Abstract

We study principal component analysis (PCA), where given a dataset in $\mathbb{R}^d$ from a distribution, the task is to find a unit vector $v$ that approximately maximizes the variance of the distribution after being projected along $v$. Despite being a classical task, standard estimators fail drastically if the data contains even a small fraction of outliers, motivating the problem of robust PCA. Recent work has developed computationally-efficient algorithms for robust PCA that either take super-linear time or have sub-optimal error guarantees. Our main contribution is to develop a nearly-linear time algorithm for robust PCA with near-optimal error guarantees. We also develop a single-pass streaming algorithm for robust PCA with memory usage nearly-linear in the dimension.
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1 Introduction

Principal component analysis (PCA) is a central subroutine in dimension reduction and data visualization. PCA is typically used to identify directions of large variance in the data, which are considered to be the most informative aspects of the data. In the classical setting, we observe a set $S$ of $n$ i.i.d. points in $\mathbb{R}^d$ from a (subgaussian) distribution with covariance $\Sigma$. Then, classical estimators, for e.g., the leading eigenvector of the empirical covariance of $S$, output a direction $v$ with the property that $v^T \Sigma v$ approaches $\|\Sigma\|_{\text{op}}$ as $n$ increases. Importantly, these estimators can be computed fast — namely, they have nearly linear runtime.\(^1\)

However, access to i.i.d. samples is often an unrealistic assumption, and data may contain a small number of outliers as formalized below:

**Definition 1.1 (Strong Contamination Model).** Given a parameter $0 < \epsilon < 1/2$ and a class of distributions $\mathcal{D}$, the strong adversary operates as follows: The algorithm specifies a number of samples $n$, then the adversary draws a set of $n$ i.i.d. samples from some $D \in \mathcal{D}$ and after inspecting them, removes up to $\epsilon n$ of them and replaces them with arbitrary points. The resulting set is given as input to the learning algorithm. We call a set $\epsilon$-corrupted if it has been generated by the above process.

Unfortunately, outliers can skew the results of classical estimators, by artificially increasing the variance along low-variance directions, which renders the output of these estimators unreliable.\(^2\) To address this, a robust algorithm must be able to effectively remove outliers from the data, while still being computationally efficient in high dimensions.

Recent works have developed computationally-efficient algorithms for robust PCA [JLT20, KSKO20]. These works fall under the umbrella of algorithmic robust statistics, where the goal is to develop computationally-efficient algorithms for high-dimensional statistical estimation that are robust to a constant fraction of outliers, see, e.g., [DKK+16, LRV16, DK19, DK23]. However, these polynomial-time algorithms [JLT20, KSKO20] either (i) achieve near-optimal error but take super-linear time (scaling with $nd^2$) or (ii) run in nearly linear time but achieve sub-optimal error and require additional assumptions. As modern datasets continue to expand in both sample size and dimension, there is a growing need for nearly linear time algorithms and streaming algorithms with near-optimal error guarantees. Our work addresses this need by developing robust PCA algorithms that meet these demands.

1.1 Our Results

The main result of this paper is a nearly-linear time algorithm to identify a direction of large variance from a corrupted high-dimensional dataset. We state our results below for subgaussian distributions (Definition 2.1), and note that the same results hold for a more general class of “stable” distributions (Definition 2.12).

**Theorem 1.2 (Nearly Linear Time Robust PCA).** Let $\epsilon \in (0,c)$ for a small constant $c > 0$. Let $D$ be a subgaussian distribution with mean zero and (unknown) covariance matrix $\Sigma$. For a sufficiently large $C > 0$, let $S$ be an $\epsilon$-corrupted set of $n \geq Cd/\epsilon^2$ samples from $D$ in the strong contamination model (Definition 1.1). There exists a $C \geq 0$, let $S$ be an $\epsilon$-corrupted set of $n \geq Cd/\epsilon^2$ samples from $D$ in the strong contamination model (Definition 1.1). There exists an algorithm that takes as inputs $S$ and $\epsilon$, runs in $O(nd/\epsilon^2)$ time and with probability at least 0.99 outputs a unit vector $u$ such that $u^T \Sigma u \geq (1 - O(\epsilon \log(1/\epsilon))) \|\Sigma\|_{\text{op}}$.

\(^1\) By the term “nearly linear time” algorithms, we mean runtime scaling linearly in the size of input $nd$, up to logarithmic factors.

\(^2\) Naive techniques such as naive pruning and random projection are also susceptible to outliers.
We note that the error guarantee of our algorithm is near-optimal up to logarithmic factors (cf. Appendix A.1). The work most closely related to ours is [JLT20, Theorem 2], and our algorithm improves upon it in three significant ways:

- (Near-optimal error) The output of Theorem 1.2 achieves the near-optimal \((1 - O(\epsilon \log(1/\epsilon)))\) approximation of \(\|\Sigma\|_{op}\), whereas the result of [JLT20] achieves the sub-optimal approximation of \((1 - O(\sqrt{\epsilon \log(1/\epsilon) \log d}))\).

- (Dependence on \(\epsilon\) in runtime) The runtime of our algorithm is \(\tilde{O}(nd/\epsilon^2)\), in contrast to the runtime of [JLT20], which is \(\tilde{O}(nd/\epsilon^{4.5}+ndm/\epsilon^{1.5})\), where \(m\) is a parameter discussed below that belongs in \([2,d]\). Thus, the runtime of our algorithm improves upon theirs in all cases.

- (Eigenvalue separation) The algorithm of [JLT20] requires the assumption that the \(m\)-th largest eigenvalue of \(\Sigma\) is smaller than \(\|\Sigma\|_{op}\) by \((1 - \Theta(\epsilon))\) for some \(m \in [2,d]\), which appears in the runtime. Thus, [JLT20] obtains nearly linear time only if \(m=\text{polylog}(d/\epsilon)\). Theorem 1.2 has no such restriction.

We finally note that, even without corruptions, \(\tilde{\Omega}(d/\epsilon^2)\) samples are necessary for \((1 - O(\epsilon \log(1/\epsilon)))\)-approximation of \(\|\Sigma\|_{op}\). Moreover, even if there are no corruptions, the standard Lanczos algorithm achieving such a guarantee runs in time \(\tilde{O}(\sqrt{\epsilon / \epsilon} + \epsilon)\) [SV14, Theorem 10.1]. Thus, our result considerably reduces the price of robustness in PCA, and brings it much closer to the clean data setting.

### Streaming Algorithm

The distributed nature of modern data science applications and large-scale datasets often impose the restriction that the complete dataset cannot be stored in the memory. In such scenarios, the streaming model, defined below, is a much more realistic setting:

**Definition 1.3** (Single-Pass Streaming Model). Let \(S\) be a fixed set. In the one-pass streaming model, the elements of \(S\) are revealed one at a time to the algorithm, and the algorithm is allowed a single pass over these points.

In the streaming model, the algorithm also needs to optimize the amount of memory that it uses. We still consider corruptions in the data. This means that the input \(S\) above is not comprised of i.i.d. samples from a distribution, but comes from a “corrupted” distribution, formalized below:

**Definition 1.4** (TV-contamination). Given a parameter \(\epsilon \in (0,1/2)\) and a distribution class \(\mathcal{D}\), the adversary specifies a distribution \(D'\) such that there exists \(D \in \mathcal{D}\) with \(d_{TV}(D,D') \leq \epsilon\). Then the algorithm draws i.i.d. samples from \(D'\). We say that the distribution \(D'\) is an \(\epsilon\)-corrupted version of the distribution \(D\) in total variation distance.

All prior algorithms for robust PCA needed to store the entire dataset in memory, leading to space usage of at least \(\tilde{\Omega}(d^2/\epsilon^2)\). Building on Theorem 1.2, we present the first algorithm for robust PCA that uses \(\tilde{O}(d)\) memory.

**Theorem 1.5** (Streaming Algorithm for Robust PCA; informal version). Let \(\epsilon \in (0,c)\) for a small constant \(c > 0\). Let \(D\) be a subgaussian distribution with mean zero and (unknown) covariance matrix \(\Sigma\). Let \(P\) be an \(\epsilon\)-corrupted version of \(D\) in total variation distance (Definition 1.4). There is a single-pass streaming algorithm that given \(\epsilon\), it reads \(\text{poly}(d/\epsilon)\) many samples from \(P\), and with probability 0.99, returns a unit vector \(u\) such that \(u^\top \Sigma u \geq (1 - O(\epsilon \log(1/\epsilon))))\|\Sigma\|_{op}\). Moreover, the algorithm uses memory \((d/\epsilon) \cdot \text{polylog}(d/\epsilon)\) and runs in time \((nd/\epsilon^2)\text{polylog}(d/\epsilon)\).

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3In particular, the latter result requires that \(\epsilon \to 0\) as \(d \to \infty\) to obtain non-vacuous guarantees.

4In experiments, memory usage has been highlighted as a key bottleneck in robust estimation; see [DKK⁺ 17].
Observe that the asymptotic error of the algorithm is near-optimal and the memory usage is nearly linear in \( d \) (the size of the output). We refer the reader to Section 4.2.1 for further discussion on bit complexity.

1.2 Our Techniques

Our approach is to combine the “easy” version of the robust PCA algorithm from [JLT20, KSKO20] with the fast mean estimation algorithm of [DKPP22] (see also [DKK+22b]). This “easy” algorithm works essentially by computing the top eigenvector, \( v \), of the empirical covariance matrix. It then projects the samples onto the \( v \)-direction and estimates the true covariance in the \( v \)-direction. If this is close to the empirical covariance, then this and the fact that errors cannot substantially decrease the empirical covariance in any direction implies that \( v \) is close to a principal eigenvector of the true covariance matrix (cf. Lemma 2.15). Otherwise, some small number of corruptions must account for the difference, and these can be algorithmically filtered out. One then repeats this process of filtering out outliers until an approximate principal eigenvector is found.

The issue with this simple algorithm is its runtime. Although each filtering step can be performed in nearly linear time, there is no guarantee that the number of these steps will be small. It is entirely possible that each filtering step removes only a tiny fraction of corruptions with very large projections in the leading eigenvector direction \( v \). To fix this, we leverage ideas from [DKPP22] and make \( v \) essentially a random linear combination of the few largest eigenvectors of \( \Sigma_t \) (the empirical covariance at step \( t \)), by defining it to be \( \Sigma_t^p w \) (instead of the principal eigenvector of the empirical covariance matrix), where \( w \) is a random Gaussian vector and \( p \) is a suitably large integer. This prevents an adversary from “hiding” a corruption by making it orthogonal to some particular \( v \). Instead, any corruption substantially contributing to one of the largest eigenvalues of \( \Sigma_t \) is reasonably likely to be filtered out. Our approach is to keep track of the potential function \( \text{tr}(\Sigma_t^{2p+1}) \) and show: (i) small value of the potential certifies that a solution vector can be recovered from the empirical covariance, (ii) whenever this certification is not possible, we can reduce the potential by a multiplicative factor of \( (1 - \Omega(\epsilon)) \). This approach presents two main challenges.

The first key challenge, pertinent to the “certification” that was mentioned before, is to obtain optimal error without any restriction on the spectrum of \( \Sigma \) (recall that the near-linear time result of [JLT20] imposes restrictions on the spectrum of \( \Sigma \)). A natural stopping condition is when the current \( \Sigma_t \) is comparable (in an appropriate sense) to \( \Sigma \) up to \( (1 \pm \epsilon) \) factor. [JLT20] used Schatten-\( p \) norm for large enough \( p \) to compare \( \Sigma \) and \( \Sigma_t \). However, a simple example (Example 3.2) shows that, even then, any deterministic algorithm relying on \( \Sigma_t \) must take \( nd^2 \) time. Our two-pronged solution is to (i) perform a white-box analysis of robust PCA to enforce a stronger stopping condition, and (ii) output a random leading eigenvector of \( \Sigma_t \) (cf. Section 3.1).

Second, unlike in [DKPP22], we cannot quite achieve a runtime of \( \tilde{O}(nd) \). This is essentially because of the choice of \( p \). Assuming that \( \| \Sigma \|_{op} = 1 \), naïve filtering can guarantee that our initial \( \Sigma_t \) at \( t = 0 \) has eigenvalues at most \( \text{poly}(d) \). Thus, the starting value of our potential is \( d^{O(p)} \), and requires \( \tilde{O}(p/\epsilon) \) rounds of filtering (each of which takes \( O(pmd) \) time since we need to do \( p \) many matrix-vector products with \( \Sigma \)’), for a total runtime of \( \tilde{O}(ndp^2/\epsilon) \). Now, our algorithm requires \( p \) to be at least \( \log(d)/\epsilon \) to distinguish between eigenvalues that differ by a \( (1 + \epsilon) \)-factor, as opposed to \( p = O(\log(d)) \) in [DKPP22], resulting in a runtime of \( \tilde{O}(nd/\epsilon^3) \).

The runtime can be improved to \( \tilde{O}(nd/\epsilon^2) \) by noting that such fine differences in the eigenvalues become relevant only at the last stage (“certification stage”) of our algorithm. Until then, we can use a smaller value of \( p \). More formally, for any \( p' \geq \log d \), we can still decrease the potential

\[ \text{As all quantities scale with } \| \Sigma \|_{op}, \text{ we use this normalization for the sake of simplifying presentation.} \]
multiplicatively until our stopping condition is satisfied. Although this no longer certifies that we have a good solution, we can use it to upper bound the potential by $\text{poly}(d)$. Thus, we run our method in multiple stages: in stage $k$, we take $p_k = 2^k \log d$ and reduce the potential until it becomes $d^C$, which takes $\tilde{O}(ndp_k/\epsilon)$ time. By the time we reach $p = \log(d)/\epsilon$, since that stage also starts with potential at most $d^C$, the algorithm terminates in $\tilde{O}(nd/\epsilon^2)$ time, overall.

