May 13, 1998

Dynamic Logic[1]

1. Introduction

In this paper I shall develop a dynamic variant of formal logic. The motivation for this is the need for a logic that can describe processes. In fact, the present work originally had a very practical purpose, namely to construct a logic that would be helpful in designing interactive multimedia systems (Øhrstrøm & Bøgh Andersen 1994).

The present paper, however, is broader and more radical in scope, and does not take its point of departure in formal temporal logic (Øhrstrøm & Hasle 1995). Instead, the paper inscribes itself in the setting of non-linear dynamic systems theory (see e.g. Ott 1993, Smith & Thelen eds., 1993, Thom 1989, 1990, Peitgen, Jürgens & Saupe 1992, Maturana & Varela 1980, Luhmann 1984, 1991, Mingers 1995). Aspects of this setting is described in Bøgh Andersen (1996). I cannot present arguments for adopting a dynamic systems approach here, only sketch the necessary background for the ideas of the paper. A systematic treatment can be found in Bøgh Andersen (in preparation).

The basic concept is the operationally closed recursive system. A system is something that is delimited from its environment and which displays an autonomous mode of operation. It is recursive if its mode of operation is based on a recursive process that uses its own output as input. It is operationally closed if its unity is maintained by a circular process, where ultimately every process indirectly contributes to its own input. Examples are biological systems such as bodies, social systems such as organisations, and psychic systems. Although such systems are closed in the sense that the environment cannot directly interfere with their reproductive process, they are open in the sense that the recursive process can be perturbed by the environment, i.e. environmental influences can cause the process to take another trajectory. For example, although we control our train of thoughts ourselves, our environment can certainly make it take a new direction. Similarly, although only money “goes on” in the economic system, the trajectory of payments is certainly influenced by texts from the legal system; and, conversely, although the legal system only produces legality through legislation and court orders, it is influence by changes in the economic system. We say that the systems can *compensate* for the environmental *perturbation*. Descriptions employing these concepts are said to use the *perturbed recursion schema*. It is possible for two systems to act as each other's environment, i.e. to perturb each other mutually. For example, the social linguistic system leaves it mark upon the individual psychic system, e.g. by ordering our thoughts, but the psychic system animates our language by charging it with desires and emotions.

Although the exact nature of the recursive processes is different for different systems, I only use only one type of process in this paper, namely a *gradient dynamics*. The process itself is specified by a parametrized complex potential and the trajectory of the system follows the gradient of the potential. The parameters represent the influence of the environment on the system. Parameter changes can cause catastrophes in the system in the sense that they cause abrupt changes of equilibrium conditions. For example, a wrong intonation in a conversation can completely change the emotional dynamics of the speakers.

The basic unit of description is the *phase-space* (sometimes called a property space). A phase-space consists of one or more dimensions, and the state of a system is given by a particular value of each dimension. For example, the location of a physical system is given by its three spatial coordinates. If a dynamic is defined on the phase-space I call it a *dynamic phase space.* Physical systems are often described by dynamic phase spaces, but there is a tradition for applying dynamic concepts in psychology (e.g. the Freudian tradition) and economics (e.g. the Marxist tradition).

Both society and psyche are analysed as a collection of autonomous subsystems, i.e. they are *decentered*. The subsystems are not apriori regulated by any "boards of directors" with intentionality, but have emerged in the course of history. Luhmann advocates this view in sociology; in cognitive science, the point of view can be found in Varela, Thompson & Rosch (1991), and evidence from experimental psychology is provided by Engelkamp & Zimmer (1994).

The simple phasespace only represents *episodes*, i.e. discrete or continuous changes of a system. It can be used to describe *episodic* memory, but this is clearly insufficient for describing texts which refer to possible worlds of obligation, intension, and possibility. In order to solve this problem, I adopt the idea in Ryan (1991) of "systems of reality". A *system of reality* consists of an actual world (an episodic semantic system) surrounded by possible world satellites representing intensions, obligations, possibilities, and values concerning the episodes of the actual world.

The last concept we need is the distinction between an *unmarked* and a *marked* dynamics. The unmarked dynamics is the normal default dynamics of a system, whereas the marked dynamics is a deviation from the default dynamics. It is needed in semantics to account for the fact that large parts of the meaning of a text is tacitly presupposed. It corresponds to the notion of *schemas* or *frames*. The result of combining the marked and the unmarked dynamics is simply their sum.

The basic point in this paper is that logic, like the semantic and syntactic system, is an autonomous system with its own laws. Still it can perturb and be perturbed by the icons of the semantic systems and by the symbols of the syntactic system. And, conversely, the syntactic system can be perturbed by the symbols of the logical system.

Although the logic system contains its own laws, it shares elements with the syntactic system. Many words from natural syntax also occurs in logical notations, but the syntax, word classes, and dynamics are different. For example, a basic distinction in logic runs between predicates and arguments. Predicates correspond to nouns, adjectives, and verbs in language, and arguments resemble pronouns. Thus, some distinctions from natural syntax are cancelled in logic and vice versa.

Quantifiers provide an example of differences in syntax. In natural language, quantifiers are modifiers to nouns, whereas they are modifiers to propositional functions in logic. Thus,

(1) All children have some toy

is translated into

(2) $\forall x\ (child(x) \Rightarrow \exists y\ (toy(y) \wedge owns(x,y))$

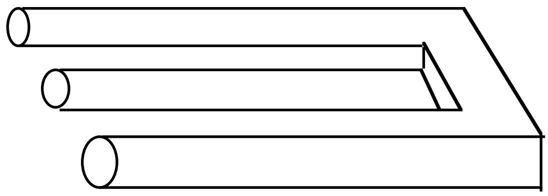
“For all x, if x is a child, then there exists an y, such that y is a toy and x owns y”. Although the logical formula can be translated into natural language, the translation is awkward[2].

Like the syntactic system, the logical system contains symbols, not icons as the semantic system. But unlike syntax, logic must to a large degree be consciously learned, and is in fact taught as a part of several professions, such as computer programmers and electricians.

The status of logic is much debated. One position holds that logic is artificial and has very little to do with human thoughts. And indeed, the meaning of conjunctions like *and* and *or* is different in language and in logic. But the opposite opinion has also been voiced, that the semantics of language is identical to a logical representation. This view was first systematically explored in Reichenbach (1947) and was later incorporated into Chomsky’s transformational grammar.

The view advanced here disagrees with both positions: logic is an independent system that is different from the semantic iconic system and the syntactic symbolic system, but can influence both systems. Logic is as real and natural as are any other learned skill: if people can learn to apply logic and in fact do it, then it is a part of the human mental faculties.

This view predicts that conflicts between systems can arise, and this is indeed the case. We experience conflicts between the visual system and the logical system in the so-called "impossible drawings" one of which is shown in Fig. 1.1.



$\forall x\ (\text{swan}(x) \supset \text{white}(x))$

(6) $\exists x\ (\text{house}(x) \wedge \text{white}(x))$

Part of the motivation for introducing transformations in transformational grammar is the difference between deep-structure, which is assumed to be some kind of logical representation, and surface-structure which must be identical to actual sentences.

Thus, the interaction between logic on the one hand and the semantic and syntactic system on the other hand is of the same kind as any other interaction between two systems, e.g. between semantics and syntax, or between the marked and unmarked dynamics in a single system, namely: there are two or more sources generating two or more sets of potentials, and the resultant potential that determines the actual attractors of the system is simply the sum of the competing potentials.

In fact, the logic system seems to belong to the "world"-category described in Ryan (1991): most of the sentences of the syntactic system occurs in the logic system but the laws governing them is unique to logic. Logic seems to be yet another satellite orbiting the Actual World of episodes.

As in the case of semantics and syntax, the balance of power between the competing sources can be varied, so that one source becomes stronger than the other. Actual applications of formal logic can be described as situations where the influence from the syntactic system has been disconnected so that only the operations of the logical systems are active. Although this is a situation that is difficult to bring about for many people, it is clearly a skill that can be learned.

This account of the interference between logic, semantics and syntax requires logic to be analysed in the same manner of the other two systems. Otherwise the result of their interaction cannot be defined. The purpose of this chapter is therefore to construct a theory of dynamic logic within the general framework set up so far.

One place to look for inspirations for a dynamic versions of logic is in the analysis of narratives by A. J. Greimas and others.

According to structural semiotics, the driving force of any discourse — its deep structure — is an opposition. Spy and detective novels exploit the opposition between Being and Seeming, Tarzan stories exploit the difference between Nature and Culture, and innumerable soap operas excel in Love versus Hate, Rich versus Poor, Good versus Evil. Negating the two opposites gives us the four terms that constitute Greimas' semiotic square, *the fundamental semantics* of a narrative.

