2 + (-1)^n/n$

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Sequence Cagematch

Given two sequences a_n and b_n , which grows faster?

- Polynomial of degree n versus polynomial of degree $n-1$?
- Exponential versus polynomial?
- Polynomial versus logarithmic?

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Identifying a Sequence

The Online Encyclopedia of Integer Sequences

- <http://www.research.att.com/~njas/sequences/index.html>

Enter part of a sequence:

- 0 1 3 6 10 15 21

Get back a list of known related sequences

ID Number: [A000217](#) (Formerly M2535 and N1002)
URL: <http://www.research.att.com/projects/OEIS?Anum=A000217>
Sequence: 0, 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105, 120, 136, 153, 171, 190, 210, 231, 253, 276, 300, 325, 351, 378, 406, 435, 465, 496, 528, 561, 595, 630, 666, 703, 741, 780, 820, 861, 903, 946, 990, 1035, 1081, 1128, 1176, 1225, 1275, 1326, 1378, 1431
Name: Triangular numbers: $a(n) = C(n+1, 2) = n(n+1)/2 = 0+1+2+\dots+n$.
Comments: Number of edges in complete graph of order n , K_n .
Number of legal ways to insert a pair of parentheses in a string of n letters. E.g. there are 6 ways for three letters: (a)bc, (ab)c, (abc), a(b)c, a(bc), ab(c). [Proof: there are $C(n+2, 2)$ ways to choose where the parentheses might go, but $n+1$ of them are illegal because the parentheses are adjacent.] Cf. [A002415](#).
For $n \geq 1$ $a(n) = n(n+1)/2$ is also the genus of a nonsingular curve of degree $n+2$ like the Fermat curve $x^{n+2} + y^{n+2} = 1$ - Ahmed Fares (ahmedfares(AT)my-deja.com), Feb 21 2001
 $a(n)$ is the number of ways in which $n+2$ can be written as a sum of three positive integers if representations differing in the order of the terms are considered to be different. In other words $a(n)$ is the number of positive integral solutions of the equation $x + y + z = n+2$. - Amarnath Murthy (amarnath_murthy(AT)yahoo.com), Apr 22 2001

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