Finally, as in [DKPP22], the above described algorithm can be turned into a single-pass streaming algorithm. Since we use a small number of filters, where each filter removes all samples $x$ with $|v^\top x| > T$ (for some carefully chosen $v$ and $T$), we can implement this with low memory by storing just the $v$’s and $T$’s of the previously created filters, along with a minimal set of temporary variables for necessary calculations.

1.3 Related Work

Our work is situated within the field of algorithmic robust statistics, where the goal is to develop computationally-efficient algorithms for high-dimensional estimation problems that are robust to outliers. Since the publication of [DKK+16, LRV16], computationally-efficient robust algorithms have been developed for many tasks, including mean estimation [KSS18, CDG19, DHL19, DL22, DKP20, HLLZ20, HR21, CTBJ22, ZJS21], sparse estimation [BDLS17, DKK+19b, CDK+22, DKK+22a, DKLPLP22], covariance estimation [CDGW19], linear regression [KKM18, DKS19, PJJL20, CAT+20], clustering and list-decodable learning [DKS18, HL18, KSS18, KKK19, LM21, BDJ+22], and stochastic optimization [DKK+19a, PSBR20]. We refer the reader to the recent book [DK23] for an overview of this field.

The study of near-linear time algorithms for high-dimensional robust estimation tasks was initiated by [CDG19] in the context of mean estimation. Since this work, a number of improvements and generalizations have been obtained in various contexts. The most related line of works to the current paper is the work on high-dimensional robust PCA, starting from [XCM13]. While this early work did not obtain dimension-independent error guarantees in polynomial time, polynomial-time algorithms with near-optimal dependence on $\epsilon$ in the error guarantee were subsequently developed in [JLT20, KSKO20]. We note that these algorithms have runtimes at least $\Omega(nd^2)$. Additionally, [JLT20] developed a nearly linear time algorithm for robust PCA; however, their algorithm runs in nearly linear-time only in certain cases, and the dependence on $\epsilon$ in the error guarantee is sub-optimal and dimension-dependent. Finally, our streaming algorithm is inspired by [DKPP22], who presented the first streaming algorithms for high-dimensional robust mean estimation and related tasks.

We remark that robust PCA is closely related to the problem of robust covariance estimation in the operator norm, where the algorithm is required to output an estimate $\hat{\Sigma}$ given a corrupted set of samples such that $\|\Sigma - \hat{\Sigma}\|_{\text{op}}$ is small. It is easy to see that the top eigenvector of $\hat{\Sigma}$ satisfies the guarantees of robust PCA. In this paragraph, we restrict our attention to Gaussian data and when $\epsilon$ is a small enough absolute constant. Although the sample complexity of robust covariance estimation in operator norm is still $\tilde{O}(d)$, [DKS17, Theorem 1.5] has shown that the problem is computationally hard (in the Statistical Query model) unless one takes $\Omega(d^2)$ samples. With $\Omega(d^2)$ samples, one can estimate the covariance in the relative Frobenius norm (and thus in operator norm); see the discussion in [DKS17]. Thus, robust PCA is an easier task, both computationally and statistically, than robust covariance estimation in operator norm, while still being useful.

Finally, we emphasize that the focus of our work (and that of [JLT20, KSKO20]) is somewhat orthogonal to other works with similar terminology [CLMW11, MMA18, BJSY22], as the definition of robustness is significantly different.
2 Preliminaries

Notation. We denote \([n] := \{1, \ldots, n\}\). For a vector \(v\), we let \(\|v\|_2\) and \(\|v\|_\infty\) denote its \(\ell_2\) and infinity-norm respectively. We use boldface capital letters for matrices. We use \(I\) for the identity matrix. For two matrices \(A\) and \(B\), we use \(\text{tr}(A)\) for the trace of \(A\), and \((A, B) := \text{tr}(A^\top B)\) for the trace-inner product. We use \(\|A\|_F, \|A\|_{op}, \|A\|_p\) for the Frobenius, operator, and Schatten \(p\)-norm of a matrix \(A\) for \(p \geq 1\). In particular, if \(A\) is a \(d \times d\) symmetric matrix with eigenvalues \(\lambda_1, \ldots, \lambda_d\), then \(\|A\|_p := (\sum_i |\lambda_i|^p)^{1/p}\). We say that a square symmetric matrix \(A\) is PSD (positive semidefinite), and write \(A \succeq 0\), if \(x^\top Ax \geq 0\) for all \(x \in \mathbb{R}^d\). We write \(A \preceq B\) when \(B - A\) is PSD. We write \(a \lesssim b\) when there exists an absolute universal constant \(C > 0\) such that \(a \leq Cb\).

For a distribution \(D\) over a domain \(\mathcal{X}\) and a weight function \(w : \mathcal{X} \rightarrow \mathbb{R}_+\), we use \(D_w\) to denote the distribution over \(\mathbb{R}^d\) with pdf \(D_w(x) := D(x)w(x) / \int D(x)w(x)\). We denote the second moment of \(D\) by \(\Sigma_D := \mathbb{E}_{X \sim D}[XX^\top]\). We will repeatedly use that \(\mathbb{E}_{X \sim D}[\|UX\|_2^2] = (U^\top U, \Sigma_D)\).

We use the following notion of subgaussianity that was also used in [JLT20].

**Definition 2.1.** For a parameter \(r \geq 1\), we say a distribution \(D\) on \(\mathbb{R}^d\) with mean zero and covariance \(\Sigma\) is \(r\)-subgaussian if for all unit vectors \(v\) and \(t > 0\),

\[
\mathbb{E}_{X \sim D}[\exp(tv^\top X)] \leq \exp \left( t^2 r(v^\top \Sigma v)/2 \right)
\]

Observe that this notion of subgaussianity is stronger than [Ver18, Definition 3.4.1] because the sub-gaussian proxy in the direction \(v\) scales with \(v^\top \Sigma v\).

2.1 Miscellaneous Facts

We now collect some facts that we shall use in the paper.

Linear Algebraic Facts. We first begin by noting the following useful properties of trace inner product for PSD matrices:

**Fact 2.2.** If \(A, B, C\) are symmetric \(d \times d\) matrices with \(A \succeq 0\) and \(B \preceq C\) then \(\text{tr}(AB) \leq \text{tr}(AC)\).

A proof of the following fact can be found, e.g., in [JLT20].

**Fact 2.3.** Let \(A, B\) be PSD matrices with \(B \preceq A\) and \(p \in \mathbb{N}\). Then, \(\text{tr} (B^p) \leq \text{tr} (A^{p-1}B)\).

The following fact shows that a PSD matrix continues to be PSD if both pre-multiplied and post-multiplied:

**Fact 2.4.** If \(A \in \mathbb{R}^{d \times d}\) is a PSD matrix, for any other \(B \in \mathbb{R}^{d \times d}\) it holds \(BABA^\top \succeq 0\).

**Proof.** For any vector \(x\), denoting \(y := B^\top x\), we have that \(x^\top (BABA^\top) x = y^\top Ay \geq 0\).

We shall use the following fact to compare \(A^m\) for some \(m \geq 1\) and \(A\) along certain directions:

**Fact 2.5.** Let \(A\) be a PSD matrix. Then for any unit vector \(x\) and \(m \in \mathbb{N}\), \(x^\top A^m x \geq (x^\top A x)^m\).

**Proof.** By spectral decomposition, \(A\) is equal to \(\sum_i \lambda_i u_i u_i^\top\) for \(\lambda_i \geq 0\), and orthonormal vectors \(u_i\). Furthermore, \(A^m = \sum_i \lambda_i^m u_i u_i^\top\). For any unit vector \(x\), the orthonormality of \(u_i\)'s imply that \(\sum_i (u_i^\top x)^2 = 1\).

We now define the non-negative random variable \(Z\) that is equal to \(\lambda_i\) with probability \((u_i^\top x)^2\), which is a valid probability assignment since it is non-negative and sums up to 1. In this definition, we have that \(\mathbb{E}[Z] = \sum_i \lambda_i (u_i^\top x)^2 = x^\top A x\). Moreover, \(\mathbb{E}[Z^m] = \sum_i \lambda_i^m (u_i^\top x)^2 = x^\top (\sum_i \lambda_i^m u_i u_i^\top) x = x^\top A^m x\). Thus, the desired result is equivalent to saying \(\mathbb{E}[Z^m] \geq (\mathbb{E}[Z])^m\) for the non-negative random variable \(Z\), which is satisfied by Jensen’s inequality. \(\square\)
We now turn our attention to Schatten norms: for a symmetric matrix $A$, we have that $\|A\|_2 = \|A\|_F$, $\|A\|_\infty = \|A\|_{\text{op}}$, and for a PSD matrix $A$, $\|A\|_1 = \text{tr}(A)$. Schatten norms satisfy Hölder inequality for trace inner product:

**Fact 2.6** (Hölder Inequality for Schatten Norms). Let $A, B$ be two matrices with dimensions $m \times n$. Let $p \geq 1$. Define $q = \frac{p}{p-1}$ if $p > 1$ and $\infty$ otherwise, then $\langle A, B \rangle \leq \|A\|_p \|B\|_q$. In particular, $\langle A, B \rangle \leq \|A\|_F \|B\|_F$ and $\langle A, B \rangle \leq \|A\|_{\text{op}} \|B\|_1$.

Moreover, Schatten-$p$ norms decrease with $p$ and are close to each other if $p$ is large.

**Fact 2.7.** If $A \in \mathbb{R}^{d \times d}$ is symmetric and $p \geq 1$, the Schatten norms of $A$ satisfy the following: $\|A\|_{p+1} \leq \|A\|_p \leq \|A\|_{p+1} d^{\frac{1}{p(p+1)}}$.

In particular, we will frequently use the following special case:

**Fact 2.8.** If $A$ is a PSD matrix, $\|A\|_F^2 \leq \text{tr}(A)^2$.

**Proof.** Note that $\|A\|_F = \|A\|_2$ and $\text{tr}(A) = \|A\|_1$ since $A$ is PSD. The result then follows by Fact 2.7. \qed

**Probability Facts** The quadratic polynomial of Guassians will play an important role in our analysis, i.e., $z^\top Az$ for $z \sim \mathcal{N}(0, I)$, and we will repeatedly use the following fact.

**Fact 2.9.** For any symmetric $d \times d$ matrix $A$, we have $\text{Var}_{z \sim \mathcal{N}(0, I)}[z^\top Az] = 2\|A\|_F^2$. If $A$ is a PSD matrix, then for any $\beta > 0$, it holds $\Pr_{z \sim \mathcal{N}(0, I)}[z^\top Az \geq \beta \text{tr}(A)] \geq 1 - \sqrt{e\beta}$.

For the streaming section of the paper, we will use the following result on the concentration of the empirical second moment matrix:

**Fact 2.10** (see, e.g., [Ver10]). Consider a distribution $D$ on $\mathbb{R}^d$ that is supported in an $\ell_2$-ball of radius $R$ from the origin. Let $\Sigma$ be its second moment matrix and $\Sigma_N = (1/N)\sum_{i=1}^N X_i X_i^\top$ be the empirical second moment matrix using $N$ i.i.d. samples $X_i \sim D$. There is a constant $C$ such that for any $0 < \epsilon < 1$ and $0 < \tau < 1$, if $N > C e^{-2\|\Sigma\|_2^{-1}R^2 \log(d/\tau)}$, we have that $\|\Sigma - \Sigma_N\|_{\text{op}} \leq \epsilon \|\Sigma\|_{\text{op}}$, with probability at least $1 - \tau$.

Finally, we record the guarantee of power iteration:

**Fact 2.11** (Power iteration). For any PSD matrix $A \in \mathbb{R}^{d \times d}$ and $\epsilon, \delta \in (0, 1)$, if $p > \frac{C}{\epsilon} \log(d/(\epsilon \delta))$ for a sufficiently large constant $C$, and $u := A^p z$ for $z \sim \mathcal{N}(0, I)$, then

$$\Pr\left[u^\top Au/\|u\|_2^2 \geq (1 - \epsilon)\|A\|_{\text{op}}\right] \geq 1 - \delta.$$

### 2.2 Stability Condition on Inliers

Recall that in our notation we use $\Sigma_D = \mathbb{E}_{X \sim D}[XX^\top]$ for the second moment of a distribution $D$, and $D_w(x) := D(x)w(x)/\int D(x)w(x)$ for the distribution $D$ re-weighted by $w(x)$. Our algorithm will rely on the following condition.

**Definition 2.12** (Stability Condition). Let $0 < \epsilon < 1/2$ and $\epsilon \leq \gamma < 1$. A distribution $G$ on $\mathbb{R}^d$ is called $(\epsilon, \gamma)$-stable with respect to a PSD matrix $\Sigma \in \mathbb{R}^{d \times d}$, if for every weight function $w : \mathbb{R}^d \to [0, 1]$ with $\mathbb{E}_{X \sim G}[w(X)] \geq 1 - \epsilon$, the weighted second moment matrix, $\Sigma_{G_w} := \mathbb{E}_{X \sim G}[w(X)XX^\top]/\mathbb{E}_{X \sim G}[w(X)]$, satisfies that $(1 - \gamma)\Sigma \preceq \Sigma_{G_w} \preceq (1 + \gamma)\Sigma$. 