The square in Fig. 2.2, borrowed from Greimas & Courtés (1979), has been exploited skilfully by John Le Carré, e.g. in "Tinker, Taylor, Soldier, Spy". Our hero George Smiley *is* Loyal but does not *seem* so, since he has recently been sacked. Whereas Smiley belongs to the Secret axis, other characters occupy the Lie side; they *seem* Loyal but *turns out* to be double agents — like the new chief of the British Intelligence Agency who turns out to be a mole.

Truth

	Being	Seeming	
Secret	Not Seeming	Not Being	Lie

Falsity

1.1. Narratives need a dynamic logic

The square, however, does not specify any stories, since it is a static *paradigm*. We need to convert it into linear *syntagms*, to unfold it in time, to create a *generative trajectory,* as Greimas calls it.

Smiley and his antagonist must be made to move around in the square. In the beginning of the book, the mole is chief of the British Intelligence, so he starts by Seeming (loyal) but not Being it. In the end he is exposed as a traitor, and changes to neither Being nor Seeming. Smiley, on the other hand, has just been fired when the book starts, but is approached and hired to make a discreet investigation. From the square of secrets, Being but not Seeming, Smiley succeeds in both Being and Seeming at last. His secret is disclosed, and he is made (temporary) chief of the spy organisation.

The operations moving subjects around among the predicates of the semiotic square are called *syntactic operations* by Greimas. They are quasi-logical of nature and essentially consist in negating or asserting predicates.

But since the task of the semiotic square is to explain dynamic phenomena, its static nature is somewhat awkward, and it seems worthwhile searching for concepts that are inherently dynamic and do not need logical rules imposed from the outside. Petitot (1985) investigates catastrophe theory with a similar purpose in mind. Although the present paper stays closer to classical Boolean logic, it has the same objective as Petitot's book: the idea is to conceive of logic as a dynamic phenomenon.

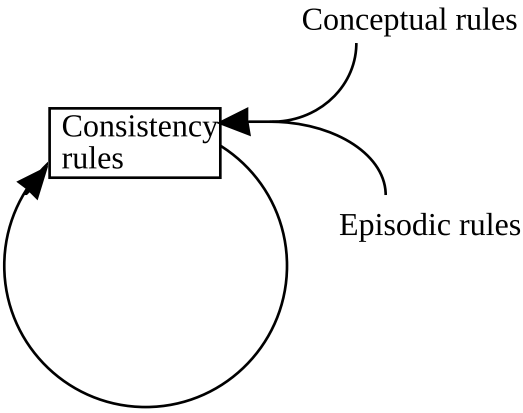
Informed by Greimas' original proposal, we start by defining 4 components:

1. *A state-description.* It represents the current state of the actors. In the beginning, we would have: *Be(Smiley, Loyal), ¬Seem(Smiley, Loyal), ¬Be(Mole, Loyal), Seem(Mole, Loyal),*
2. *Consistency Rules,* corresponding to Greimas' fundamental semantics. Fast "housekeeping" rules ensuring consistency with the laws of the fictive universe at every stage. For example, if Carré's cold war universe is constituted by the law "nothing is what it seems", $\forall x,y{:}Seem(x,y) \rightarrow \neg Be(x,y)$

treasure) into *¬Has(King, treasure),* and the deed of the hero changes it back into *Has(King, treasure)*. In the Smiley example, the disclosure of the mole is effected by a negation converting *Seem(Mole, Loyal)* into *¬Seem(Mole, Loyal).*

4. *Conceptual Rules.* Many narratives contain Conceptual Rules changing the conditions under which (2), the fundamental semantics, operates. One example is mediation, also called syncretism, in linguistics. Mediating stories often start with a pair of antonyms — Seeming versus Being, Culture versus Nature — and the point of the story is to construct a character who, alone of all, is able to span both predicates, in spite of their contrariety. Tarzan is such a figure, and possibly Smiley: in the end of the novel he is recognised as Loyal by all while really being so. But his outward recognition — his Seeming— is only temporary, and he will probably be replaced by a younger person. Conceptual Rules change the relationships between concepts; in the Smiley example they change $\forall x, y{:} Seem(x,y) \rightarrow \neg Be(x,y)$ $\exists x, y{:} Seem(x,y) \wedge Be(x,y)$

are similar to the compensation rules, and that the Episodic and Conceptual Rules both act as perturbations of the system.



ignore them as deviant cases, as we normally do in logic. In Carré's paranoid cold war atmosphere, loyalty to one power entails non-loyalty to other powers:

(1) $\forall x, y, z: Loyal(x, y) \wedge (z \neq y) \rightarrow \neg Loyal(x, z)$

$\neg(A \wedge \neg A)$ among their axioms and uses reductio ad absurdum as a method of proof, Carré is quite happy with having $Loyal \wedge \neg Loyal$ in his book.

If we extend semiotic analysis to cover culture and daily life, the inadequate handling of dilemmas becomes even more problematic. The reason is that our daily world is full of dilemmas which we handle routinely and non-arbitrarily. A common way of coping with dilemmas is to unfold them in time, so that no instant contains a contradiction, even if the truth-values of the problematic sentences change cyclically[3].

For example, parents should spend most of their energy on work if they want to be good workers, but if they want to qualify as good parents they must spend it on their children. If they want both, they must spend the major part of their time on work and on their children, and if these two activities do not contain overlapping actions, they are faced with a contradiction, $A \wedge \neg A$. Unfolding the contradiction in time simply means to sometimes neglect job and take care of children, sometimes the opposite. The truth value of A will neither be constant nor random, but oscillate regularly between true and false.

Apart from historically contingent dilemmas, like the work-children dilemma, there are also genuine logical dilemmas. The liar-paradoxes like “This sentence is false” have haunted logicians since ancient Greece. The problem is that it seems to oscillate between two truth-values, true and false, much in the manner of the haunted parents above. In the framework presented here, these paradoxes can be given a rational description in terms of stability and periodicity, cf. Gupta & Belnap (1993).

Instability due to self-reference is a characteristic of some philosophical positions.

One example is the dilemma of *critical philosophers.* They claim that all members of capitalist societies suffer from false consciousness, which must mean that the assertions made by those members are false or at least perverted. But critical philosophers are members of a capitalist society so they too must suffer from false consciousness. Hence the theory they have just asserted must be false or perverted; but if it is false that all members suffer from false consciousness, then at least one member of a capitalist society must exist that does not have false consciousness[4].

Radical constructivism has the same self-destructive tendency (Bøgh Andersen 1995): if no necessary sentence exists, then that very statement must also be contingent, entailing that it is possible that necessary statements exist after all[5]!

In both cases we have three elements:

(1) A proposition *a* asserting a general feature of propositions.
(2) Self-application of *a* on *a* produces a new version of *a* contradicting the original version.
(3) But the contradiction opens a possibility of keeping *a* intact and maintaining the universe defined by *a*.

The interesting thing is not (1) and (2), that these world views are inherently inconsistent, but rather (3), the way they manage to overcome the inconsistency and maintain themselves, namely to spot and occupy the vacant exception opened by the inherent contradiction (the critical philosopher escapes miraculously from the ideological distortion of capitalism, the radical constructivist occupies the only position where necessary truth is possible).

Some authors consider such contradictions to be a necessary ingredient for systems to be able to adapt to change requirements of the environment:

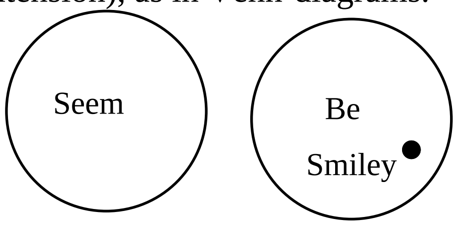
> So such systems ensure that they have something in reserve, a repertory of possible behaviours, ways of interacting, that gives them some flexibility, some plasticity of response. Some of their internal interactions work counter to perfect self-regulation: these systems embody contradictions to enlarge their range of behaviours, enhance their resilience [...]. *Lemke 1995: 176.*

1.3. Construction of a dynamic logic

Dynamic logics can be built in many ways. In the remaining part of the paper I shall sketch a simple version that incorporates the four kinds of rules from Section 1.1 in one framework, and which is able to cope with contradictions. The construction is intended to be close to ordinary Boolean logic, but — unfortunately — seems to deviate on some points.

The construction runs as follows: we use a phase-space in which our symbols, predicates and their arguments, are scattered. For illustrative purposes, I shall only use a plane or a line as spaces

The location of a *predicate* represents its *extension* (or rather, as we shall see, the centre of its extension), as in Venn-diagrams.



whereas its marked part typically concerns only one. The total field is specified by combining predicates by means of the logical conjunctions. The actual dynamics is built recursively by assigning simple fields to the predicates themselves, and then modulate the simple fields according to the meaning of the conjunctions that unite the predicates.

