### 8.3.1 The Image Brightness Constancy Equation

It is common experience that, under most circumstances, the apparent brightness of moving objects remains constant. We have seen in Chapter 2 that the image irradiance is proportional to the scene radiance in the direction of the optical axis of the camera; if we assume that the proportionality factor is the same across the entire image plane, the constancy of the apparent brightness of the observed scene can be written as the stationarity of the image brightness $E$ over time:

$$\frac{dE}{dt} = 0. \tag{8.15}$$

☞ In (8.15), the image brightness, $E$, should be regarded as a function of both the spatial coordinates of the image plane, $x$ and $y$, and of time, that is, $E = E(x, y, t)$. Since $x$ and $y$ are in turn functions of $t$, the *total* derivative in (8.15) should not be confused with the *partial* derivative $\partial E/\partial t$.

Via the chain rule of differentiation, the total temporal derivative reads

$$\frac{dE(x(t), y(t), t)}{dt} = \frac{\partial E}{\partial x}\frac{dx}{dt} + \frac{\partial E}{\partial y}\frac{dy}{dt} + \frac{\partial E}{\partial t} = 0. \tag{8.16}$$

The partial spatial derivatives of the image brightness are simply the components of the spatial image gradient, $\nabla E$, and the temporal derivatives, $dx/dt$ and $dy/dt$, the components of the motion field, $\mathbf{v}$. Using these facts, we can rewrite (8.16) as the image brightness constancy equation.

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**The Image Brightness Constancy Equation**

Given the image brightness, $E = E(x, y, t)$, and the motion field, $\mathbf{v}$,

$$(\nabla E)^{\top}\mathbf{v} + E_t = 0. \tag{8.17}$$

The subscript $t$ denotes partial differentiation with respect to time.

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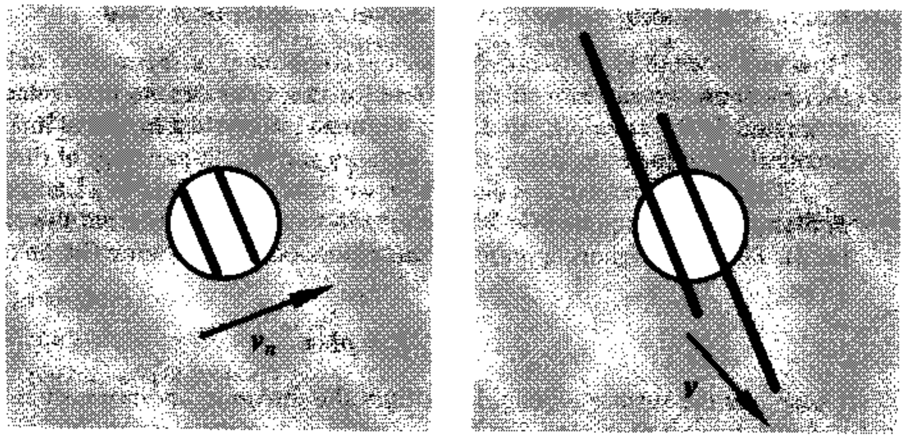
We shall now discuss the relevance and applicability of this equation for the estimation of the motion field.

### 8.3.2 The Aperture Problem

How much of the motion field can be determined through (8.17)? *Only its component in the direction of the spatial image gradient,*[9] $v_n$. We can see this analytically by isolating the measurable quantities in (8.17):

$$-\frac{E_t}{\|\nabla E\|} = \frac{(\nabla E)^{\top}\mathbf{v}}{\|\nabla E\|} = v_n \tag{8.18}$$

[9] This component is sometimes called the *normal component*, because the spatial image gradient is normal to the spatial direction along which image intensity remains constant.




(a) (b)

**Figure 8.7** The aperture problem: the black and grey lines show two positions of the same image line in two consecutive frames. The image velocity perceived in (a) through the small aperture, $\mathbf{v}_n$, is only the component parallel to the image gradient of the true image velocity, $\mathbf{v}$, revealed in (b).

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**The Aperture Problem**

The component of the motion field in the direction *orthogonal* to the spatial image gradient is not constrained by the image brightness constancy equation.

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The aperture problem can be visualized as follows. Imagine to observe a thin, black rectangle moving against a white background through a small aperture. "Small" means that the corners of the rectangle are not visible through the aperture (Figure 8.7($a$)); the small aperture simulates the narrow support of a differential method. Clearly, there are many, actually infinite, motions of the rectangle compatible with what you see through the aperture (Figure 8.7($b$)); the visual information available is only sufficient to determine the velocity in the direction *orthogonal* to the visible side of the rectangle; the velocity in the *parallel* direction cannot be estimated.

☞ Notice that the parallel between (8.17) and Figure 8.7 is not perfect. Equation (8.17) relates the image gradient and the motion field at the *same* image point, thereby establishing a constraint on an *infinitely small* spatial support; instead, Figure 8.7 describes a state of affairs over a *small but finite* spatial region. This immediately suggests that a possible strategy for solving the aperture problem is to look at the spatial and temporal variations of the image brightness over a neighborhood of each point.[10]

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[10] Incidentally, this strategy appears to be adopted by the visual system of primates.