In particular, the second moment matrix is stable under deletion of $\epsilon$-fraction. Some remarks are in order: (i) The definition above is intended for distributions with zero mean; This can be assumed without loss of generality since we can always work with pairwise differences, (ii) The uniform distribution over a set of $n = C d / (\epsilon^2 \log(1/\epsilon))$ i.i.d. samples from a $O(1)$-subgaussian distribution (cf. Definition 2.1) with covariance $\Sigma$ is w.h.p. $(\epsilon, \gamma)$-stable with $\gamma = O(\epsilon \log(1/\epsilon))$ [JLT20, Lemma 11], and (iii) As stated in Theorem 1.2, the algorithm takes as input an $\epsilon$-corrupted set of samples from a subgaussian distribution. However, for notational convenience, we will consider the input to be a distribution $P = (1-\epsilon)G + \epsilon B$ where $G$ is an (unknown) stable distribution and $B$ is arbitrary. $P$ is meant to be the uniform distribution over the $\epsilon$-corrupted set of samples, $G$ will be the part from the remaining inliers, and $B$ that of outliers.\footnote{To be precise, the claim mentioned in (iii) needs a proof; see, e.g., Lemma 2.12 in [DKPP22].} We emphasize that the identity of outliers is unknown to the algorithm.

Our algorithm will remove points iteratively, so we will use binary weights $w(x) \in \{0,1\}$ to distinguish between the points that we keep ($w(x) = 1$) and the ones that have been removed ($w(x) = 0$). Throughout the run of our algorithm, we will ensure that the weights satisfy the following setting:

**Setting 2.13.** Let $0 < 20\epsilon \leq \gamma < \gamma_0$ for a small constant $\gamma_0$. Let $P$ be a mixture distribution $P = (1-\epsilon)G + \epsilon B$ on $\mathbb{R}^d$, where $G$ is $(20\epsilon, \gamma)$-stable distribution with respect to a PSD $d \times d$ matrix $\Sigma$, and $B$ is arbitrary. Let $w : \mathbb{R}^d \rightarrow \{0,1\}$ be a weight function with $E_{X \sim G}[w(X)] \geq 1 - 3\epsilon$.

As we will use projections of points to filter outliers, we need the following implication of stability, concerning the average value of these projections (and thresholded projections), as well as their quantiles and their robust estimates.

**Lemma 2.14.** In Setting 2.13, define the following: (i) $g(x) := \|UX\|_2^2$ for some $U \in \mathbb{R}^{n \times d}$, (ii) $L$ be the top $3\epsilon$-quantile of $g(x)$ under $P_w$ ($P$ re-weighted by $w$) and $S := \{x : x > L\}$, (iii) $\hat{\sigma} := E_{X \sim P}[w(X)g(X)1(g(X) \leq L)]$. Then,

(i) $\left| E_{X \sim G}[w(X)g(X)] - \langle U^\top U, \Sigma \rangle \right| \leq 2\gamma \langle U^\top U, \Sigma \rangle$,

(ii) $E_{X \sim G}[w(X)g(X)1\{X \in S\}] \leq 2.35\gamma \langle U^\top U, \Sigma \rangle$,

(iii) $L \leq 1.65(\gamma/\epsilon)\langle U^\top U, \Sigma \rangle$,

(iv) $|\hat{\sigma} - \langle U^\top U, \Sigma \rangle| \leq 4\gamma \langle U^\top U, \Sigma \rangle$,

(v) $Pr_{X \sim G}[\|X\|_2^2 > \frac{2d}{\epsilon}\|\Sigma\|_{op}] \leq \epsilon$.

As discussed earlier, the goal is to filter out points until a top eigenvector $u$ of the data set’s empirical second moment $\Sigma_{P_w}$ approximately maximizes the true variance. As in previous work, an easy way to detect when this happens is by comparing the empirical variance along $u$ to the true variance along the same direction (which although unknown, it can be easily robustly estimated using Item 4 above). The following is implicit in [XCM13, JLT20, KSKO20]:

**Lemma 2.15 (Basic Certificate Lemma).** Consider Setting 2.13. Let $u$ be a unit vector with $u^\top \Sigma_{P_w} u \geq (1 - O(\gamma)) \|\Sigma_{P_w}\|_{op}$. If $u^\top \Sigma u \geq (1 - O(\gamma))u^\top \Sigma_{P_w} u$, then we have that $u^\top \Sigma u / \|u\|_2^2 \geq (1 - O(\gamma)) \|\Sigma\|_{op}$.

For completeness, its proof can be found in Appendix A.3, where the lemma is restated as Lemma A.7.
2.3 Filtering

We now recall the standard filtering routine from algorithmic robust statistics literature. The filter uses a score \( \tau(x) \) for each point \( x \), indicating how atypical the point is for the distribution. Given a (known) upper bound \( T \) for the average scores over only the inliers \( \mathbb{E}_{X \sim G}[w(X)\tau(X)] \), if the average score of all (corrupted) points, \( \mathbb{E}_{X \sim P}[w(X)\tau(X)] \), is much bigger than \( T \), then points with large scores are likely to be outliers. Thus, Algorithm 1 removes all points with scores greater than a (random) threshold.\(^7\) We also note the following:

(i) Instead of storing a bit \( w(x) \) for every \( x \), we can store a more succinct description of the filters, i.e., just the \( r_\ell \)'s of Line 4, and calculate \( w(x) \) whenever needed. This modification is amenable to the streaming setting.

(ii) Since every time we remove all points with \( \tau(x) > r_\ell \) and \( r_\ell \sim \mathcal{U}([0, r_{\ell-1}]) \), this (in expectation) halves the range of \( \tau(x) \), and thus Algorithm 1 terminates after \( O(\log(\max_x \tau(x)/\min_x \tau(x))) \) steps in expectation.

The following result states that Algorithm 1 removes more outliers than inliers in expectation.

Algorithm 1 HARDThresholdingFilter

1: **Input**: Distribution \( P = (1-\epsilon)G + \epsilon B \), weights \( w \), scores \( \tau \), parameters \( T, R \).
2: \( w_0(x) \leftarrow w(x) \), and \( r_0 \leftarrow R \), \( \ell \leftarrow 1 \).
3: **while** \( \mathbb{E}_{X \sim P}[w(X)\tau(X)] > \frac{3}{2}T \)
4: \( \text{Draw } r_\ell \sim \mathcal{U}([0, r_{\ell-1}]) \).
5: \( w_{\ell+1}(x) \leftarrow w_\ell(x) \cdot 1(\tau(x) > r_\ell) \).
6: \( \ell \leftarrow \ell + 1 \).
7: **return** \( w_\ell(x) \).

Lemma 2.16 (Guarantees of Filtering). Let \( T \) be such that \((1-\epsilon)\mathbb{E}_{X \sim G}[w(X)\tau(X)] < T \) and \( \hat{T} \) such that \(|\hat{T} - T| < T/5 \). Denote by \( F \) the randomness of HARDThresholdingFilter (i.e., the collection of the random thresholds \( r_1, r_2 \ldots \) used). Let \( w' \) be the weight function returned. Then,

(i) \( \mathbb{E}_{X \sim P}[w'(X)\tau(X)] \leq 3T \) almost surely, and

(ii) \( \mathbb{E}_F[\epsilon \mathbb{E}_{X \sim B}[w(X) - w'(X)]] > (1-\epsilon)\mathbb{E}_{X \sim G}[w(X) - w'(X)] \).

The proof of Lemma 2.16 can be found in Appendix A.4. Our main algorithm will repeatedly call HARDThresholdingFilter with appropriate \( \tau(x) \) and other parameters. From a technical standpoint, if the second part of Lemma 2.16 (which states that more mass is removed from outliers than inliers) were true deterministically, we would have that \( \mathbb{E}_{X \sim G}[w(X)] \geq 1 - \epsilon \) no matter how many times the filter is called. This would mean that Setting 2.13 would be maintained throughout the main algorithm. However, part (ii) of Lemma 2.16 holds only in expectation (with respect to \( F \)), but one can still show via a martingale argument that a relaxed condition of \( \mathbb{E}_{X \sim G}[w(X)] \geq 1 - 3\epsilon \) will still be true throughout the main algorithm with high probability; further details can be found in Appendix A.4 (Lemma A.9). For the first reading, one can simply think that Setting 2.13 will always hold.

\(^7\)Consequently, for each point \( x \), \( \mathbb{P}(x \text{ is removed}) \propto \tau(x) \).
3 Robust PCA in Nearly-Linear Time

Organization In this section we prove Theorem 3.1, which is a more general version of the Theorem 1.2 that was stated in Section 1.1. To simplify the presentation, Sections 3.1 and 3.2 will focus on establishing a weaker version of Theorem 3.1, with a runtime of $O(nd/\gamma^2)$ (where $\gamma$ is the stability parameter; recall that $\gamma = O(\epsilon \log(1/\epsilon))$ for subgaussians). This still improves upon prior work in terms of both runtime and error guarantee, while effectively showing key aspects of our approach. Next, in Section 3.3, we will outline an improvement that reduces the runtime to $O(nd/\gamma^3)$. Finally, in Section 3.4, we will provide a formal proof of Theorem 3.1 with all the details.

We now present the main result of our paper, which establishes Theorem 1.2 since $\gamma = \tilde{O}(\epsilon)$ for subgaussian distributions:

**Theorem 3.1.** Let $\gamma_0$ be a sufficiently small positive constant. Let $d > 2$ be an integer, and $\epsilon, \gamma \in (0, 1)$ such that $0 < 20\epsilon < \gamma < \gamma_0$. Let $P$ be the uniform distribution over a set of $n$ points in $\mathbb{R}^d$, that can be decomposed as $P = (1 - \epsilon)G + \epsilon B$, where $G$ is a $(20\epsilon, \gamma)$-stable distribution (Definition 2.12) with respect to a PSD matrix $\Sigma \in \mathbb{R}^{d \times d}$. There exists an algorithm that takes as input $P, \epsilon, \gamma$, runs for $O\left(\frac{nd}{\gamma^2} \log^3(d/\epsilon)\right)$ time, and with probability at least 0.99, outputs a unit vector $u$ such that $u^\top \Sigma u \geq (1 - O(\gamma))\|\Sigma\|_{\text{op}}$.

As mentioned earlier in Sections 1.2 and 2, we run an iterative algorithm, where in each iteration $t$, we assign a score to each point $x$, and use these scores to removes points. These scores are usually projections of points along a direction, where outliers have much larger projections than the inliers. Under the stability condition, we can reliably filter outliers as long as the empirical (corrupted) variance is $(1 + O(\gamma))$ times larger than the true variance in a direction.

As each round of filtering decreases the variance, we see that $\Sigma_t$ should decrease with $t$ (after appropriate normalization). As noted in prior work, using scores that involve projections of the samples on the top eigenvector $v_t$ of the empirical second moment $\Sigma_t$ does not necessarily make good progress because the filtering removes points only in a single (fixed) direction of the largest variance (cf. Section 1.2 for details). Instead, one needs to filter along many of these large variance directions simultaneously (on average), which we do by filtering along the direction $v_t = M_t z$ for $z \sim \mathcal{N}(0, I)$ and $M_t = \Sigma^{op}_t$ for a large integer $p = \frac{C \log(d/\gamma)}{\gamma^2}$.

To track progress of the algorithm, we use the potential function $\phi_t := \text{tr}(\Sigma^{2p+1}_t)$. This potential tracks the contribution from all large eigenvalues (and not just the largest one), and has been used in prior work for robust mean estimation. In our setting, we will show that (I) small enough $\phi_t$ (e.g., $\phi_t \leq \|\Sigma\|_{\text{op}}^{2p+1}/\text{poly}(d^p)$) implies that $\Sigma$ and $\Sigma_t$ share the leading eigenspace without any restriction on the spectrum of $\Sigma$, which in turn, certifies that a good solution vector can be produced, and (II) if the spectrum of $\Sigma$ and $\Sigma_t$ disagree, then we can decrease $\phi_t$ multiplicatively (on average). In particular, we will show that $\phi_{t+1} \leq (1 - \gamma)\phi_t$ on average. Combining this with the fact that a simple pruning can always ensure $\phi_0 \leq \text{poly}(d^p)\|\Sigma\|_{\text{op}}^{2p+1}$, we can obtain $\phi_t < \|\Sigma\|_{\text{op}}^{2p+1}/\text{poly}(d)$ after just $t = O(p \log(d)/\gamma)$ rounds.

We now explain the items (I) and (II) above: Section 3.1 formalizes the notion of “shared leading eigenspaces”, and Section 3.2 outlines the decrease of the potential function. The formal proof is then given in Section 3.4.

3.1 Advanced Certificate Lemma

We begin by summarizing the algorithmic framework of [JLT20]. In particular, we highlight the roadblocks in their framework to remove the eigenvalue separation condition, and our proposed modifications that rely on formalizing the notion of “shared leading eigenspaces”.

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Roughly speaking, their iterative algorithm stops at an iteration $t$ such that $\|\Sigma_t\|_p^p \leq (1 + C\gamma)^p \|\Sigma\|_p^p$, where $\gamma$ is the stability parameter.\footnote{While comparing norms under stability, $(1 + C\gamma)$ is necessary.} Then, their algorithm tries the top-$m$ eigenvectors of $\Sigma_t$ for some $m \in \mathbb{N}$ and returns the (normalized) eigenvector $v$ that attains the best possible $v^\top \Sigma v$ (which can be estimated up to small error); see Lemma 2.15. In particular, the last step takes $O_{\gamma}(ndm)$ time. However, as the following example shows (by taking $r = d/2$ as $m = \Omega(d)$), this approach cannot give a nearly-linear time algorithm.