The unmarked dynamics represents the *intension* of the concepts. In formal logic intension is normally defined by meaning postulates; for example, antonymity can be defined as $\forall x,y{:}\,Seem(x,y) \rightarrow \neg Be(x,y)$

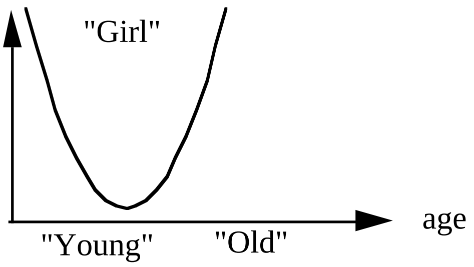


Fig. 1.5. Stability. “The girl is young”

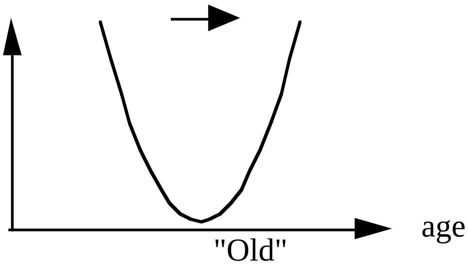


Fig. 1.6. Instability. “The girl grows old”

The forces pushing subjects towards equilibria — if any exists — correspond to the fast *Consistency Rules (2)* in Section 1.1. The slower *Episodic Rules (3)* are represented by changes of the arguments. For example, moving the potential controlling “girl” to the right along the age dimension represents an ageing of the girl. As shown in Fig. 1.6, this perturbation will make “Girl” unstable and cause it to slide to the new equilibrium, above “old”.

In the following I indicate how number 4, the *Conceptual Rules,* can be described. I assume that there is no friction on the surfaces, and that the subject is only influenced by their gradient. On the other hand, I assume that the subject may possess a certain inertia, so that very small forces may not move it. The critical limit for movement is denoted ε. The limit is needed because we sometimes want the subject to rest, even if it is in fact influenced by very small forces.

2. Logic

This section presents one way of defining the force fields in the previous section. In the following I shall denote the force associated to an arbitrary expression E by f(E). There is a bit of mathematics involved in the following, but most of it has been put into footnotes in order not to bother readers with math fobia.

2.1. The logical connectives

Logical *or*, *P v Q,* is represented by a function consisting of the smallest values of f(P) and f(Q)[6].

Fig. 2.1 shows two predicates located in the same space but placed at different locations. The result of always taking the smallest value of the two curves, *P v Q,* is shown in Fig. 2.2. Visual inspection shows clearly that the dynamic meaning of *P v Q* is that the subject can rest in P, it can rest in Q, or in both if their topology permits.

Fig. 2.1. Two concepts, P and Q.

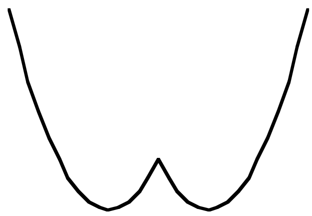


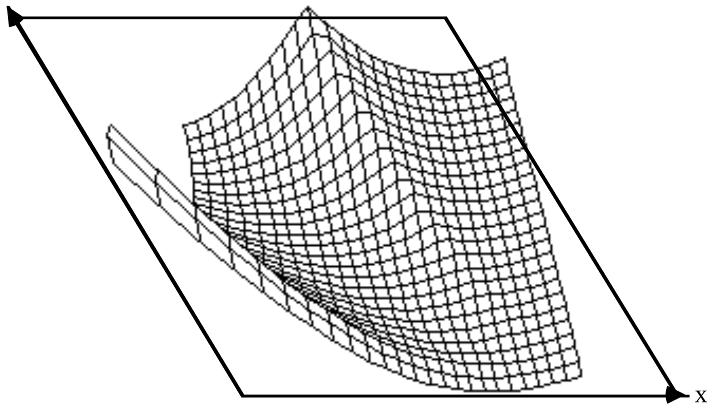
Fig. 2.2. The disjunction, P ∨ Q

Fig. 2.2 represents the general law that subjects must either be P or Q or both, but cannot avoid one of them. For example, persons must either be young or old or both, but cannot escape one of the predicates.

We can now exemplify the last process mentioned in Section 1.1, namely the *Conceptual Rules (4).* Conceptual Rules do not move objects, but change the force fields. In the dynamic setting, these rules are represented by changes of the “energy” landscape in Fig. 2.2.

The changes can either concern the location of the concepts, i.e. the surface they cover (change of *extension*) or it can involve the relation between the concepts (change of *intension*). The former describes a change of reality (material change), whereas the latter represents a change of interpretation (ideological change).

An example of the former is shown in Fig. 2.3, where we vary the distance between the two concepts but keep the *or*-operator constant. In the foreground of the drawing, P and Q have the same location, and we have only one minimum, so the subject can rest in both P and Q. As Q is displaced, a repulsor grows between them, marking the area covered by neither P nor Q. If $P \vee Q$ is to be true, the subject may not rest in this area, and this is indeed the case in our topology: neither the slopes nor the top of the ridge are equilibrium locations, so the subject will slide down into one of the valleys.



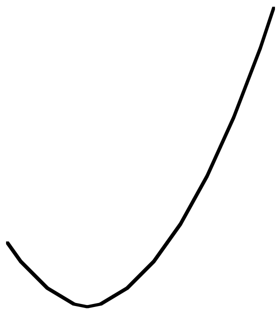


Fig. 2.4. $P \wedge Q$, P and Q close. There is an area of equilibrium in the bottom.

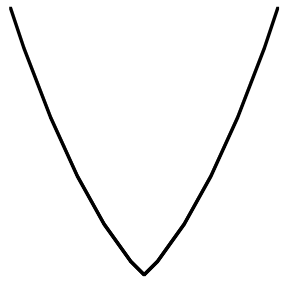


Fig. 2.5. $P \wedge Q$, P and Q distant. No equilibrium.

An example of the latter, i.e. change of *intension*, implies that one conjunction is continuously changing into another, for example *or* into *and*. Before we can give an example of this, we need to define logical *and*.

Logical *and* is represented by the maximum values of the component functions [8]. Look again at Fig. 2.1. The $\wedge$ operator is produced by always taking the highest value of the two curves. If they are close together, we get Fig. 2.4. Here the gradients at the bottom are so small that it works as a stable minimum representing the intersection of the two predicates.

However, if we displace the predicates, the intersection disappears, the bottom gets pointed, with gradients so large that it no longer functions as a stable location (Fig. 2.5). In this situation the two predicates have no common space where the subject can rest. Placing a subject in the bottom will make it slide up and down the slopes, never reaching equilibrium (remember that there is no friction and that the subject is only influenced by the gradient of the surfaces which never becomes zero).

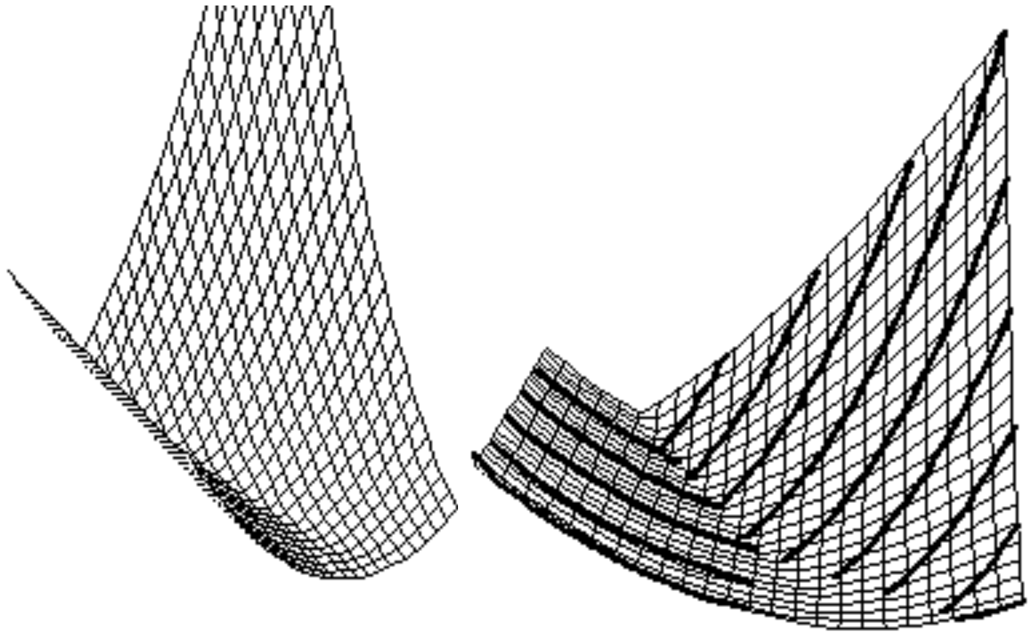
In the example with parents and workers the conjunction reads: “I am a good worker and a good parent”. If the two concepts are disjunct in reality, it is not possible to satisfy both requirements at the same time, and our subjects do what many parents do: in one week they try to be good parents, but have a bad conscience towards their workplace (“I am a good worker” is a lie), the next week the situation is reversed (“I am a good parent” is a lie).

