

CSE 20—Discrete Math

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Day 10
Sets

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Lexicographic Ordering

“Dictionary” ordering

- Words
 - $\text{ban} < \text{bat} < \text{car} < \text{cat} < \dots < \text{rat}$
- Numbers (add leading zeros so that each number has the same number of digits)
 - $000 < 009 < 010 < 157 < 158 < 999$
- Lists (elements within the list need a defined ordering)
 - $\text{red} < \text{green} < \text{blue}$
 - $A < AA < AAA$
 - $05 < 10 < 12$
 - $(\text{red}, A, 05) < (\text{red}, A, 10) < (\text{red}, A, 12) < (\text{red}, AA, 05) < (\text{red}, AA, 10) < (\text{red}, AA, 12) < (\text{red}, AAA, 05) < (\text{red}, AAA, 10) < (\text{red}, AAA, 12) < \dots < (\text{blue}, AAA, 12)$

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Lexicographic Ordering

What if we have items of varying lengths?

- If two items have different lengths, the shorter one sorts before the longer one if the shorter one is a prefix of the longer one
- Words in dictionary
 - $A < AA < AAA < B < BAT < \dots < Z < ZY < ZZ < ZZZ < ZZZZZZZZ$
- Tuples of varying lengths
 - $S^0 = \epsilon$ (empty string)
 - $S^1 = S$
 - $S^n = SxSx\dots xS$ (n times)
 - $S^* = \text{union of } S^0, S^1, S^2, \dots$
- How to sort S^* ?
 - ϵ sorts before any other string
 - $(a_0, \dots, a_m) \leq (b_0, \dots, b_n)$ if:
 - $m \leq n$ and $\forall i, a_i = b_i$
 - or, $m = n$ and $\exists j: a_j < b_j, \forall i < j, a_i = b_i$

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Counting

Given the set $S = \{x, y, z\}$

How many ways to list (without duplicates) elements of S ?

- Permutations of S

How many ways to list (with duplicates) elements of S ?

How many ways to construct a k -list of distinct elements from S ?

How many ways to construct a 2-set of (distinct) elements?

How many ways to construct a 2-multiset of elements?

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Permutations and Combinations

Permutation of n objects: ordering of all n objects

r -permutation: ordering of r of n objects

- Order matters
- $P(n, 2) = n(n-1)$
 $P(n, 3) = n(n-1)(n-2)$
 $P(n, r) = n(n-1)(n-2)\dots(r-1)$
 $P(n, n) = n(n-1)\dots(2)(1) = n!$
- $P(n, r) = n(n-1)(n-2)\dots(n-(r-1)) = n!/(n-r)! = n^r_rP$

r -combination of n objects: choosing r of n objects (unordered)

- $C(n, r) = \# \text{ of } r\text{-permutations} / \# \text{ of ways to order } r \text{ items}$
 $= P(n, r) / P(r, r)$
 $= n!/(n-r)! / r!$
 $= n!/(n-r)!r!$

- $C(n, r) = C(n, n-r)$

$$\binom{n}{r} = C(n, r) = \frac{n(n-1)\dots(n-r+1)}{r!} = \frac{n!}{r!(n-r)!}$$

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Example Problem

How many ways are there to rank n candidates for the job of chief wizard?

If rankings are chosen at random, what is the probability that the fifth candidate is in second place?

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Example Problem

If a 5-card hand is chosen at random, what is the probability of a flush (all 5 cards of the same suit)?

- How many 5-card hands are there?
- How many 5-card hands are flushes?

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Binomial Theorem

Let's look at $(x+y)^n$

- $(x+y)^3 = xxx + xxy + xyx + xyy + yxx + yxy + yyx + yyy$

$$= x^3 + 3x^2y + 3xy^2 + y^3$$

- How many terms of the form x^ky^{3-k} in $(x+y)^3$?

How many ways to choose k x s and $(3-k)$ y s from 3 total?
 - $C(3, k)$

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Binomial Theorem

Let's look at $(x+y)^n$

- What is the coefficient of $x^k y^{n-k}$ in $(x+y)^n$?

$$\binom{n}{k}$$

Binomial Theorem:

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \dots + \binom{n}{k}x^{n-k}y^k + \dots + \binom{n}{n}y^n$$

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Binomial Identities

Symmetry Identity

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \binom{n}{n-k}$$

Fundamental Identity

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

- Algebraic Proof
- Proof by Pascal's triangle
- Proof by combinatorial argument

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Pascal's Triangle

Number of paths equals $C(n, k)$

	k=0	k=1	k=2	k=3	k=4
n=0					
n=1					
n=2					
n=3					
n=4					

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Representing Subsets as (0, 1) vectors

Given S , an n -set, a subset T can be specified by:

- Listing each element of S
- or, listing, for every element of S , whether or not it belongs to T
- Characteristic function, $\chi_T(x) = 1$ iff $x \in T$ (domain: S , range: $\{0, 1\}$)
- General notation:
 - $[A] = \chi(A) = 1$ if A is true, 0 if A is false

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Powersets and Binomial Coefficients

How many subsets of an n -set are there total?

How many k -subsets of an n -set are there?

Therefore:

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Partitions of a Set

Break a set S into a *partition*: set of subsets $\{S_1, \dots, S_k\}$ such that:

- $S_1 \cup S_2 \cup \dots \cup S_k = S$ ($\{S_1, \dots, S_k\} \subseteq P(S)$)
- $\forall i, S_i \neq \emptyset$
- $\forall i, \forall j \neq i, S_i \cap S_j = \emptyset$

How many partitions of a set of size n ?

- At least:
- No more than:
- B_n = number of partitions of a set of size n (Bell number)
- $B_1 = 1$
 $B_2 = 2$
 $B_3 = 5$
 $B_4 = 15$

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Refinement of a Partition

Given a partition T , of S

- Choose an element e , of size > 1 of T
- Let $T' = T$ but replacing e with the elements of a partition of e
- T' is a *refinement* of T

Example:

- Refinement of $\{\{1, 2, 3\}, \{5, 8\}\}$

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