**Example 3.2** (Hard Example for Schatten $p$-Norm Stopping Condition). Let $\Sigma$ be a PSD projection matrix of rank $r$, then $\Sigma_t = I$ satisfies the condition $\|\Sigma_t\|_p^p \leq (1 + C\gamma)^p \|\Sigma\|_p^p$ as long as $p \geq \log(d/r)/\gamma$. However, any deterministic algorithm to identify a good direction of $\Sigma$ from the eigenvectors of $\Sigma_t$ must take $m = \Theta(d - r)$.

Thus, [JLT20] imposes an additional assumption that the largest eigenvalue and the $m$-th largest eigenvalue of $\Sigma$ are separated for some $m$. Consequently, their result and their version of certificate are inherently restricted to have dependence on $m$; see Proposition 4 therein.

We make two crucial changes in our algorithm (for technical reasons, we also make a change of variable to use $2p + 1$ instead of $p$). The first change is to strengthen the stopping condition: instead of reducing the Schatten norm, which is equivalent to $\langle \Sigma_t, \Sigma_t^{2p} \rangle \leq (1 + C\gamma)^{2p+1} \langle \Sigma, \Sigma^{2p} \rangle$, we stop at a stronger condition\footnote{Please see Appendix A.2 for why this is stronger.} $\langle \Sigma_t, \Sigma_t^{2p} \rangle \leq (1 + C\gamma) \langle \Sigma, \Sigma^{2p} \rangle$. That is, we stop whenever the empirical variance in directions of $\Sigma_t^p$, $\Sigma_t$, is comparable to the true variance, $\langle \Sigma, \Sigma^{2p} \rangle$, up to $(1 + C\gamma)$ factor (cf. Lemma 2.14 with $U = \Sigma_t^p$). Observe that this $(1 + C\gamma)$ factor is the best possible factor that we can achieve while comparing the robust and empirical variances. We are able to achieve this stronger stopping condition because until this condition is satisfied, our algorithm will remove outliers along $\Sigma_t^p$, which will decrease the potential by a multiplicative factor (this will be the topic of the next subsection).

Still, the hard example from Example 3.2 continues to satisfies this stronger condition: one must take $m = \Theta(d\gamma)$ if $r = d/(1 + \gamma)$. Thus, any deterministic algorithm would require $m = \Omega(nd^2\gamma)$ time, which is quadratic in $d$. Our second main insight is to randomize the output of the algorithm. That is, even though, a deterministic algorithm would need to try $\Omega(\gamma d)$ many top eigenvectors of $\Sigma_t$ before finding a high-variance direction of $\Sigma$, a top eigenvector sampled randomly from the spectrum of $\Sigma_t$ will suffice with a large constant probability. In particular, we take a Gaussian sample and iteratively multiply it by $\Sigma_t$ in the style of power iteration. Formally, we prove the following:

**Lemma 3.3.** Let a sufficiently large constant $C$. Let $0 < 20\epsilon \leq \gamma < \gamma_0$ for a sufficiently small absolute constant $\gamma_0$. Let $P = (1 - \epsilon)G + \epsilon B$, where $G$ is a $(20\epsilon, \gamma)$-stable distribution with respect to $\Sigma$. Let $w : \mathbb{R}^d \to [0, 1]$ with $E_{X \sim G}[w(X)] \geq 1 - 3\epsilon$, $p > C \log(d/\gamma)/\gamma$, $M := (E_{X \sim P}[w(X)XX^\top])^p$, and assume $\langle \Sigma, M^2 \rangle \geq (1 - 250\gamma)\langle \Sigma_{P_w}, M^2 \rangle$. If $u := Mz$ for a random $z \sim \mathcal{N}(0, I)$, then with probability at least $0.9$ (over the random selection of $z$) we have that:

(i) $u^\top \Sigma_{P_w} u / \|u\|_2^2 \geq (1 - \gamma) \|\Sigma_{P_w}\|_{op}$,

(ii) $u^\top \Sigma u > (1 - O(\gamma)) u^\top \Sigma_{P_w} u$.

The above result states that whenever $\langle \Sigma, M^2 \rangle \geq (1 - 250\gamma)\langle \Sigma_{P_w}, M^2 \rangle$ for large $p$, we can use Algorithm 2 to obtain a vector $u$ that will satisfy the “basic certificate” from Lemma 2.15 and thus be a good solution. Our main algorithm will thus call Algorithm 2 to output a vector.
**Algorithm 2**  
**SampleTopEigenvector**

1. **Input:** Distribution $P$, weights $w$, parameters $\epsilon, \gamma, \delta$.
2. $y \leftarrow \Sigma_{Pw}^p g$ for $p' = \frac{d}{\epsilon} \log \left( \frac{d}{\epsilon \delta} \right)$ and $g \sim \mathcal{N}(0, I)$.
3. $\hat{r} \leftarrow y^\top \Sigma_{Pw} y / \|y\|^2_2$. ▶ cf. Fact 2.11
4. Let $M:= (E_{X \sim P} [w(X)XX^\top])^p$ for $p= C \log (d/\epsilon) / \gamma$.
5. $u \leftarrow Mz$ for $z \sim \mathcal{N}(0, I)$.
6. Find $\hat{\sigma}_u$ such that $|\hat{\sigma}_u - u^\top \Sigma u| \leq 4 \gamma u^\top \Sigma u$.
7. If $\hat{\sigma}_u \geq (1 - C \gamma) u^\top \Sigma_{Pw} u$ and $\frac{u^\top \Sigma_{Pw} u}{\|y\|^2_2} \geq (1 - \gamma) \hat{r}$ then
8. return $u / \|u\|^2_2$. ▶ cf. Lemma 2.15
9. end if

**Proof of Lemma 3.3.** The part of the conclusion that $u^\top \Sigma_{Pw} u / \|u\|^2 \geq (1 - \gamma) \|\Sigma_{Pw}\|^\text{op}$ follows directly by the guarantee of power iteration (Fact 2.11 with probability of failure being less than 0.001).

We now focus on showing that $u^\top \Sigma u > (1 - O(\gamma)) u^\top \Sigma_{Pw} u$. Let $A := \Sigma - (1 - 250 \gamma) \Sigma_{Pw}$ with high probability. We analyze the random variable $Y := z^\top \Sigma A z$ for $z \sim \mathcal{N}(0, I)$. The assumption $\langle \Sigma, M^2 \rangle \geq (1 - 250 \gamma) \langle \Sigma_{Pw}, M^2 \rangle$ means that $\langle M^2, \Sigma - (1 - 250 \gamma) \Sigma_{Pw} \rangle = \langle M^2, A \rangle \geq 0$. Noting that $E[Y] = \text{tr}(MAM) = \langle M^2, A \rangle$, we have $E[Y] \geq 0$. Thus, if variance of $Y$ is not too large, we will have that $Y$ is not too negative with large probability.

We prove the following technical result on the variance of $Y$ in Section 3.1.1.

**Claim 3.4** (Variance of $Y$). $\text{Var}(Y) \lesssim \gamma^2 \|\Sigma_{Pw}\|_\text{op}^2 \|M\|^2_1$.

**Proof sketch of Claim 3.4.** By Fact 2.9: $\text{Var}(Y) = 2 \|MAM\|^2_F$, and thus we need to show $\|MAM\|^2_F \lesssim \gamma \|\Sigma_{Pw}\|_\text{op} \|M\|^2_1$. Recall the definition of $A$. We can decompose $\Sigma_{Pw}$ into the contribution from inliers (which is very close to $\Sigma$ by stability) and the contribution due to outliers $\Sigma_B := E_{X \sim B} [w(X)XX^\top]$. Roughly speaking, we have $\Sigma_{Pw} \approx (1 - \epsilon) \Sigma + \epsilon \Sigma_B$, which implies that $A \approx \gamma \Sigma + \epsilon \Sigma_B$ as $\epsilon \leq \gamma$.

By triangle inequality, $\|MAM\|^2_F \leq \gamma \|M\Sigma M\|^2_F + \epsilon \|M\Sigma_B M\|^2_F$. The first term is easy to upper bound using PSD property of $\Sigma$ and $M$ as follows: $\|MAM\|^2_F \lesssim \|\Sigma\|_\text{op} \|M^2\|^2_F \lesssim \|\Sigma_{Pw}\|_\text{op} \|M\|^2_1$ by Fact 2.7 and stability. The second term is more involved. Using the decomposition of $\Sigma_{Pw}$ in the condition $\langle \Sigma, M^2 \rangle \geq (1 - O(\gamma)) \langle \Sigma_{Pw}, M^2 \rangle$, we obtain that $\langle \Sigma_B, M^2 \rangle \lesssim (\gamma / \epsilon) \langle \Sigma, M^2 \rangle$. Finally, as $\Sigma_B$ is a PSD matrix, we obtain $\epsilon \|M\Sigma_B M\|^2_F \lesssim \epsilon \gamma \langle \Sigma_B, M^2 \rangle \lesssim \gamma \langle \Sigma, M^2 \rangle \lesssim \gamma \|\Sigma\|_\text{op} \|M\|^2_1$, leading to the result.

Using Claim 3.4 along with Chebyshev’s inequality, we get that with probability at least 0.999, $Y$ satisfies the following lower bound:

$$Y \geq E[Y] - \sqrt{1000 \text{Var}_{z \sim \mathcal{N}(0, I)}[Y]} = -O \left( \gamma \|M\|^2_1 \|\Sigma_{Pw}\|_\text{op} \right). \tag{1}$$

We need one more intermediate result. For the vector $u := Mz$ for $z \sim \mathcal{N}(0, I)$, an application of Fact 2.9 with $A = M^2$ and $\beta = 10^{-4} / \epsilon$ yields

$$\Pr \left[ \|M\|^2_1 / \|u\|^2_2 \leq 1 / \beta \right] = \Pr_{z \sim \mathcal{N}(0, I)} \left[ z^\top M^2 z \geq \beta \text{tr}(M^2) \right] \geq 1 - \sqrt{\epsilon / \beta} = 0.99. \tag{2}$$

We can now complete the proof of Lemma 3.3. In what follows, we condition on the following events: (i) $\|M\|^2_1 / \|u\|^2_2 \leq 1 / \beta \leq 10^5$, (ii) the inequality of Equation (1) holds, and (iii) the conclusion from part one of the lemma. Since all of these events happen individually with probability 0.99
at least, we have that the intersection of the three events has probability at least 0.97 by a union bound. Recalling that \( Y = z^\top \text{MAM} z = u^\top \Sigma u - (1 - 250\gamma)u^\top \Sigma_{P_w} u \) and dividing by \( \|u\|_2^2 \) both sides of Equation (1), we have the following:

\[
\frac{u^\top \Sigma u}{\|u\|_2^2} \geq (1 - 250\gamma) \frac{u^\top \Sigma_{P_w} u}{\|u\|_2^2} - O \left( \gamma \left\| \frac{M}{F} \right\|_F^2 \left\| \Sigma_{P_w} \right\|_{op} \right)
\]

\[
\geq (1 - 250\gamma) \frac{u^\top \Sigma_{P_w} u}{\|u\|_2^2} - O \left( \gamma \left\| \Sigma_{P_w} \right\|_{op} \right) \quad \text{(using that the event from Equation (2) holds)}
\]

\[
\geq (1 - 250\gamma) \frac{u^\top \Sigma_{P_w} u}{\|u\|_2^2} - O \left( \frac{\gamma}{1 - \gamma} \frac{u^\top \Sigma_{P_w} u}{\|u\|_2^2} \right) \quad \text{({\( u^\top \Sigma_{P_w} u/\|u\|_2^2 \geq (1 - \gamma)\| \Sigma_{P_w} \|_{op} \))}
\]

\[
= (1 - O(\gamma)) \frac{u^\top \Sigma_{P_w} u}{\|u\|_2^2}. \quad (\gamma < 1/20)
\]

Thus, we have shown that the second bullet of Lemma 3.3 holds with probability 0.97. The first bullet holds with probability 0.99 when \( C \) is appropriately large constant. Thus, by a union bound, both bullets hold with probability at least 0.90. This completes the proof of Lemma 3.3. \( \square \)

3.1.1 Proof of Claim 3.4: Bound on the Variance of \( Y \)

We now prove the following result on the bound of the variance that was omitted earlier.

**Claim 3.4** (Variance of \( Y \)). \( \text{Var}(Y) \leq \gamma^2 \left\| \Sigma_{P_w} \right\|_{op}^2 \| M \|_F^4 \).

**Proof.** As \( Y = z^\top \text{MAM} z \), Fact 2.9 implies that \( \text{Var}(Y) = 2\| \text{MAM} \|_F^2 \). We will, thus, upper bound \( \| \text{MAM} \|_F \) in the remainder of the proof.