This is the general method of representing dilemmas: situations where no equilibria exists and the subject must move for ever without rest. The pattern of oscillation occurs in every case where we try to achieve goals that are incompatible. For example, governments want to keep inflation and unemployment low, but according to the Philips curve these two magnitudes are inversely related,

(1) $$inflation \approx \frac{1}{unemployment}$$

$\wedge$ Q true.

If we travel from the background into the foreground, we experience that the oscillation between the two poles gradually diminishes and finally stops. This process is called *mediation* or *syncretism* and can be found in myths and popular stories that narrate how irreconcilable opposites approach and finally offer a resting ground for the hero. Stories of this kind need the 3D representation in Fig. 2.6 in order to represent both the ordinary actants and the mediator. In the Tarzan stories, only Tarzan can live in the foreground where equilibria exist between culture and nature: the animals, Negroes and white villains all live in the unstable background where each must chose either to belong to civilisation or to the jungle.

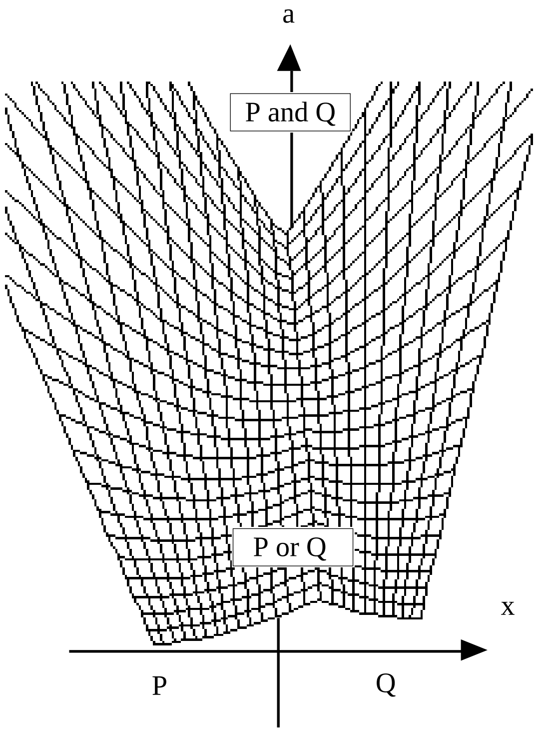


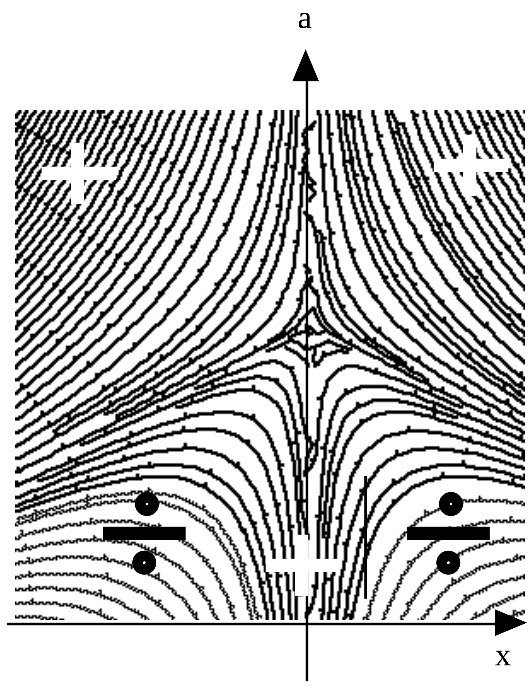
$\wedge$ to that of $\vee$. Since $\wedge$ was represented by max(f(P),f(Q)) and $\vee$ = min(f(P),f(Q)), where P and Q are predicates, we need to define a function F(a) that will be identical to Max for one value of the parameter *a,* and is identical to Min for another value of *a.* Fig. 2.8 and 2.9 depicts the dynamics of the formula $z = \min(x^2,y^2) + a(\max(x^2,y^2) - \min(x^2,y^2))$ where *a* ranges from 0 to 1, and x and y represent the locations of the predicates P and Q. When $a = 0$, $z = \min(x^2,y^2)$, that is, the formula represents a $\vee$. When $a = 1$, $z = \min(x^2,y^2) + \max(x^2,y^2) - \min(x^2,y^2) = \max(x^2,y^2)$, so the formula represents a $\wedge$.

With low *a,* the dynamics represents an *or*-relation, and there are two minima over P and Q. As the *or* morphs into an *and,* the two equilibria disappear and an unstable region appears between P and Q, cf. Fig. 2.8. Note that Fig. 2.8 does not display a bifurcation of one equilibrium into two as we decrease *a,* since the bottom of the valley in the and-section is not differentiable. There is no resting place there, the gradients are non-zero everywhere. If we place a ball in the upper end of Fig. 2.8 and move it downwards, it will start oscillating between P and Q, and then stabilise in one of the two basins as we reach the bottom.

In opposition to Fig. 2.7 that represents the effects of changing the extension (“reality”) and keeping the intension (“interpretation”) constant, Fig.

2.8 represents effects of changing the interpretation while keeping the reality intact. If we reuse the example of parents and workers, Fig. 2.7 depicts a situation where the interpretation is kept constant (“one ought to be a good parent *and* worker”) and reality is changed so that the two predicates move closer together (for example by laws changing the working times and offering leave for parents) and thereby create a place where both demands can be fulfilled. Fig. 2.8, in opposition, retains the actual working and family conditions, but changes the interpretation of the two concepts “worker” and “parent”. With large *a*, there is a law that both predicates must be true, but this requirement is relaxed as *a* decreases, and in the bottom the law is that one needs only fulfil one of the obligations.



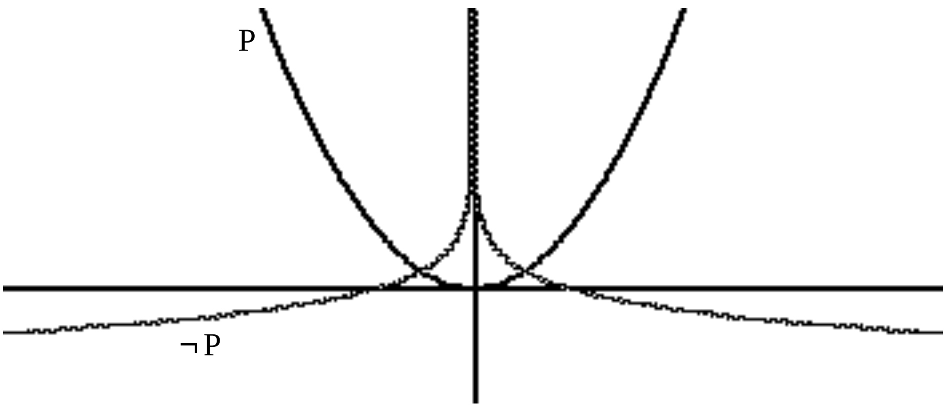


placed between the two concepts (are vira living or dead? Is a person dead when the heart has stopped functioning or should we look at the brain activity to get a useful criteria). As we shall see later, this location is an unstable position.

The next question is obviously: where does the a-parameter come from? That is, where are the sources of the perturbations that change the extension and intension of the concepts of the logical system? We shall offer an answer in Section 3.2, but now we need to finish the translation of formal logic into dynamic logic.

How should we represent negation? Suppose that a person A believes that the earth is a flat plane and is told it is not a plane but a ball. In this case, A suffers a major conceptual disaster, the subject “earth” quickly flees from the predicate “flat” and travels to “round”. The example shows that a negation must turn an attractor into a repulsor, that is: it must change the sign of the gradients.

In addition, the strength of the new vector must be inversely related to the old one. If the gradient of “earth” in A’s conception was near 0 when placed on “flat”, the new information must turn the near-zero gradient into a large one that can effect the major conceptual reorganisation. If A had not really believed that the earth was flat, and therefore had placed “earth” well away from “flat” — for example over “ellipsoid” — the new information should not affect him fundamentally, since a minor adaptation suffices. Therefore negation should turn a large gradient into a small one [9].



cate, P, is black whereas its negation, ¬P, is grey. Note that the slope of the negation has the opposite sign of the original, and that the absolute value of its gradient increases as the original’s decreases, and conversely.

