# Chernoff Bound

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We will prove a fairly general form of the Chernoff bound. This proof was given by Van Vu at the University of California, San Diego.

**Theorem 1.** *Let $X_1, \ldots, X_n$ be discrete, independent random variables such that $\mathrm{E}[X_i] = 0$ and $|X_i| \leq 1$ for all $i$. Let $X = \sum_{i=1}^{n} X_i$ and $\sigma^2$ be the variance of $X$. Then*

$$\Pr[|X| \geq \lambda\sigma] \leq 2e^{-\lambda^2/4}$$

*for any $0 \leq \lambda \leq 2\sigma$.*

*Proof.* We will prove

$$\Pr[X \geq \lambda\sigma] \leq e^{-\lambda^2/4}.$$

The argument is symmetric for $\Pr[-X \geq \lambda\sigma]$. Let $t$ be a real number between 0 and 1, to be determined later. Note that

$$\begin{aligned} \Pr[X \geq \lambda\sigma] &= \Pr[tX \geq t\lambda\sigma] \\ &= \Pr[e^{tX} \geq e^{t\lambda\sigma}] \\ &\leq \frac{\mathrm{E}[e^{tX}]}{e^{t\lambda\sigma}}, \end{aligned}$$

the last step by the Markov inequality.

Before going any further, we establish a bound on $\mathrm{E}[e^{tZ}]$, where $-1 \leq Z \leq 1$ and $\mathrm{E}[Z] = 0$. Additionally, let $t \leq 1$. By the definition of expectation,

$$\begin{aligned} \mathrm{E}[e^{tZ}] &= \sum_{j=1}^{m} p_j e^{tz_j} \\ &= \sum_{j=1}^{m} p_j \big(1 + tz_j + \frac{1}{2!}(tz_j)^2 + \frac{1}{3!}(tz_j)^3 + \cdots\big) \\ &= \underbrace{\sum_{j=1}^{m} p_j}_{A} + t\underbrace{\sum_{j=1}^{m} p_j z_j}_{B} + \underbrace{\sum_{j=1}^{m} p_j \big(\frac{1}{2!}(tz_j)^2 + \frac{1}{3!}(tz_j)^3 + \cdots\big)}_{C}. \end{aligned}$$

Summation $A$ is the sum of all probabilities, so $A = 1$. Summation $B$ is the expectation of $Z$, so $B = 0$. Since $|tz_j| \leq 1$, we can upper-bound $C$ by

$$\sum_{j=1}^{m} p_j (tz_j)^2 \left(\frac{1}{2!} + \frac{1}{3!} + \cdots\right) \leq t^2 \sum_{j=1}^{m} p_j z_j^2.$$

But the above summation is just the variance of $Z$, giving

$$\mathrm{E}[e^{tZ}] \leq 1 + t^2 \mathrm{Var}[Z].$$

Returning to our claim,

$$\begin{aligned}
\mathrm{E}[e^{tX}] &= \mathrm{E}[e^{t(X_1+X_2+\cdots+X_n)}] \\
&= \mathrm{E}\big[\textstyle\prod_{i=1}^n e^{tX_i}\big] \\
&= \textstyle\prod_{i=1}^n \mathrm{E}[e^{tX_i}] && \text{by the independence of } X_i \\
&\leq \textstyle\prod_{i=1}^n \big(1 + t^2\mathrm{Var}[X_i]\big) \\
&\leq \textstyle\prod_{i=1}^n e^{t^2\mathrm{Var}[X_i]} && \text{since } 1+\alpha \leq e^\alpha \text{ for } \alpha \geq 0 \\
&= e^{t^2\sigma^2}. && \text{by the independence of } X_i
\end{aligned}$$

Thus,

$$\begin{aligned}
\Pr[X \geq \lambda\sigma] &\leq \frac{e^{t^2\sigma^2}}{e^{t\lambda\sigma}} \\
&= e^{t\sigma(t\sigma-\lambda)}.
\end{aligned}$$

Optimizing $t$ we get $t = \lambda/2\sigma$, which gives

$$\Pr[X \geq \lambda\sigma] \leq e^{-\lambda^2/4}.$$

□