Since \( P = (1 - \epsilon)G + \epsilon B \), we have that \( \Sigma_{P_w} = (1 - \epsilon)\Sigma^G_{P_w} + \epsilon \Sigma^B_{P_w} \), with the components being \( \Sigma^G_{P_w} := \mathbb{E}_{X \sim G}[w(X)XX^\top]/\mathbb{E}_{X \sim P}[w(X)] \), and \( \Sigma^B_{P_w} := \mathbb{E}_{X \sim B}[w(X)XX^\top]/\mathbb{E}_{X \sim P}[w(X)] \). Using this decomposition, we obtain the following expression for \( A \):

\[
A = \Sigma - (1 - 250\gamma) \left( (1 - \epsilon)\Sigma^G_{P_w} + \epsilon \Sigma^B_{P_w} \right)
\]

\[
= \left( \Sigma - \Sigma^G_{P_w} (1 - 250\gamma - \epsilon + 250\gamma \epsilon) \right) - \epsilon (1 - 250\gamma) \Sigma^B_{P_w}.
\]

In order to simplify notation, let us define \( \Delta := \Sigma - \Sigma^G_{P_w} (1 - 250\gamma - \epsilon + 250\gamma \epsilon) \), and \( \bar{\epsilon} := \epsilon (1 - 250\gamma) \). We thus have \( A = \Delta - \epsilon \Sigma^B_{P_w} \). Our approach will be to upper bound \( \| M \Delta \|_F \) and \( \| M \Sigma^B_{P_w} \|_F \) separately. The following intermediate result, that easily follows from the stability the of distribution \( G \), will be useful:

**Claim 3.5.** We have that (i) \( 0 \leq \Delta \leq 270\gamma \Sigma_{P_w} \), (ii) \( \Sigma - (1 - \epsilon) \Sigma^G_{P_w} \leq 1.2 \gamma \Sigma \), and (iii) \( \Sigma \leq (1/(1 - 1.2\gamma)) \Sigma_{P_w} \).

**Proof.** We start with establishing the PSD property of \( \Delta \). Using the stability, we first obtain the following upper bound on \( \Sigma^G_{P_w} \):

\[
(1 - 4\epsilon) \Sigma^G_{P_w} \leq (1 - \epsilon)(1 - 3\epsilon) \Sigma^G_{P_w} \leq \frac{\mathbb{E}_{X \sim P}[w(X)]}{\mathbb{E}_{X \sim G}[w(X)]} \Sigma^G_{P_w} \leq \frac{\mathbb{E}_X[w(X)]}{\mathbb{E}_X[w(X)]} \leq (1 + \gamma) \Sigma,
\]

where the second inequality uses \( w(x) \leq 1 \) for the denominator and the assumption that \( \mathbb{E}_{X \sim P}[w(X)] \geq (1 - \epsilon) \mathbb{E}_{X \sim G}[w(X)] \geq (1 - \epsilon)(1 - 3\epsilon) \) for the numerator, and the last inequality uses the stability of \( G \). Rearranging the inequality above, we obtain the following:

\[
\Sigma^G_{P_w} \leq \frac{1 + \gamma}{1 - 4\epsilon} \Sigma \leq \frac{1}{1 - \epsilon - 250\gamma + 250\epsilon \gamma} \Sigma,
\]

\[
\text{Var}(Y) \leq \gamma^2 \left\| \Sigma_{P_w} \right\|_{op}^2 \| M \|_F^4.
\]
where the first step uses Equation (3) and the second is true for $\epsilon < \gamma/20$. Plugging this inequality in the definition of $\Delta$ implies that $\Delta \succeq 0$.

We now show that $\Delta \succeq 270\gamma \Sigma P_w$. First, we note that

$$
\Sigma P_w \succeq (1-\epsilon)\Sigma^G P_w = (1-\epsilon)\frac{E_{X \sim G}[w(X)]}{E_{X \sim P}[w(X)]} \frac{E_X[XX^\top]}{X \sim G_w},
$$

where the last inequality is triangle inequality and

$$
(1-\epsilon)(1-3\epsilon)(1-\gamma)\Sigma \succeq (1-1.2\gamma)\Sigma, \quad (4)
$$

where the first inequality uses the decomposition $\Sigma P_w = (1-\epsilon)\Sigma^G P_w + \epsilon \Sigma^B P_w$, the second is a rewriting, the third uses stability of $G$ along with $E_{X \sim G}[w(X)] \geq 1 - 3\epsilon$, and the last one uses that $\epsilon < \gamma/20$.

We can now complete our proof:

$$
\Delta = \Sigma - (1-\epsilon - 250\gamma + 250\gamma\epsilon)\Sigma^G P_w
\succeq \Sigma - (1-\epsilon - 250\gamma + 250\gamma\epsilon)(1-3\epsilon)(1-\gamma)\Sigma \quad \text{(using middle part of Equation (4))}
\succeq \Sigma - (1-\epsilon - 250\gamma + 250\gamma\epsilon)(1-1.15\gamma)\Sigma \quad \text{($\epsilon \leq \gamma/20$)}
\succeq 252\gamma \Sigma \quad \text{(using $\epsilon \leq \gamma/20$ and $\gamma \leq 1/20$)}
\succeq \frac{252\gamma}{1-1.2\gamma} \Sigma P_w \succeq 270\gamma \Sigma P_w. \quad \text{(using Equation (4) and $\gamma \leq 1/20$)}
$$

Combining this with $\Delta \succeq 0$ completes the proof of the part (i) of the claim. Part (ii) of the claim, i.e., $\Sigma - (1-\epsilon)\Sigma^G P_w \succeq 1.2\gamma \Sigma$ can be extracted from the middle part of Equation (4) after some rearranging. Part (iii) also follows immediately from Equation (4).

We are now ready to upper bound the variance of $Y$: Using Fact 2.9, we have that

$$
\text{Var}[Y] = 2\|MAM\|_F^2 \leq 4\|M\Delta M\|_F^2 + 4\epsilon^2 \|M\Sigma^B P_w M\|_F^2, \quad (5)
$$

where the last inequality is triangle inequality and $(a+b)^2 \leq 2a+2b$. We upper bound each term separately. Using Claim 3.5, we bound the first variance term as follows:

$$
\|M\Delta M\|_F^2 = \text{tr}(M^2 \Delta M^2 \Delta)
\quad \text{(using $\|A\|_F^2 = \langle A, A \rangle$ for symmetric $A$ and cyclic property of trace)}
\lesssim \gamma \text{tr}(M^2 \Delta M^2 \Sigma P_w) \quad \text{(using Claim 3.5 and Fact 2.2)}
= \gamma \text{tr}(M^2 \Sigma P_w M^2 \Delta) \quad \text{(cyclic property of trace)}
\lesssim \gamma^2 \text{tr}(\Sigma P_w M^2 \Sigma P_w M^2) \quad \text{(using Claim 3.5 and Fact 2.2)}
\leq \gamma^2 \|\Sigma P_w\|_{op}^2 \text{tr}(M^4) \quad \text{(using Fact 2.2)}
= \gamma^2 \|\Sigma P_w\|_{op}^2\|M\|_4^4
\leq \gamma^2 \|\Sigma P_w\|_{op}^2\|M\|_2^4 \quad \text{(using $\|A\|_{q+1} \leq \|A\|_q$)}
= \gamma^2 \|\Sigma P_w\|_{op}^2\|M\|_4^4, \quad (6)
$$

where the first application of Fact 2.2 above required that $M^2 \Delta M^2 \succeq 0$ which is indeed satisfied because we have shown that $\Delta \succeq 0$ in Claim 3.5 and thus Fact 2.4 implies $M^2 \Delta M^2 \succeq 0$. A similar argument was used in the second application of Fact 2.2.

It remains to bound the second term of Equation (5). To this end, we have that

$$
\epsilon \langle \Sigma^B P_w, M^2 \rangle = \langle \Sigma P_w - (1-\epsilon)\Sigma^G P_w, M^2 \rangle
\leq ((1+O(\gamma)) \langle \Sigma, M^2 \rangle - (1-\epsilon) \langle \Sigma^G P_w, M^2 \rangle)
$$

where the first step uses Equation (5) and the last is true for $\epsilon < \gamma/20$.
whenever the condition is true we can output a good solution, (ii) if the condition is false, then $\phi_{t+1} \leq (1 - \gamma) \phi_t$ on average, (iii) if $\phi_t \leq \|\Sigma_t\|_{op}^{2p+1}/\text{poly}(d)$ then the condition from (i) is necessarily true.  

We have already shown (i) in the previous subsection, and we will state and prove (iii) in the Section 3.4.1 (Lemma 3.8). In this section, we focus on establishing (ii) in the form of Lemma 3.6 below.  

Before proving Lemma 3.6, we outline how these claims imply a version of Theorem 1.2 with runtime $\tilde{O}(nd/\gamma^3)$: By the pruning\(^{11}\) of Line 4, $\phi_0 \leq (d/\epsilon)^{O(p)} \|\Sigma\|_{op}^{2p+1}$. Thus, there exists $t_{\text{end}} = O(\log^2 (d/\epsilon)/\gamma^2)$ such that after $t_{\text{end}}$ rounds, $\phi_{t_{\text{end}}} \leq (1 - \frac{C}{\gamma})^t \phi_0 \leq \|\Sigma\|_{op}^{2p+1}/\text{poly}(d)$, which would cause SAMPLETOPEIGENVECTOR to output a good solution. Now, each iteration of the loop can be implemented in $\tilde{O}(ndp)$ time: The calculation of $M_t z$ involves $p$ multiplications of $z$ with the empirical second moment matrix, each of which can be done in $O(nd)$ time as $\Sigma_t z = \sum_{x} x (x^\top z)$. Thus, the total runtime is $\tilde{O}(t_{\text{end}} d p) = \tilde{O}(nd/\gamma^3)$. A detailed proof can be found in Section 3.4.  

\(^{11}\)Naïve pruning removes only a few inliers; see Lemma 2.14.(v)
Algorithm 3 Robust PCA in Small Number of Iterations

1: **Input:** \( P, \epsilon, \gamma \). 
2: Let \( p = \frac{C \log(d/\epsilon)}{\gamma} \), \( t_{\text{end}} = \frac{C \log^2(d/\epsilon)}{\gamma^2} \) for large enough \( C \). 
3: Find estimator \( \hat{\sigma}_{op} \in (0.8|\Sigma|_{\text{op}}, 2d|\Sigma|_{\text{op}}) \). \( \triangleright \) c.f. Lemma 2.14.(iv) with \( U = I \) 
4: Initialize \( w_{1,1}(x) = \mathbb{I}(\|x\|^2_2 \leq 10\hat{\sigma}_{op}(d/\epsilon)) \).
5: for \( t = 1, \ldots, t_{\text{end}} \) do 
6: Call \textsc{SampleTopEigenvector} \( (P, w_t, \epsilon, \gamma, \frac{1}{t_{\text{end}}} \). 
7: Let \( P_t \) be the distribution of \( P \) weighted by \( w_t: P(x)w_t(x)/E_{X \sim P}[w_t(X)] \). 
8: Let \( B_t := E_{X \sim P}[w_t(X)XX^\top] \) and \( M_t := B_t^p \). \( \triangleright \) \( M_t \) does not need to be explicitly computed. 
9: Let \( g_t(x) := \|M_t x\|^2_2 \). 
10: \( v_t \leftarrow M_t z_t \), where \( z_t \sim \mathcal{N}(0, I) \). 
11: Let \( f_t(x) = (v_t^\top x)^2 \). 
12: Let \( L_t \) be the 3e-quantile of \( f_t(\cdot) \) under \( P_t \). 
13: \( L_t \leftarrow \max\{L_t, (0.1/d)\hat{\sigma}_{op}\|v_t\|^2_2\} \).\(^{12}\) 
14: Let \( \tau_t(x) = f_t(x)I(f_t(x) > L_t) \) 
15: Find \( \hat{\sigma}_t \) such that \( |\hat{\sigma}_t - v_t^\top \Sigma v_t| \leq 4\gamma v_t^\top \Sigma v_t \). (e.g., \( \hat{\sigma}_t := E_{X \sim P}[w_t(X)f_t(X)I(f_t(X) \leq L_t)] \)). \( \triangleright \) c.f. Lemma 2.14.(iv) with \( U = v_t^\top \)
16: \( \hat{T}_t \leftarrow 2.35\gamma \hat{\sigma}_t \). 
17: \( w_{k,t+1} \leftarrow \text{HardThresholdingFilter}(P, w_t, \tau_t, \hat{T}_t, R) \). 
18: end for 
19: return \text{FAIL}.

**Lemma 3.6** (Informal). In Setting 2.13, if \( \langle \Sigma, M_t^2 \rangle \geq (1 - C\gamma)\langle \Sigma_t, M_t^2 \rangle \), then \( E_t[\phi_{t+1}] \leq (1 - \gamma)\phi_t \) in Algorithm 3, where \( E_t \) is the conditional expectation up to \( t \)-th iteration.

**Proof Sketch of Lemma 3.6.** For simplicity, we assume that scores \( f_t(x) \) used by the algorithm (Line 11) are the same as the scores \( g_t(x) \). Observe that computing \( g_t(x) = \|M_t x\|^2_2 \) for all \( n \) points would be computationally costly. This is why we use the one-dimensional random projections \( f_t(x) \), which are unbiased estimates of \( g_t(x) \). A complete proof that uses the estimates \( f_t \) is significantly more involved and can be found in Section 3.4.2.