By means of negation we can now represent antonyms like “old” and “young”. Antonyms obey the rule that the same subject cannot rest on both predicates at the same time, viz. that $\neg(P \wedge Q) = (\neg P \vee \neg Q)$ must be true. ¬P and ¬Q are shown in Fig. 2.11.

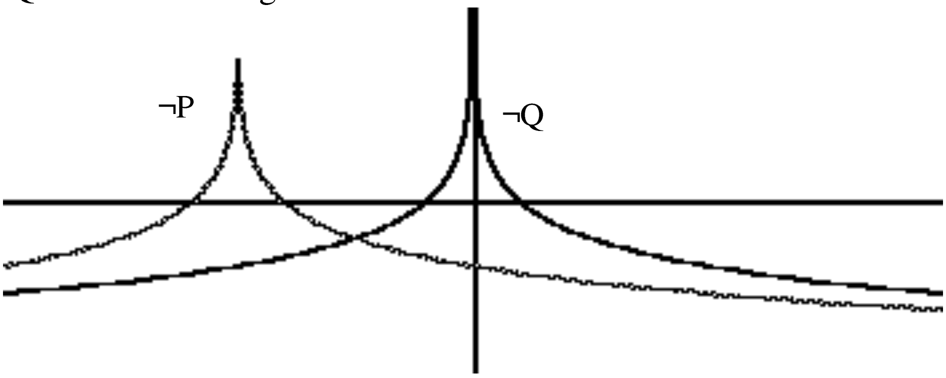


Fig. 2.11. ¬P, ¬Q

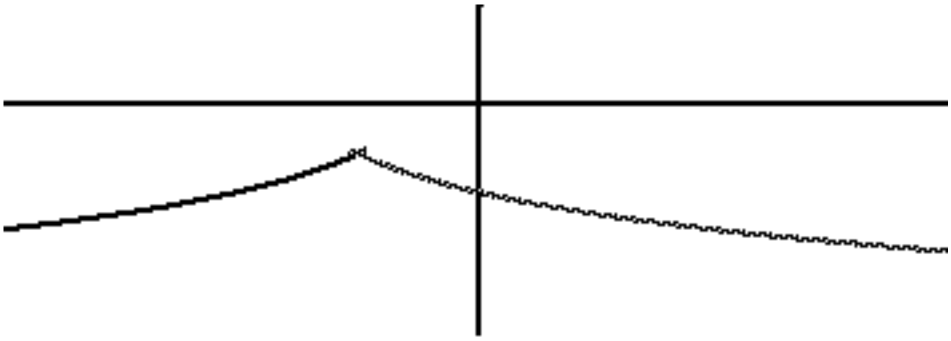


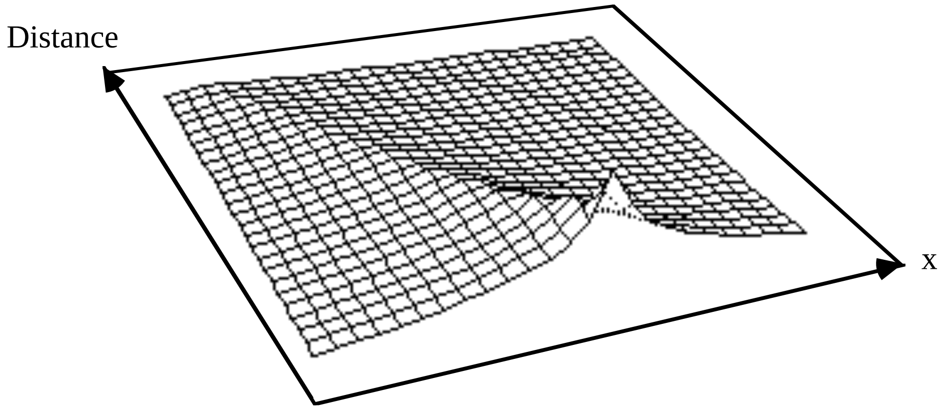
Fig. 2.12. $\neg P \vee \neg Q$

We see that both antonyms act as repulsors, and since we are concerned with an *or*-operation, it is the lowest values of both curves that influence stability. This gives us Fig. 2.12 as a representation of two antonyms.

The ridge between them prevents the subject from resting on both predicates at the same time; however, to the left and to the right in the diagram resting places will occur when the gradient becomes sufficiently small, allowing the subject to be either P or Q.

Fig. 2.13 shows what happens when we vary the distance between the two predicates: in the foreground their extensions cover each other and this results in a large forbidden area. In the back they are far from each other, with only a very small unstable area as the result.

A journey from the back to the front of the picture could for example signify the real approach of phenomena that ideology wants to keep apart. An example could be racial segregation. In the background the two races lives in different locations, so at this time no one notices that Black and White are really conceived as antonyms. But then one of them is transported into the country of the other, either voluntarily or by force. As the two races approach in reality, the antonymness of Black and White becomes visible (the ridge in the foreground), and a racist ideology surfaces and take measures to enforce the segregation more and more explicitly.



→ *Q*, which is the basis of well-known deduction rules like modus ponens and tollendo ponens:

P → Q	P ∨ Q
P	¬P
--------------------	--------------------
Consequent: Q	Consequent: Q

Since it is easiest to illustrate tollendo ponens, I shall only discuss this type of deduction. Fig. 2.14 displays the first antecedent, and Fig. 2.15 shows what happens when we add the second one. We see that the previous stable area over P has got a spike in it that will effectively expel any subject that seeks rest there. The only remaining resting place is over Q, which is exactly what the deduction says.

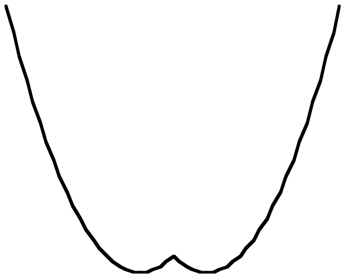


Fig. 2.14. P ∨ Q

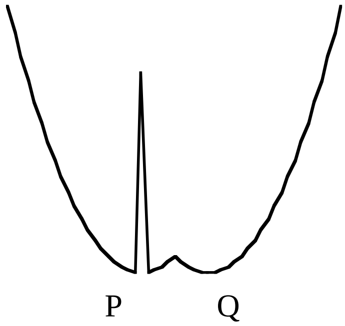


Fig. 2.15. (P ∨ Q) ∧ ¬P

Finally, let us look at contradiction and tautology.

P ∧ ¬*P* creates Fig. 2.16, which has no stable places — no subject can find peace here, as expected.

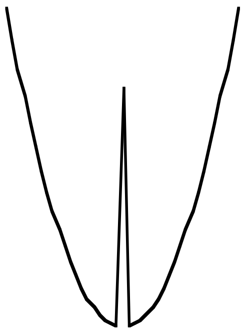


Fig. 2.16. Contradiction. P ∧ ¬P

Fig. 2.17. Tautology. P ∨ ¬P

However, *P v* ¬*P* does not give the desired result — zero gradient everywhere — since there is a crater over P where the gradient might be too strong for rest (Fig. 2.17). I don't know what to make out of it [10].

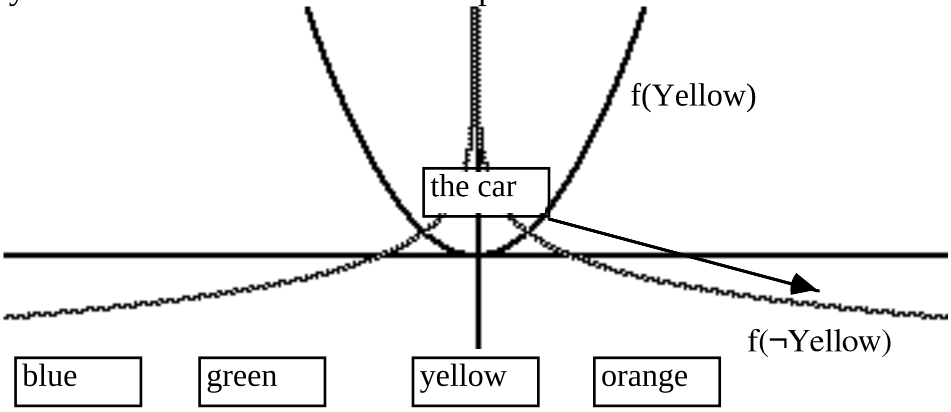
We have now seen that it is possible to design topologies and changes of topologies that closely resembles an ordinary Boolean algebra. However, there are some welcome differences.