We describe a differential technique that gives good results. The basic assumption is that the motion field is well approximated by a *constant* vector field, $\mathbf{v}$, within any small region of the image plane.[11]

**Assumptions**

1. The image brightness constancy equation yields a good approximation of the normal component of the motion field.
2. The motion field is well approximated by a *constant* vector field within any small patch of the image plane.

***An Optical Flow Algorithm.*** Given Assumption 1, for each point $\mathbf{p}_i$ within a small, $N \times N$ patch, $Q$, we can write

$$(\nabla E)^\top \mathbf{v} + E_t = 0$$

where the spatial and temporal derivatives of the image brightness are computed at $\mathbf{p}_1, \mathbf{p}_2 \dots \mathbf{p}_{N^2}$.

☞ A typical size of the "small patch" is $5 \times 5$.

Therefore, the optical flow can be estimated within $Q$ as the constant vector, $\bar{\mathbf{v}}$, that minimizes the functional

$$\Psi[\mathbf{v}] = \sum_{\mathbf{p}_i \in Q} \left[ (\nabla E)^\top \mathbf{v} + E_t \right]^2 .$$

The solution to this least squares problem can be found by solving the linear system

$$A^\top A \mathbf{v} = A^\top \mathbf{b}. \tag{8.22}$$

The $i$-th row of the $N^2 \times 2$ matrix $A$ is the spatial image gradient evaluated at point $\mathbf{p}_i$:

$$A = \begin{bmatrix} \nabla E(\mathbf{p}_1) \\ \nabla E(\mathbf{p}_2) \\ \vdots \\ \nabla E(\mathbf{p}_{N \times N}) \end{bmatrix}, \tag{8.23}$$

and $\mathbf{b}$ is the $N^2$-dimensional vector of the partial temporal derivatives of the image brightness, evaluated at $\mathbf{p}_1, \dots \mathbf{p}_{N^2}$, after a sign change:

$$\mathbf{b} = -\left[ E_t(\mathbf{p}_1), \dots, E_t(\mathbf{p}_{N \times N}) \right]^\top . \tag{8.24}$$

[11] Notice that this is in agreement with the first conclusion of section 8.2.3 (motion field of moving planes) regarding the approximation of smooth motion fields.

The least squares solution of the overconstrained system (8.22) can be obtained as[12]

$$\bar{\mathbf{v}} = (A^\top A)^{-1} A^\top \mathbf{b}. \tag{8.25}$$

$\bar{\mathbf{v}}$ is the optical flow (the estimate of the motion field) at the center of patch $Q$; repeating this procedure for all image points, we obtain a dense optical flow. We summarize the algorithm as follows:

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**Algorithm CONSTANT_FLOW**

The input is a time-varying sequence of $n$ images, $E_1, E_2, \ldots E_n$. Let $Q$ be a square region of $N \times N$ pixels (typically, $N = 5$).

1. Filter each image of the sequence with a Gaussian filter of standard deviation equal to $\sigma_s$ (typically $\sigma_s = 1.5$ pixels) along each spatial dimension.
2. Filter each image of the sequence along the temporal dimension with a Gaussian filter of standard deviation $\sigma_t$ (typically $\sigma_t = 1.5$ frames). If $2k + 1$ is the size of the temporal filter, leave out the first and last $k$ images.
3. For each pixel of each image of the sequence:
   (a) compute the matrix $A$ and the vector $\mathbf{b}$ using (8.23) and (8.24)
   (b) compute the optical flow using (8.25)

The output is the optical flow computed in the last step.

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☞ The purpose of spatial filtering is to attenuate noise in the estimation of the spatial image gradient; temporal filtering prevents aliasing in the time domain. For the implementation of the temporal filtering, imagine to stack the images one on top of the other, and filter sequences of pixels having the same coordinates. Note that the size of the temporal filter is linked to the maximum speed that can be "measured" by the algorithm.

***An Improved Optical Flow Algorithm.*** We can improve CONSTANT_FLOW by observing that the error made by approximating the motion field at $\mathbf{p}$ with its estimate at the center of a patch increases with the distance of $\mathbf{p}$ from the center itself. This suggests a *weighted* least-square algorithm, in which the points close to the center of the patch are given more weight than those at the periphery. If $W$ is the weight matrix, the solution, $\bar{\mathbf{v}}_w$, is given by

$$\bar{\mathbf{v}}_w = (A^\top W^2 A)^{-1} A^\top W^2 \mathbf{b}.$$

***Concluding Remarks on Optical Flow Methods.*** It is instructive to examine the image locations at which CONSTANT_FLOW fails. As we have seen in Chapter 4, the $2 \times 2$ matrix

$$A^\top A = \begin{pmatrix} \sum E_x^2 & \sum E_x E_y \\ \sum E_x E_y & \sum E_y^2 \end{pmatrix}, \tag{8.26}$$

[12] See Appendix, section A.6 for alternative ways of solving overconstrained linear systems.