Using our definitions, we calculate the decrease in potential at the \((t+1)\)-th step:

\[
\phi_{t+1} = \text{tr} \left( B_{t+1}^{2p+1} \right) \leq \text{tr} \left( B_t^p B_{t+1} B_t^p \right) = \text{tr}(M_t B_{t+1} M_t) \\
= E_{X \sim P}[w_{t+1}(X)g_t(X)] \\
= (1-\epsilon) E_{X \sim G}[w_{t+1}(X)g_t(X)] + \epsilon E_{X \sim B}[w_{t+1}(X)g_t(X)] \\
\leq E_{X \sim G}[w_{t+1}(X)g_t(X)] + \epsilon E_{X \sim B}[w_{t+1}(X)g_t(X)], \tag{10}
\]

where the first inequality uses that \( B_{t+1} \preceq B_t \) is decreasing in PSD order (along with Fact 2.3) and the last inequality uses \( w_{t+1} \leq w_t \). We argue that the RHS above is \((1 + O(\gamma))\langle \Sigma, M_t^2 \rangle\): The first term in Equation (10), which corresponds to inliers, can be upper-bounded by \((1 + 2\gamma)\langle \Sigma, M_t^2 \rangle\) using stability (Lemma 2.14.(i)) For the second term in Equation (10), we use \( \tau_t(x) \leq g_t(x) + L_t \leq g_t(x) + 1.65(\gamma/\epsilon)\langle \Sigma, M_t^2 \rangle \), where the last inequality uses Lemma 2.14.(iii). Moreover, filtering ensures

\(^{12}\)This ensures a lower bound for \( \tau(x) \) whenever it is non-zero. See second bullet in Section 2.3 for why this is needed.
that $\epsilon E_{X \sim B}[w_{t+1}(X)\tau_t(X)] \leq 2.35\gamma(S, M^2_t)$, where we use Lemma 2.16 with $T = 2.35\gamma(S, M^2_t)$, with this choice of $T$ justified by Lemma 2.14(ii). Combining these bounds, we obtain

$$\phi_{t+1} \leq (1 + O(\gamma))(S, M^2_t).$$

Now, using our assumption $(S, M^2_t) < (1 - C\gamma)(S_t, M^2_t)$, we complete the proof as follows:

$$\phi_{t+1} < (1 + O(\gamma))(S, M^2_t) \leq (1 + O(\gamma))(1 - C\gamma)(S_t, M^2_t)$$

$$\leq \frac{(1 + O(\gamma))(1 - C\gamma)}{E_{X \sim P}[w_t(X)]}(B_t, M^2_t) \leq (1 - \gamma)\phi_t,$$

using $C \gg 1$, $(B_t, M^2_t) = \phi_t$, and $E_{X \sim P}[w_t(X)] \geq 1 - O(\epsilon)$, as filtering does not delete more than $O(\epsilon)$-mass from the inliers (see discussion below Lemma 2.16).

The above sketch, while demonstrating the main idea of the argument, is rather simplistic, and the formal proof is given in Section 3.4.2. We note that Lemma 3.6 holds for all $p \geq \log d$; however, Lemma 3.3 requires large $p$.

### 3.3 Improving the Dependence on $\epsilon$ in Runtime

We first describe the improvement in high-level and then move to its formal proof in Section 3.4. In the simpler version of the algorithm, the main reason for a large runtime was that (i) the initial potential was $\text{poly}(\epsilon^p)\|S\|_{2p}^{2p+1}$, (ii) the value of $p$ was $\frac{C}{\gamma} \log\left(\frac{d}{\epsilon}\right)$, (iii) the potential decreases by only $(1 - \gamma)$ factor, and (iv) the algorithm terminates when $\phi_t \leq \|S\|_{2p}^{2p+1}/\text{poly}(d)$. Since (ii)-(iv) are most likely needed, we modify the algorithm to ensure that the initial potential is much smaller: $\text{poly}(\epsilon^p)\|S\|_{2p}^{2p+1}$. If we are able to ensure this cheaply, the algorithm would need only $\text{polylog}(d/\epsilon)/\gamma$ rounds, saving one factor of $1/\gamma$ from the runtime.

The idea is that while we do need (ii), it is only necessary when the algorithm is about to produce a final solution. Suppose we run Algorithm 3 with $p = \log d$, where the initial potential is indeed $\text{poly}(\epsilon^p)\|S\|_{2p}^{2p+1}$. In Section 3.2, we guaranteed a $(1 - \gamma)$-reduction in potential until the condition $(S, M^2_t) \geq (1 - C\gamma)(S_t, M^2_t)$ gets activated. However, even if this condition gets activated, we do not know if Algorithm 2 will succeed as $p \ll \log(d/\gamma)$ (c.f. Lemma 3.3).

Nonetheless, when this condition gets activated, we see that the final potential has become much smaller: $\phi_t = \langle B_t, M^2_t \rangle \leq (S_t, M^2_t) \leq (1 + C\gamma)(S_t, M^2_t) \leq (1 + C\gamma)\|S\|_{2p}^{2p+1}\|B_t\|_{2p}^{2p}$, where the second step uses that the condition is violated. Using Fact 2.7 to relate $\|B_t\|_{2p}^{2p}$ and $\phi_t = \|B_t\|_{2p+1}^{2p+1}$, we obtain $\phi_t \leq \text{poly}(\epsilon^p)\|S\|_{2p+1}^{2p+1}$.

So far, we have shown that starting with $\phi_0 \leq \text{poly}(\epsilon^p)\|S\|_{2p}^{2p+1}$, we can reduce $\phi_t$ to $\phi_t \leq \text{poly}(\epsilon)\|S\|_{2p}^{2p+1}$ after $t \asymp \log^2(d/\epsilon)/\gamma$ rounds. We can then double the value of $p$ to $p' = 2p$ and restart the counter $t$. The new potential $\phi'_0$ will be upper bounded using Fact 2.7 as follows:

$$\phi'_0 = \|B_t\|_{2p+1}^{2p+1} \leq \|B_t\|_{2p+1}^{2p+1} = (\phi_t)^{2p+1} = (\phi_t)^{2p+1},$$

i.e., it is still $\text{poly}(\epsilon)\|S\|_{2p+1}^{2p+1}$. We run the same procedure repetitively until $p$ reaches its final value of $\Theta(\log^2(d/\epsilon)/\gamma)$, where the analysis of the previous two sections are applicable, returning a good vector. This leads to the nested-loop algorithm outlined in Algorithm 4. This will be the algorithm realizing Theorem 3.1.

---

13Lemma 3.3 (Certificate lemma) requires $p > C\log(d/\gamma)/\gamma$. 

---

16
We now provide the formal proof of Theorem 3.1 and provide the details omitted from the earlier sketches, such as the fact that the scores $f_t(x)$ computed by the algorithm do not coincide with $g_t(x)$. We will also incorporate the arguments of Section 3.3 for the improved runtime.

As discussed earlier, the proof consists of three main claims: (i) whenever the condition $\langle \Sigma, M^2 \rangle \geq (1 - C)\langle \Sigma, M^2 \rangle$ is true (for a large enough value of $p$), we can output a good solution with \textsc{SampleTopEigenvector} , (ii) if the condition is false, then $\phi_{t+1} \leq (1 - \gamma)\phi_t$ on average, and (iii) if $\phi_t \leq \|\Sigma\|_{op}^2 + 1/poly(d)$, then the condition from (i) is necessarily true. The first part has been shown in Lemma 3.3, the last part is a fairly straightforward implication of stability, which is shown in Section 3.4.1 (Lemma 3.8), and the result that corresponds to (ii) is shown in Section 3.4.2. These pieces are then put together in Section 3.4.3 to complete the proof of the theorem. A detailed runtime analysis is provided in Section 3.4.4.

In terms of notation, we will follow the notations defined in Algorithm 4. In particular, many relevant quantities will have two subscripts such as $\Sigma_{k,t}$: here $k$ refers to the outer loop and $\ell$ refers to the inner loop.

For the formal proof, we will use that the scores $f_{k,t}(x)$ are a constant factor approximation of the scores $g_{k,t}(x)$ with constant absolute probability (a detail that we omitted in the sketch of the
Assumption 3.7. The random selection of the vector \( v_{k,t} \) in Line 13 of Algorithm 4 is such that for every point \( x \) in our domain,

\[
\Pr_{v_{k,t}} \left[ f_{k,t}(x) > \frac{g_{k,t}(x)}{10} - 0.01\frac{\gamma}{\epsilon} \|M_{k,t}\|_F^2 \|\Sigma\|_{op} \right] \geq 0.4.
\]

We emphasize again that Fact 2.9 implies that Assumption 3.7 holds for Algorithm 4.

3.4.1 Small Potential Implies that Stopping Condition Holds

In this section, we formally show the result that whenever the potential function \( \phi := \text{tr}(B^{2p+1}) \) is smaller than \( \|\Sigma\|_{op}^{2p+1}/\text{poly}(d/\epsilon) \) for a large enough value of \( p \), then the condition of the previous lemma gets satisfied. Note that the factor \((1-2\gamma)^{2p}\) mentioned in the conclusion of the following lemma is indeed bigger than \(1/\text{poly}(d/\epsilon)\) since \( p = O(\log(d/\gamma)/\gamma)\).

Lemma 3.8. Let \( 0 < 20\epsilon \leq \gamma < 1/20 \), and a positive integer \( p \). Let \( P = (1-\epsilon)G + \epsilon \mathcal{B} \), where \( G \) is a \((20\epsilon, \gamma)\)-stable distribution with respect to \( \Sigma \), and let \( w : \mathbb{R}^d \to [0,1] \) with \( \mathbb{E}_{X \sim G}[w(X)] \geq 1-3\epsilon \). Define \( \mathcal{B} := \mathbb{E}_{X \sim P}[w(X)XX^\top], \mathcal{M} := \mathcal{B}^p \), and \( \phi := \text{tr}(\mathcal{B}^{2p+1}) \). If \( \phi \leq \mathbb{E}_{X \sim P}[w(X)]^{(1-2\gamma)^{2p}} \|\Sigma\|_{op}^{2p+1} \), then 
\[
\langle \Sigma, M^2 \rangle \geq (1-250\gamma)\langle \Sigma_{P_{w}}, M^2 \rangle.
\]

Proof. We claim that for the statement to be true, it suffices to show that

\[
\langle M^2, \Sigma \rangle \geq (1-2\gamma)^{2p} \|\Sigma\|_{op}^{2p+1}.
\]

This is indeed sufficient, because if we had Equation (12) at hand, then whenever it holds \( \phi \leq \mathbb{E}_{X \sim P}[w(X)]^{(1-2\gamma)^{2p}} \|\Sigma\|_{op}^{2p+1} \), then we have that

\[
\langle M^2, \Sigma_{P_{w}} \rangle = \frac{\phi}{\mathbb{E}_{X \sim P}[w(X)]} \leq \frac{(1-2\gamma)^{2p}}{1-250\gamma} \|\Sigma\|_{op}^{2p+1} \leq \frac{1}{1-250\gamma} \langle M^2, \Sigma \rangle,
\]

where the first step uses the definitions \( \phi = \text{tr}(\mathcal{B}^{2p+1}) = \langle M^2, \mathcal{B} \rangle = \mathbb{E}_{X \sim P}[w(X)]\langle M^2, \Sigma_{P_{w}} \rangle \), the second step uses \( \phi \leq \mathbb{E}_{X \sim P}[w(X)]^{(1-2\gamma)^{2p}} \|\Sigma\|_{op}^{2p+1} \), and the last step uses Equation (12).

In the reminder, we establish Equation (12). Consider the spectral decomposition \( \Sigma = \sum_{i=1}^{d} \lambda_i u_i u_i^\top \) with \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d \geq 0 \). We will lower bound \( \langle M^2, \Sigma \rangle \) using \( \langle M^2, \lambda_1 u_1 u_1^\top \rangle \) as follows:

\[
\langle M^2, \Sigma \rangle = \sum_{i=1}^{d} \lambda_i u_i u_i^\top = \sum_{i=1}^{d} \lambda_i u_i^\top (M^2 - \|\Sigma\|_{op} u_1 u_1^\top) \geq \lambda_1 u_1^\top M^2 u_1 = \|\Sigma\|_{op} u_1^\top M^2 u_1. \tag{13}
\]

In the remainder of this proof, we will show that \( u_1^\top M^2 u_1 \geq (1-2\gamma)^{2p} \|\Sigma\|_{op}^{2p+1} \), which will complete the proof. We begin by lower bounding \( \mathcal{B} \) by \( \Sigma \) in the PSD sense below:

\[
\mathcal{B} = \mathbb{E}_{X \sim P}[w(X)XX^\top] \geq (1-\epsilon) \mathbb{E}_{X \sim G}[w(X)]\mathbb{E}_{X \sim G}[w(X)XX^\top] \mathbb{E}_{X \sim G}[w(X)]^{-1} \geq (1-\epsilon)(1-3\epsilon)(1-\gamma)\Sigma \geq (1-2\gamma)\Sigma, \tag{14}
\]
where the first line uses that \( P = (1 - \epsilon)G + \epsilon B \), and the last line uses \( \mathbb{E}_{X \sim G}[w(X)] \geq 1 - 3\epsilon \), and stability condition for the second moment of the normalized distribution \( G_w \), and the last step uses that \( \epsilon < \gamma / 20 \). Since \( M = B^p \) and \( B \) is at least \( \Sigma \), we obtain the following lower bound on \( u_1^\top M^2 u_1 \):

\[
u_1^\top M^2 u_1 = u_1^\top B^p u_1 \geq (u_1^\top B u_1)^{2p} \geq ((1 - 2\gamma)u_1^\top \Sigma u_1)^{2p} = (1 - 2\gamma)^{2p} \| \Sigma \|^2_{\text{op}},
\]

where the first inequality uses Fact 2.5, and the second inequality uses Equation (14).

\[
\square
\]

### 3.4.2 Formal Proof of Multiplicative Decrease of the Potential

In this section, we provide a formal proof that establishes the following: If the condition \( \langle \Sigma, M_{k,t}^2 \rangle \geq (1 - 250\gamma)(\Sigma_{k,t}, M_{k,t}^2) \) is not satisfied, then, on average, the potential function decreases multiplicatively; Recall that \((k,t)\) denotes the double index of the iteration under analysis. First, we establish Lemma 3.9, which analyzes a single iteration of our algorithm’s inner loop, conditioned on the previous ones. Then, we state Corollary 3.11 to show that, when the entire loop finishes, the potential function is reduced by a factor of \((1 - \gamma)^t\) on expectation.