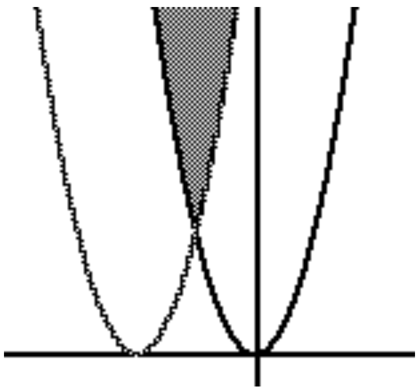
2.3. Non-standard phenomena

We have already seen that the simple act of adding time to logic generates new phenomena that have a clearly human touch: child families will easily recognise the dynamics of Fig. 2.7. In this section we shall give a few more examples of this type.

According to the conventions we use, if we negate a predicate (Fig. 2.18), changing e.g. “The car is yellow” to “The car is not yellow”, the subject will slide down the newly created ridge, and rest at the nearest available stability, giving us for example “The car is orange”, “The car is green”, etc.

In formal logic, we cannot immediately infer anything from a negation in isolation, but in our kind of dynamic topological logic, negation is a process taking place in a space with a basic metric of distance, which causes the subject to find a resting place “near” the original place. This seems to be paralleled in humans: if we hear “She is not blonde”, we may guess that “She is dark”, but not that “She is sick” or “She is director of the IBM”. We seem to stay within the same section of our space.





∧ Q: the shaded area represents P ∧ Q.

Cyclic values were described in Section 1.2 and 2.1. Our example was P = good parent, Q = good worker, where P and Q are in fact disjunct. In this case the formula P ∧ Q creates a ravine that is not differentiable at the bottom, so the subject will keep oscillating down there, never reaching a place with a sufficiently small gradient.

Non-cyclic values can be *stable* or *unstable*. The truth is *stable* when the subject rests at the bottom of a minimum, and *unstable* if it comes to rest on top of a maximum or saddlepoint (where the gradient is 0 too). Stable truth values stay true under large perturbations, whereas unstable ones change under small perturbations. Again the notion of stability of truths seem easily interpretable in humans. “The earth is round” stays true under large perturbations, whereas “I am a nice guy” may be endangered by very small insults (“I have never tried to cheat anyone!”).

The analysis of paradoxes as cyclic truth-values seems to fit human behaviour rather well: we do not rest immobile contemplating the paradox, but oscillate, trying at one time to satisfy one demand, at another time the other contradictory demand, or we try to change reality. In the next section we shall look at a slightly more elaborated example of the latter possibility.

3. The “Schopenhauer” system

I start by describing the logic of a Romantic world view which I call “Schopenhauer” (the quotes means that the world view does not necessarily reflect the opinion of the real philosopher):

(1) $(\neg\text{life} \vee \neg\text{death}) \wedge (\neg\text{peace} \vee \neg\text{suffering}) \wedge (\text{life} \rightarrow \text{suffering}) \wedge (\text{peace} \rightarrow \text{death})$

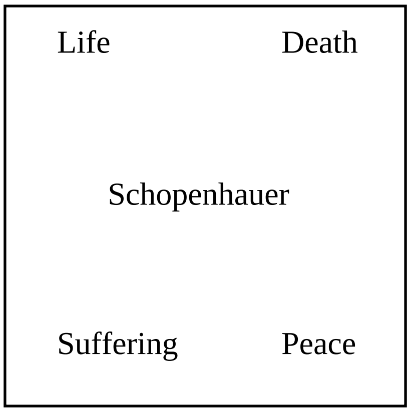
Life and Death, Peace and Suffering are antonyms, so it is not possible for the romantic subject to rest on both of them at the same time. In addition, Life implies Suffering, and Peace implies Death. Since the philosophy refers only to the force-fields of the concepts and not to their actual location, the philosophy concerns the intension (ideology), not the extension (reality) of the concepts.

3.1. Investigating the dynamics

In order to investigate the system, I built a small computer simulation where the predicates are placed in a two-dimensional plane.

As usual, each predicate has associated a potential of the form x^2, with origin in the middle of the predicate, so without the logic there would be four basins, one in each corner. The logical formula modulates this original potential according to “Schopenhauer”.

If we experiment with this logic, it turns out that the only stable place for a Subject is either on the Suffering or Death predicate or completely outside all the predicates — a conclusion the real Schopenhauer in fact correctly drew, at least in theory: if Death and Suffering must be avoided, the only way out of the dilemma is the Buddhist nirvana that is characterised by the absence of all distinctions.



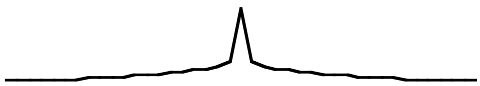


Fig. 3.2. Suffering and Life disjunct.

Fig. 3.3. Life inside Suffering.

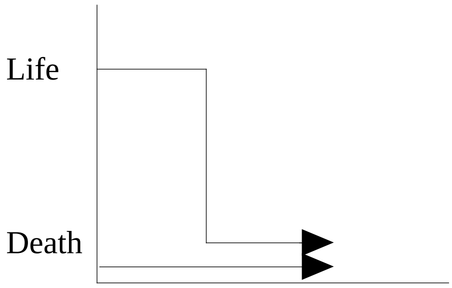
3.2. Interpreting the dynamics

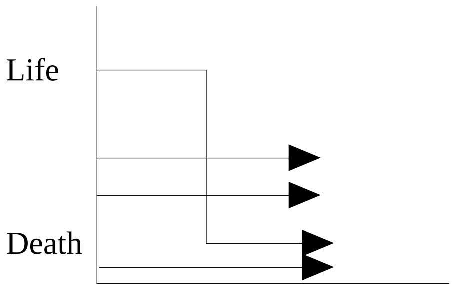
The possibility of moving the subject "Schopenhauer" around in the landscape is a useful facility for getting a hands-on experience of the dynamics of the constructed world. It works like a logical cousin to Papert's Turtle world that enables children to explore a mathematical world. In the model, predicates can be dragged too. The practical purpose is to enable the user to learn how the total field changes as the extension of the predicates changes.

Since the locations of the predicates in the plane represent their actual extensions, moving the “Schopenhauer” *argument* means to change the truth-value of sentences of the form *Predicate(Schopenhauer),* i.e. to assert or negate propositions. Moving the *predicates* means to change the material conditions of life. Finally, changing *conjunctions* that create the force-field means to change the interpretation or ideology.

What is the relation between these logical processes and the episodic semantic system?

It seems sensible to say that a particular trajectory in the episodic system, e.g. the Schopenhauer-system moving from life to death, is reflected in the logical system as a movement of the argument Schopenhauer from the concept of life to the concept of death.





The increased density of trajectories in the middle part change the extension of the two predicates and causes them to draw nearer. In this way the occurrence of trajectories in the middle part of the dimension is summarised.

But this change of extensions will create instability in the intension, i.e. the logical modulation of the two predicates, since the intension is still one of antonymity: arguments will keep moving into the unstable zone of Fig. 2.6, and this may finally cause the modulation — the intension of the concepts — to change from one of antonymi (not both) to alternativity (either one or the other or both).

The relationship between the trajectories (episodes) of the semantic system and the symbols of the logic system seems strange: what exists as a trajectory in the semantic system, shows up as entities in the logical system. The same phenomenon lives as a process in one system, but as an entity in another. However, this phenomenon can be observed in quite everyday conversations too. Some types of conversations treat processes as event unfolding in time, whereas other types focus on the *relation* between events which themselves are treated as entities, i.e. endpoints of relations (see Bøgh Andersen 1990: III.2). We can relate this to the concept of world introduced in Section 1. Events exist as trajectories in the Actual World, but as entities in the "Obligation"-world .

We can finally note a relation between the speaker's role in the event and the world: the entity-version living in the "Obligation"-world is normally used by speakers that act as supervisors of the action, not as participants. A similar analysis seems appropriate in the domain of logic. Like other possible worlds, logic is dedicated to a specific role, namely that of reflection and analysis.

Fig. 1. summarises the relations between the logical and semantic concepts.

Narrative concepts	Computational representation	Traditional logic concepts	Phase-space concepts
Episodic rules	Movement of argument	Assertion and negation of proposition.	Single trajectory of a system
Conceptual rules	Movement of predicate	Change of extension (reality)	Change of distribution of trajectories
	Change of conjunction	Change of intension (ideology)	

Fig. 1. Summary of logical and dynamic concepts.