**Lemma 3.9.** Consider the notation of Algorithm 4, where the input distribution is the mixture \( P = (1 - \epsilon)G + \epsilon B \), with \( G \) being a \((20\epsilon, \gamma)\)-stable distribution with respect to \( \Sigma \) for \( 0 < 20\epsilon \leq \gamma \leq \gamma_0 \) for a sufficiently small positive constant \( \gamma_0 \). Make Assumption 3.7. Let \( E_{k,t} = E_{k,t}^{(1)} \cap E_{k,t}^{(2)} \) denote the intersection of the following two events:

(i) \( E_{k,t}^{(1)}: (1 - \epsilon)\mathbb{E}_{X \sim G}[1 - w_{k',t'}(X)] + \epsilon \mathbb{E}_{X \sim B}[w_{k',t'}(X)] \leq 3\epsilon \), for every iteration \((k', t')\) prior to (and including) \((k, t)\).

(ii) \( E_{k,t}^{(2)}: \langle \Sigma, M_{k', t'}^2 \rangle < (1 - 250\gamma)(\Sigma_{k', t'}, M_{k', t'}^2) \) for every iteration \((k', t')\) from \((k, 1)\) up to (and including) \((k, t)\).

Define the potential function \( \phi_{k,t} := \text{tr}(B_{k,t}^{2p_{k,t} + 1}) \). Let \( F_{k,t} \) be the randomness used by HARDTHRESHOLDINGFILTER during the \((k, t)\)-th loop of ROBUSTPCA (i.e., \( F_{k,t} \) is the collection of the random variables \( \{v_{1}, v_{2}, \ldots \text{ used by the filter}\}, and let \( v_{k,t} \) be the random vectors used in Line 13 of ROBUSTPCA (Algorithm 4). Also, define \( F_{k,t} = \{F_{k', t'} : k' \leq k - 1, t' \leq t_{\text{end}}\} \cup \{F_{k, t'} : t' \leq t\} \), i.e., the entire history of filters up to (and including) the iteration \((k, t)\). Define \( V_{k,t} \) similarly for \( v_{k,t} \).

\[
\text{Note that event } E_{k,t}^{(2)} \text{ implies that the algorithm has not ended yet (since the first time it happens that } \langle \Sigma, M_{k,t}^2 \rangle \geq (1 - 250\gamma)(\Sigma_{k,t}, M_{k,t}^2), \text{ we know by Lemma 3.3 that the algorithm will stop w.h.p.).}
\]

\[
\text{We will show later on that this event indeed holds through the algorithm, as an invariant condition.}
\]
Then, the following is true for every conditioning on $\mathcal{F}_{k,t}, \mathcal{V}_{k,t-1}$:

$$
\mathbb{E}_{v_{k,t}}[\phi_{k,t+1}(\mathcal{E}_{k,t}) \mid \mathcal{F}_{k,t}, \mathcal{V}_{k,t-1}] \leq (1 - \gamma) \phi_{k,t}(\mathcal{E}_{k,t-1}) .
$$

(15)

Proof. We analyze the $(k, t)$-th iteration of ROBUSTPCA, that is, the outer loop with index $k$ and the inner loop with index $t$. We condition on everything that has happened previously, and we are only interested in analyzing what happens in the current round of the inner loop with respect to randomness coming from $v_{k,t}$. In particular, the expectation will be conditional on the past history $\mathcal{F}_{k,t}, \mathcal{V}_{k,t-1}$ as in Equation (15) but we omit explicitly writing them for notational convenience.

In the case that $1(\mathcal{E}_{k,t}) = 0$, the conclusion of our lemma holds trivially. Thus, in the reminder of the proof, we analyze the case $1(\mathcal{E}_{k,t}) = 1$.

We will call a point $x$ full if $f_{t,k}(x) > g_{k,t}(x)/10 - 0.01(\gamma/\epsilon)\|M_{k,t}\|_F^2\|\Sigma\|_{\text{op}}$, otherwise we will call it empty. Note that $x$ being full implies that

$$
\tau_{k,t}(x) \geq f_{k,t}(x) - L_{k,t} \geq \frac{g_{k,t}(x)}{10} - L_{k,t} - 0.01\gamma\|M_{k,t}\|_F^2\|\Sigma\|_{\text{op}}
$$

$$
\geq \frac{g_{k,t}(x)}{10} - 1.65\gamma\|v_{k,t}\|_2^2\|\Sigma\|_{\text{op}} - 0.01\gamma\|M_{k,t}\|_F^2\|\Sigma\|_{\text{op}},
$$

(16)

where the first inequality uses the definition of $\tau_{k,t}$, and the last inequality uses Item (iii) of Lemma 2.14.

We now use Lemma A.8 to analyze the effect of filtering via HARD_THRESHOLDING_FILTER (Algorithm 8): We apply that lemma with $T = 2.35\gamma v_{k,t}^T\Sigma v_{k,t}$, $\hat{T} = 2.35\gamma\hat{\sigma}_{k,t}$, and $\delta = 0$; we will now justify the choice of these parameters. The first requirement of Lemma A.8 is that $(1 - \epsilon)\mathbb{E}_{X \sim \mathcal{C}}[w_{k,t}(x)\tau_{k,t}(x)] < T$, which is satisfied by Item (ii) of Lemma 2.14 (specifically, the special case that uses $U = v_{k,t}^T$). For its second requirement of $\hat{T} > T/5$, we have that $|\hat{T} - T| = 2.35\gamma\|\hat{\sigma}_{k,t} - v_{k,t}^T\Sigma v_{k,t}\| \leq 2.35\gamma \cdot 4\gamma \cdot v_{k,t}^T\Sigma v_{k,t} \leq (1/5) \cdot 2.35\gamma v_{k,t}^T\Sigma v_{k,t} = T/5$, where the second step uses Item (iv) of Lemma 2.14 (with $U = v_{k,t}^T$) and the last inequality uses that $\gamma < 1/20$. Thus, the lemma is applicable, and it yields that

$$
\mathbb{E}_{X \sim p}[w_{k,t+1}(X)\tau_{k,t}(X)] \leq 3T' \leq 7.1\|v_{k,t}\|_2^2\|\Sigma\|_{\text{op}},
$$

(17)

with probability one. Therefore, for the bad points that are full, we get that their updated weights after the filter in the current iteration satisfy the following:

$$
\epsilon \mathbb{E}_{X \sim B}[w_{k,t+1}(X)g_{k,t}(X)1(X \text{ full})] \leq 10\epsilon \mathbb{E}_{X \sim B}[w_{k,t+1}(X)\tau_{k,t}(X)] + 16.5\|v_{k,t}\|_2^2\|\Sigma\|_{\text{op}} + 0.1\gamma\|M_{k,t}\|_F^2
$$

$$
\leq 88\gamma\|v_{k,t}\|_2^2\|\Sigma\|_{\text{op}} + 0.1\gamma\|M_{k,t}\|_F^2\|\Sigma\|_{\text{op}}.
$$

(18)

where the first step uses Equation (16) and the second uses Equation (17).

By Assumption 3.7, every point is full with probability 0.4. We will now take expectation with respect to $v_{k,t}$ (we again remind the reader that the expectation is conditioned on the past history $\mathcal{F}_{k,t}, \mathcal{V}_{k,t-1}$ as in Equation (15) but we omit explicitly writing them for notational convenience).

$$
\mathbb{E}_{v_{k,t}}[\epsilon \mathbb{E}_{X \sim B}[w_{k,t+1}(X)g_{k,t}(X)]]
$$

$$
= \mathbb{E}_{v_{k,t}}[\epsilon \mathbb{E}_{X \sim B}[w_{k,t+1}(X)g_{k,t}(X)1(X \text{ full})] + \mathbb{E}_{v_{k,t}}[\epsilon \mathbb{E}_{X \sim B}[w_{k,t+1}(X)g_{k,t}(X)1(X \text{ empty})]]
$$

$$
\leq \mathbb{E}_{v_{k,t}}[\epsilon \mathbb{E}_{X \sim B}[w_{k,t+1}(X)g_{k,t}(X)1(X \text{ full})] + \mathbb{E}_{X \sim B}[w_{k,t}(X)g_{k,t}(X)\mathbb{E}_{v_{k,t}}[1(X \text{ empty})]]
$$
\[ \leq \mathbb{E}_{v_{k,t}} \left[ 88\gamma \|v_{k,t}\|_2 \|\Sigma\|_{\text{op}}^2 + 0.1 \|M_{k,t}\|_F^2 \|\Sigma\|_{\text{op}} + \epsilon \mathbb{E}_{X \sim B} [w_{k,t}(X)g_{k,t}(X)(0.6)] \right] \\
= 88.1\gamma \|M_{k,t}\|_F^2 \|\Sigma\|_{\text{op}} + 0.6\epsilon \mathbb{E}_{X \sim B} [w_{k,t}(X)g_{k,t}(X)], \quad (19) \]

where the second step uses \( w_{k,t+1} \leq w_{k,t} \) for the second term, the third step uses Equation (18) and the fact that the probability of a point being empty is at most 0.6, and the last step uses that 
\[ \mathbb{E}_{v_{k,t}} \left[ \|v_{k,t}\|_2^2 \right] = \mathbb{E}_{z_{k,t} \sim \mathcal{N}(0,1)} \left[ z_{k,t}^T M_{k,t}^2 z_{k,t} \right] = \text{tr}(M_{k,t}^2) = \|M_{k,t}\|_F^2. \]

We now upper bound the expectation of the potential function using following series of inequalities, which are explained below:\[16\]
\[ \mathbb{E}_{v_{k,t}} \left[ \text{tr} \left( B_{k,t}^{2p_{k}+1} \right) \right] \leq \mathbb{E}_{v_{k,t}} \left[ \text{tr} \left( B_{k,t}^{p_k} B_{k,t+1} B_{k,t}^{p_k} \right) \right] \leq \mathbb{E}_{v_{k,t}} \left[ \text{tr} \left( M_{k,t} B_{k,t+1} M_{k,t} \right) \right] \leq \mathbb{E}_{v_{k,t}} \left[ \left( 1 - \epsilon \right) \mathbb{E}_{X \sim G} [w_{k,t+1}(X)g_{k,t}(X)] + \epsilon \mathbb{E}_{X \sim B} [w_{k,t+1}(X)g_{k,t}(X)] \right] \leq \left( 1 + 2\gamma \right) \langle \Sigma, M_{k,t}^2 \rangle + 89\gamma \|M_{k,t}\|_F^2 \|\Sigma\|_{\text{op}} + \epsilon \mathbb{E}_{X \sim G} [w_{k,t}(X)g_{k,t}(X)] \]
\[ \leq \left( 1 + 2\gamma \right) \langle \Sigma, M_{k,t}^2 \rangle + 89\gamma \|M_{k,t}\|_F^2 \|\Sigma\|_{\text{op}} + 0.6\phi_{k,t} - 0.6 \left( 1 - 3\gamma \right) \langle \Sigma, M_{k,t}^2 \rangle \]
\[ = 0.4 \langle \Sigma, M_{k,t}^2 \rangle + 3.8\gamma \langle \Sigma, M_{k,t}^2 \rangle + 89\gamma \|M_{k,t}\|_F^2 \|\Sigma\|_{\text{op}} + 0.6\phi_{k,t} \leq 0.4 \langle \Sigma, M_{k,t}^2 \rangle + 93\gamma \|M_{k,t}\|_F^2 \|\Sigma\|_{\text{op}} + 0.6\phi_{k,t}, \quad (28) \]

where Equation (20) uses that \( B_{k,t+1} \preceq B_{k,t} \) along with Fact 2.3 and the cyclic property of trace operator, Equation (21) uses the definition \( B_{k,t} = E_{X \sim P}[w_{k,t}(X)\Sigma_{k,t}] = E_{X \sim P}[w_{k,t}(X)XX^\top] \), Equation (22) uses that \( P = (1 - \epsilon)G + \epsilon B \), Equation (23) uses \( w_{k,t+1}(x) \leq w_{k,t}(x) \) for the first term, Equation (24) uses \( 1 - \epsilon \leq 1 \) and Item (i) of Lemma 2.14 for the first term and Equation (19) for the second term, Equation (25) uses that \( P = (1 - \epsilon)G + \epsilon B \), Equation (26) uses the definition of \( B_{k,t} \) to get \( E_{X \sim P}[w_{k,t}(X)g_{k,t}(X)] = E_{X \sim P}[w_{k,t}(X)\text{tr}(M_{k,t}^2 XX^\top)] = \text{tr}(M_{k,t}^2 E_{X \sim P}[w_{k,t}(X)XX^\top]) = \text{tr}(B_{k,t}^{2p_{k}+1}) = \phi_{k,t} \). We now explain the last three steps. Staring with Equation (27), recall that we have conditioned on the event that \( 1 - \epsilon \mathbb{E}_{X \sim G}[1 - w_{k,t}(X)] + \epsilon \mathbb{E}_{X \sim B}[w_{k,t}(X)] \leq 2.9\epsilon \). Thus, we use Item (i) of Lemma 2.14 to get \( (1 - \epsilon) \mathbb{E}_{X \sim G}[w_{k,t}(X)g_{k,t}(X)] \geq (1 - \epsilon)(1 - 2\gamma) \langle \Sigma, M_{k,t}^2 \rangle \geq (1 - 3\gamma) \langle \Sigma, M_{k,t}^2 \rangle \) (where the last inequality uses \( \epsilon < \gamma/20 \)). Finally, Equation (28) upper bounds the terms \( \gamma \langle \Sigma, M_{k,t}^2 \rangle \) by \( \gamma \|\Sigma\|_{\text{op}} \|M_{k,t}\|_F^2 \). We will now relate both \( \langle \Sigma, M_{k,t}^2 \rangle \) and \( \|\Sigma\|_{\text{op}} \|M_{k,t}\|_F^2 \) with \( \phi_{k,t} \).

\[16\] Recall that we are in the setting when \( \mathbb{I}(E_{k,t-1})=1 \), and thus we omit writing the indicator inside the expectations explicitly.
We begin with the term \( \langle \Sigma, M^2_{k,t} \rangle \): Since the event \( \mathcal{E}^{(2)}_{k,t} \) holds, we have that
\[
\langle \Sigma, M^2_{k,t} \rangle < (1 - 250\gamma)(\Sigma_{k,t}, M^2_{k,t}) = \frac{1 - 250\gamma}{\mathbb{E}_{X \sim P[w_{k,t}(X)]}} \langle B_{k,t}, M^2_{k,t} \rangle
\]
\[
\leq \frac{1 - 250\gamma}{(1 - \epsilon)(1 - 3\epsilon)} \phi_{k,t} \leq (1 - 240\gamma) \phi_{k,t},
\]
where the first inequality uses the definition of the event \( \mathcal{E}^{(2)}_{k,t} \), the next steps use that \( \mathbb{E}_{X \sim P[w_{k,t}(X)]} \geq (1 - \epsilon)(1 - 3\epsilon) \), where the last inequality here used the definition of the event \( \mathcal{E}^{(1)}_{k,t} \).