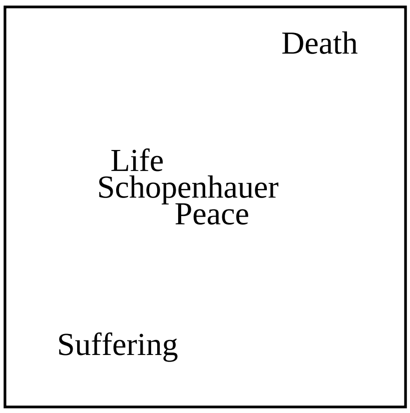
The life/death example shows that the relation between the intension and extension of predicates is not random, although there is no simple relation between them. The most relevant way of characterising them is to characterise their equilibria. We have already seen that some combinations of intension and extension have steady states, whereas others do not. Lack of equilibria can be interpreted as a mismatch between intention and extension, between ideology and reality. The picture of reality represented in the force-field of intension does not “fit” the picture of reality depicted by the location of the concepts.

Since the location of the predicates represents the actual world, some arrangements of predicates are more probable than others: for example, it is not probable that Suffering is disjunct from Life as depicted in Fig. 3.1, since according to common sense Sufferers must be alive. On the other hand, Peace needs not be contained in Death, at least not according to common sense.

An interesting question now arises: what happens when we insert material actions into the system, and let them interact with the ideological dynamics? It turns out that the dynamics and the equilibrium conditions are quite different from the static case.

According to textbooks of philosophy, Schopenhauer did not renounce worldly pleasures (women and food); although he denounced the “lust for life” theoretically, he seems to have a good deal of it himself. This can be represented as an attractor that tries to pull the “Life” predicate towards the “Schopenhauer” subject. What happens if we integrate his philosophy with his own practice in this way?

If we enter vectors that attract Life and Peace to “Schopenhauer” and keep Death and Suffering away from him, a pleasant, although rather immoral, dynamics emerges: Life and Peace cling to “Schopenhauer”, while Death and Suffering keep distance! The interpretation is that “Schopenhauer” causes material changes in the setting of the world he lives in, making arrangements that ensure him a comfortable life.



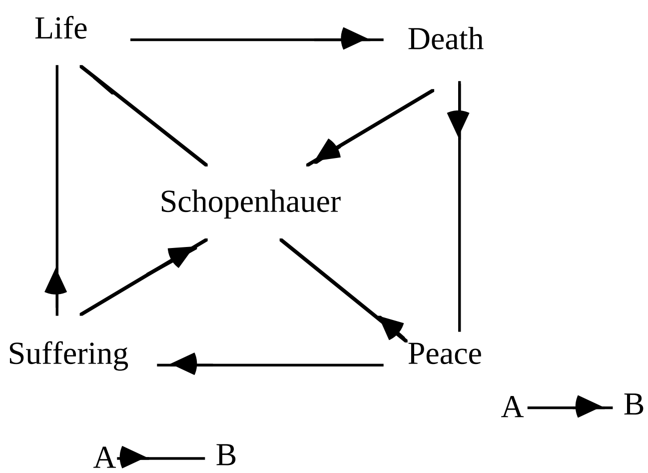
Death and Suffering. If we drag Schopenhauer to Suffering in order to remind him of his philosophy, he quickly leaves it in quest of the middle location.

The continual movements indicate that “Schopenhauer” never reaches an equilibrium where he is at rest in Life and Peace, so he does not live an honest life — as Bertrand Russell notes in his assessment of the real Schopenhauer. His life is based on a lie, but still he gets his moments of Pleasure (represented by the system states where the distance between the subject and the predicates are under a certain limit). Whereas *pure* logic defines equilibria in Nirvana or Suffering/Death, the *lived* logic (logic + desire) turns out to occasionally having a cyclic attractor at a very comfortable place: he gets by with a little help from his friends.

The reason for this is probably the following: Life and Peace cling to Schopenhauer and follows him as guardian angles, even if he is forced to suffer; when Peace approaches Suffering together with Schopenhauer, a ridge forms between them, since they are antonyms. Schopenhauer is influenced by this ridge, and moves away from Suffering.

The point of all this is that Schopenhauer’s philosophy still reigns in the world we have created. There is no cheating. The only thing we have done is adding new forces of action on top of the Schopenhauer system, but this new combination of analysis and action has completely changed the equilibrium conditions.

Still things are a bit too easy for our philosopher: we have an interpretation — the intension of the concepts — saying that Life implies Suffering, but no forces bring reality in accordance with this law. What will happen if “Schopenhauer”’s philosophy is right, i.e. that the extensions of Life and Peace are really disjunct?



What will “Schopenhauer” do if we add forces that brings reality in closer correspondence with logic? I.e. if we change the extension of the concepts according to his philosophy. This last experiment is rather complicated. The gradients of the forces are displayed in Fig. 3.7.

Now reality tries to conform to logic, so that Suffering hunts Life, and Death hunts Peace (the implier attracts the implied, since the former cannot live without the latter, whereas the converse is possible). Death shuns Life, and Suffering Peace. The tormented “Schopenhauer” hunts Life and Peace, and flees Suffering and Death.

Here are some outcomes of these forces:

- If Suffering and Death are on the same side of “Schopenhauer”, he moves forward in disgust, dragging Peace and Life with him, pursued by Death and Suffering which however never catch up with him.
- If the two dark predicates are on opposite sides, “Schopenhauer” is trapped, he oscillates on the spot near resounding Death and Suffering. Peace and Life rest immobile on him.

My impression is that even a small system like the present may turn out to be chaotic, since small changes in position and force of the vectors produce unexpected results. Whether it is really chaotic remains to be seen.

3.3. A model of the formal system.

In this section I sketch a model for the formal systems defined above. Models are theoretical constructions that provide a semantics for the formal systems, and normally they belong to the set-theoretical type where one-place predicates are mapped into sets, and n-place predicates are interpreted as sets of n-tuples, logical connectors signify operations on these sets, and constants represent members of the sets. The model used here, however, is not set-theoretic but is based on the dynamic theory developed in this paper. I suggest that the episodic semantic system for handling episodic knowledge constitutes the model for the logical system.

The *Consistency Rules (2)* of the logical system are interpreted as the basic autopoietic recursion whose function is to maintain truth by making subjects seek equilibria if any exists. *The Episodic Rules* (3) changes the locations of arguments and are caused by perturbations from individual trajectories of the episodic system. Finally, the subclass of the *Conceptual Rules* that changes the extension of the logical concepts reflects a changed distribution of trajectories in the episodic system. But, as appears from Table 1, the other subclass, the ideological rules that change the intension of the logical concepts, have no concrete basis in the episodic system. One cause of ideology change could be a desire to convert a situation of mismatch between intension and extension to one where steady states can be found. But clearly, intension can also influence extension; we see this in politics, where ideology can change the material conditions of life.

4. Applications of dynamic analysis

In this section I show how dynamic analysis can be applied to two other phenomena, *modality* and *conceptual types.*

4.1. Modality and truth

The stability of the truth of a sentence *Subject + Predicate* is defined by the gradient pulling the Subject if it is placed over the Predicate.

In line with Brandt(1989) we can define *necessity* by the condition that only 1 critical point (a location where the gradient is below ε) exists in the topology. Thus, *Subject + Predicate* is *necessary* if and only if the subject is stable above the predicate and no other critical points for the subject exist. *Subject + Predicate* is *possible* , if the Predicate is a critical point for the Subject and other critical points exist. Thus, the $\vee$-operator typically produces possible but not necessary situations (Fig. 2.2).

Necessity and possibility can develop into each other, as shown in Fig. 2.3, where two critical points merge into one. Logical inferences can produce the same effect, as shown in Fig. 2.14-2.15 where modus tollendo ponens converts the two critical points of $P \vee Q$ into the single one of $(P \vee Q) \wedge \neg P$.

Since more than one system can constitute a model for the logical system, it follows that theoretically we have more than one variant of modality, one for each system. In the literature at least two versions are recognised, namely *epistemic* and *deontic* modality, corresponding to the knowledge- and obligation-worlds of Ryan (1991: 111).

Epistemic modality is about signs since it concerns the truth of sentences: “He can be the murderer” = “Maybe he is the murderer, maybe he is not; I don’t know”. Epistemic modality belongs to the logical system.

Deontic modality is about abilities and obligations of bodies and minds: “He *can* write well if he chooses to” does not mean that I lack knowledge about his writing abilities; on the contrary, I describe his abilities with complete assurance.

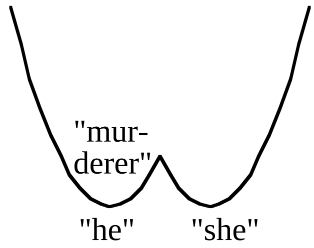


Fig. 4.1. Epistemic modality

The difference between epistemic and deontic modality is clearly brought out if we consider the different references of “is true” and “is forbidden”. “is true” is a property of a sentence, not of a state of affairs. It is the sentence “The earth is round” that is true, not the state of affairs that the earth is round. Conversely, it is the act of lying that is forbidden, not the sentence “A is lying”. Thus, the material of the epistemic system comes from the syntactic system, whereas the entities of the deontic system derive from the biological system.

The epistemic sense of the sentence “He can be the murderer” (but it is also possible that she did it) can be represented as shown in Fig. 4.1.

The hypothesis is now that deontic modality has the same definition as the epistemic one, the meaning difference between the two variants being caused by the fact the “worlds” are different.

Whereas the dynamics of the logical system is confined to logical forces of the kind we have described, the dynamics of the biological system consists of physical forces moving body and limbs (Pellionisz & Llinas (1980, 1982) present a neurophysiological account of the perception - action cycle. Arnheim (1974) analyses perception in terms of vectors).

Therefore, when we interpret deontic modality, possibility is often interpreted as the ability to overcome some physical *(can, is able to)* or social *(may)* obstacle, whereas necessity connotes the inability to resist one of these forces (Talmy 1988). “He *can* write well” is normally only uttered in a situation where something hinders the action (for example laziness or sloppiness) which is known to be weaker than the opposite force. In opposition to this, “He *must* write” only makes sense in the case where a moral obligation overrules the aforementioned obstacles. In the first case, more than 1 critical points must exist, since we cannot infer “He will write” from “He can write”. In the latter case, however, this inference must be valid, otherwise we were in error in using “must”.

The formal dynamics is common to the systems; the difference is due to the different nature of the worlds where the forces live.

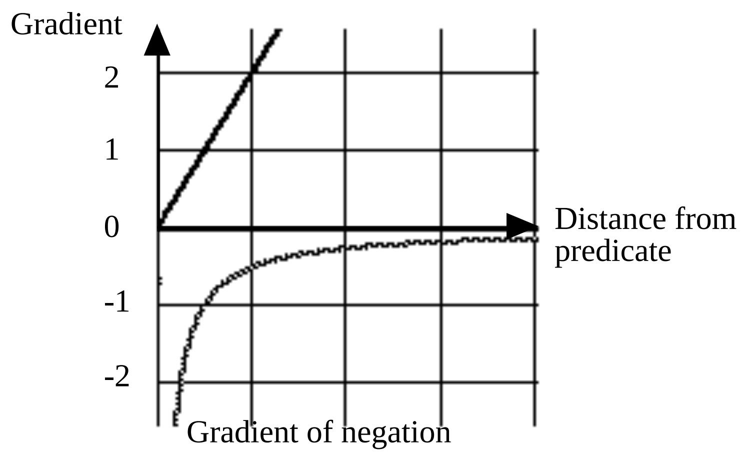
4.2. Predicate types.

Predicates come in different types. A popular distinction is that between Aristotelian and fuzzy concepts.

Aristotelian concepts typically occur in science and are characterised by sharp boundaries: either a tree is a beech or an oak. The concepts are arranged in a Porphyrean tree structured in genus, species and differentiae.

Fuzzy concepts have blurred boundaries: a particular growth can lie somewhere between a tree and a shrubbery. Fuzzy concepts are frequent in everyday language: “Is she your girlfriend? Well, yes and no....”.

This distinction is predicted by the notion of conceptual inertia . In our construction, we assumed that subjects possess inertia and that a force larger than some threshold ε is necessary to move the subject at all. If we assume that the force-field of a concept is given by x^2, then Fig. 4.2 displays the gradients working on a subject; the black curve is the positive attracting force of the predicate, whereas the grey curve is the negative repulsing force of its negation.



ε-

limit is larger than 1. The force is simply too small. If a heavy subject lies in this grey zone [12], it will stay unaffected by predicate and its negation, a behaviour characteristic of *fuzzy concepts.* One can truthfully say “This is a bush” and “This is not a bush”.

Now if ε = 1 there is only one point where negation-processes do not affect the subject, namely 0.5. To the left of this point, negation will expel the subject, and to the right the predicate will attract it. There is no "both-and" zone. This behaviour is typical of *Aristotelian concepts.*

Interestingly enough the model predicts a third type of concepts, namely light *litotes* with thresholds below 1. If ε < 1 there is an area where neither predicate nor negation are stable. For example, for ε = 1/2, the inequality |2x| $\leq \frac{1}{2}$ $\left|\frac{-1}{2x}\right| \leq \frac{1}{2}$

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[1] A version of this paper appeared in *Kodikas/Code. Ars Semeiotica. An international Journal of Semiotics* (18/4), 249-276. 1995. I am grateful to Svend Østergård for help with the mathematics.

[2] Montague 1976 offers an analysis of quantifiers that is closer to natural language.

[3] Cf. Gupta, & Belnap (1993)

[4] This argument can be traced formally as follows. The problematic sentence, "All members of capitalist society have a false consciousness" is called (a) and is formalized as follows:

(a) $\forall x: x \in CSociety \rightarrow FalseConsciousness(x)$

$$\forall x: x \in CSociety \rightarrow FalseConsciousness(x)$$

$$\forall x, y: FalseConsciousness(x) \wedge Assert(x, y) \rightarrow \neg y$$

(3) $I \in CSociety$

$\neg a$

$\neg\forall x: x \in CSociety \rightarrow FalseConsciousness(x)$ $\neg\forall p \equiv \exists p\neg$

$\exists x: x \in CSociety \wedge \neg FalseConsciousness(x)$

$\forall p, M\neg p$

$\neg(\forall p, M\neg p)$

$\exists p, \neg M\neg p)$ $\neg\forall p \equiv \exists p\neg$

$\exists p, Np)$ $\neg M\neg \equiv N$

$\vee$ Q) = min(f(P),f(Q)). In the following, the potential associated to the predicates are all rendered by the formula y = x^2. The real force field of the predicates is an empirical matter.

[7] Note that the resulting potential is not differentiable: although the peak between P and Q looks like a maximum, it is not, since the function is not differentiable at that point. This is a theoretical problem, since we use the first derivative as the force controlling the logic, and if the potential is not differentiable for all values, the space will have unmotivated "holes" in it where the gradient is undefined.

[8] f(P $\wedge$ Q) = max(f(P),f(Q)).

[9] One possibility would be that $g' = -\frac{1}{h'}$

and g' is the derivative of f(¬P). Thus, the new derivative should be the negative and reciprocal of the old derivative. This gives us the following formula for negation: $f(\neg P) = -\int \frac{1}{f'(P)}$

complicated to calculate. Why not choose simple negation, so that f(¬P) = -(f(P))? Simple negation will give us a truly Boolean logic where e.g. de Morgan's law obtains: f(¬(P $\vee$ Q)) = -f(P $\vee$ Q) = -min(f(P), f(Q)) = max(-f(P), -f(Q)) = f(¬P $\wedge$ ¬Q).

The problem is that this negation is very difficult to interpret. For example, x^2 and $-x^2$ will have the same critical point, so negating a predicate will not cause a subject whose gradient is 0 to move. Going from "The earth is flat" to "The earth is not flat" will not change our beliefs, in glaring opposition to the facts. Also, the negation will move the predicate slowly inside P, and will increase the speed as we remove ourselves from P. We would expect the opposite: if we negate a predicate P, then subjects of which P is true must change position, whereas subjects outside the stability of P can be more at ease. These are the reasons for choosing the more complicated version.

In order to find the function g in the case where f(P) = x^2, we note that

$$g = \int g' \quad \int -\frac{1}{h'} \quad -\int \frac{1}{h'} \quad -\int \frac{1}{2x} \quad \frac{-\ln|x|}{2}$$

ε. The two curves of P and ¬P intersect approximately in ±0.55. If ε = 1 then the gradient of x^2 counts as 0 in the interval -0.5... +0.5 and that of -(ln|x|)/2 counts as 0 in the intervals 0.5...∞, and -∞ ...-05. Therefore, we have only problems in the intervals -0.55...-0.50 and 0.50...0.55, where the x^2 furnishes the segment of the curve but has a too large gradient. If ε = 1.5 there is no problem, since the gradient counts as zero everywhere.

[11] The gradient of the concept is |2x|, and that of the negation is $\left|\frac{-1}{2x}\right|$ ≤ 1 and $\left|\frac{-1}{2x}\right|$ ≤ 1 has the unique solution x = ±1/2.

[12] For ε = 2 the inequality |2x| ≤ 2, $\left|\frac{-1}{2x}\right|$ ≤ 2 has solutions in the interval -1...-1/4 and l/4...+1. In this interval, subjects will remain stable when we negate a concept.