We now state the following result relating \( \phi_{k,t} \) with \( \|\Sigma\|_{op} \|M_{k,t}\|_F^2 \) for \( p_k \) sufficiently large:

**Claim 3.10.** If \( p_k \geq C \log(d) \) for a sufficiently large constant \( C \) and the event \( \mathcal{E}_{k,t} \) holds, then
\[
\text{tr} \left( B_{k,t}^{2p_k} \right) \|\Sigma\|_{op} \leq 1.01 \cdot \text{tr} \left( B_{k,t}^{2p_k+1} \right).
\]

**Proof.** We will prove the result using the relations between the Schatten \( p \)-norms and relating \( \Sigma \) with \( \Sigma_t \) using stability as follows:
\[
\text{tr} \left( B_{k,t}^{2p_k} \right) \|\Sigma\|_{op} < (1 + 3\gamma) \text{tr} \left( B_{k,t}^{2p_k} \right) \|B_{k,t}\|_{op}
\]
\[
= (1 + 3\gamma)\|B_{k,t}\|_{2p_k} \|B_{k,t}\|_\infty
\]
\[
\leq (1 + 3\gamma)\|B_{k,t}\|_{2p_k+1} \quad \text{by \cite{ Fact 2.7} (typically \( \|A\|_{\infty} \leq \|A\|_q \forall q \))}
\]
\[
\leq (1 + 3\gamma) \left( d^{2p_k(2p_k+1)} \|B_{k,t}\|_{2p_k+1} \right)^{2p_k+1}
\]
\[
= (1 + 3\gamma) d^{1/2p_k} \|B_{k,t}\|_{2p_k+1}^{2p_k+1}
\]
\[
\leq (1 + 3\gamma) \cdot 1.001 \cdot \text{tr} \left( B_{k,t}^{2p_k+1} \right) \quad \text{(\( p_k \geq C \log(d) \) for \( C \) large enough)}
\]
\[
\leq 1.01 \cdot \text{tr} \left( B_{k,t}^{2p_k+1} \right). \quad \text{(\( \gamma < \gamma_0 \) for a sufficiently small \( \gamma_0 \))}
\]
This completes the proof of Claim 3.10.

We can now upper bound the RHS of Equation (28) using Equation (29) and Claim 3.10:
\[
0.4(\Sigma, M^2_{k,t}) + 93\gamma \|M_{k,t}\|_F^2 \|\Sigma\|_{op} + 0.6\phi_{k,t} \leq 0.4(1 - 240\gamma) \phi_{k,t} + 93 \cdot 1.01 \cdot \gamma \phi_{k,t} + 0.6\phi_{k,t}
\]
\[
\leq (1 - \gamma) \phi_{k,t}.
\]
This completes the proof.

**Corollary 3.11.** In the context of Lemma 3.9, assume that \( \mathbb{E}_{\nu_{k-1,t_{\text{end}}}}[\phi_{k,1} | F_{k-1,t_{\text{end}}}] \leq R \). Then, under every conditioning \( F_{k,t-1} \) of the form \( F_{k-1,t_{\text{end}}} \cup \{ F_{k,t'} : t' \leq t-1 \} \) (i.e., every conditioning for the filter up to the \((k, t - 1)\)-th iteration that agrees with \( F_{k-1,t_{\text{end}}} \) on the first iterations up to the \((k - 1, t_{\text{end}})\)-th), we have that
\[
\mathbb{E}_{\nu_{k,t-1}}[\phi_{k,t} \mathbb{1}(\mathcal{E}_{k,t-1}) | F_{k,t-1}] \leq (1 - \gamma)^{t-1} R.
\]
Proof. We prove this by induction on $t$. The base case $t = 1$ holds trivially by assumption. Now, we assume that the claim holds for the index $(k, t)$ and we will show that it will continue to hold for the index $(k, t + 1)$. That is, suppose that $E_{\psi_{k,t-1}}[\phi_{k,t}1(\mathcal{E}_{k,t-1}) | \mathcal{F}_{k,t-1}] \leq (1 - \gamma)^{t-1}R$. Then, we know by Lemma 3.9 that

$$E_{\psi_{k,t}}[\phi_{k,t+1}1(\mathcal{E}_{k,t}) | \mathcal{F}_{k,t}, \mathcal{V}_{k,t-1}] \leq (1 - \gamma)\phi_{k,t}1(\mathcal{E}_{k,t-1}).$$

Taking expectation over $\mathcal{V}_{k,t-1}$ of both sides yields

$$E_{\psi_{k,t}}[\phi_{k,t+1}1(\mathcal{E}_{k,t}) | \mathcal{F}_{k,t}] \leq (1 - \gamma) E_{\psi_{k,t-1}}[\phi_{k,t}1(\mathcal{E}_{k,t-1}) | \mathcal{F}_{k,t-1}] \leq (1 - \gamma)^tR.$$

\qed

3.4.3 Combining Everything Together

We now put together the ingredients of the previous subsections to complete the proof of Theorem 3.1.

Proof of Theorem 3.1. It suffices to show that the conclusion of the theorem holds with a small constant probability, as this probability can be boosted arbitrarily by repeating the algorithm and selecting the output $u$ that maximizes $u^\top\Sigma u$ (more precisely, an estimate of this variance that can be obtained by Item (iv) of Lemma 2.14). For completeness, we give the bounds on runtime in Section 3.4.4.

First of all, observe that if the algorithm returns a vector using $\text{SAMPLETOPEIGENVECTOR}$, then the resulting vector satisfies the desired guarantees with high probability. We use the same notation as in the statement of Lemma 3.9. We will show that at the start of any iteration of the outer loop (i.e., for every $k \leq k_{\text{end}}$ and $t = 1$) we have that $E_{\psi_{k-1,t_{\text{end}}}}[\phi_{k,1}1(\mathcal{E}_{k-1,t_{\text{end}}}^{(1)})] \leq (d/\epsilon)C_2\log d||\Sigma||_{op}^{2p+1}$ for a large enough positive constant $C_2$. We do this by induction on $k$. At the start of the algorithm ($k = 1$), because of Line 5, every point has norm $||x||_2^2 \leq 10(d/\epsilon)\tilde{\sigma}_{op} \leq (20(d^2/\epsilon)||\Sigma||_{op}$, the potential is bounded as follows

$$\phi_{1,1} \leq d||\Sigma||_{op}^{2p+1} \leq (20(d^2/\epsilon)||\Sigma||_{op})^{2p+1} \leq (d/\epsilon)^{10p}||\Sigma||_{op}^{2p+1}$$

$$= (d/\epsilon)^{10C'\log d||\Sigma||_{op}^{2p+1}} \leq (d/\epsilon)C_2\log d||\Sigma||_{op}^{2p+1}.$$

Under the induction hypothesis, suppose the desired conclusion holds up to (and including) some $k \geq 1$. Let us first relate $\phi_{k, t_{\text{end}}+1}$ and $\phi_{k+1,1}$ as follows:

$$\phi_{k+1,1} = \text{tr}(B_{k,t_{\text{end}}+1}) = \|B_{k,t_{\text{end}}+1}\|_{op} \leq \|B_{k,t_{\text{end}}+1}\|_{op}^{2p+1}$$

$$= \text{tr}(B_{k,t_{\text{end}}+1})^{(2p_{k+1}+1)/(2p_{k+1})} = \phi_{k,t_{\text{end}}+1}^{(2p_{k+1}+1)/(2p_{k+1})}. \quad (31)$$

By induction hypothesis, at the beginning of the $k$-th iteration of the outer loop, it holds that $E_{\psi_{k-1,t_{\text{end}}}}[\phi_{k,1}1(\mathcal{E}_{k-1,t_{\text{end}}}^{(1)})] \leq (d/\epsilon)C_2\log d||\Sigma||_{op}^{2p+1}$. In the $k$-th iteration of the outerloop, we will show that the end of the inner loop (i.e., at $t = t_{\text{end}} + 1$), the potential will be much smaller. In particular, we will show that $E_{\psi_{k,t_{\text{end}}}}[\phi_{k,t_{\text{end}}+1}1(\mathcal{E}_{k,t_{\text{end}}}^{(1)})] \leq (d/\epsilon)^{0.1C_2\log d||\Sigma||_{op}^{2p_{k+1}+1}}$. Note that in the $k$-th iteration of the algorithm, two cases might happen: (i) either the condition $\langle \Sigma t, M_{k,t}^{2} \rangle < (1 - 250\gamma)||\Sigma t, M_{k,t}^{2}||$ holds for all $t_{\text{end}}$ iterations of the inner loop, which would lead to a $(1 - \gamma)^{t_{\text{end}}}$ decrease in potential, or this condition fails to hold for some $t$, which itself implies
that the potential is small. Recall that $\mathcal{E}_{k,t}^{(2)}$ corresponds to these two cases. Thus, we have the following decomposition:

$$
\mathbb{E}_{t_{end}} \left[ \phi_{k, t_{end} + 1} \mathbb{I}(\mathcal{E}_{k,t_{end}}^{(1)}) \mathbb{I}(\mathcal{E}_{k,t_{end}}^{(2)}) \right] = \mathbb{E}_{t_{end}} \left[ \phi_{k, t_{end} + 1} \mathbb{I}(\mathcal{E}_{k,t_{end}}^{(1)}) \mathbb{I}(\mathcal{E}_{k,t_{end}}^{(2)}) \right] + \mathbb{E}_{t_{end}} \left[ \phi_{k, t_{end} + 1} \mathbb{I}(\mathcal{E}_{k,t_{end}}^{(1)}) \mathbb{I}(\mathcal{E}_{k,t_{end}}^{(2)}) \right]
$$

(32)

The first term in Equation (32) can be bounded using Corollary 3.11 as follows:

$$
\mathbb{E}_{t_{end}} \left[ \phi_{k, t_{end} + 1} \mathbb{I}(\mathcal{E}_{k,t_{end}}^{(1)}) \mathbb{I}(\mathcal{E}_{k,t_{end}}^{(2)}) \right] \leq (1 - \gamma) t_{end} + 1 \left( d/\epsilon \right) C_2 \log d \| \Sigma \|_{op}^{2p_k + 1}
$$

(33)

which is less than $(d/\epsilon)^{0.05} C_2 \log d$ after $t_{end} := C \log^2 (d/\epsilon)/\gamma$ for sufficiently large $C$ (note that we have picked $C$ to be much larger than $C_2$).

We now turn our attention to the second term in Equation (32), where we show that if $\mathcal{E}_{k,t_{end}}^{(2)}$ does not hold, then the potential will be $\text{poly}(d/\epsilon) \| \Sigma \|_{op}^{2p_k + 1}$. More formally, if for some $(k, t)$ we have that $(\Sigma, M_{k,t}^2) \geq (1 - 250\gamma) (\Sigma_{k,t}, M_{k,t}^2)$, there exists $c$ such that

$$
\phi_{k,t} = \| \mathbf{B}_{k,t} \|_{2p_k + 1} = \langle \mathbf{B}_{k,t}, M_{k,t}^2 \rangle \leq \langle \Sigma_{k,t}, M_{k,t}^2 \rangle \leq \frac{1}{1 - 250\gamma} (\Sigma, M_{k,t}^2)
$$

(34)

Rearranging the inequality above implies that

$$
\| \mathbf{B}_{k,t} \|_{2p_k + 1} \leq e^{\gamma} \| \Sigma \|_{op}^{p_k} d \| \Sigma \|_{op}^{2p_k + 1} \leq (d/\epsilon)^{O(1)} \| \Sigma \|_{op}^{2p_k + 1},
$$

where the last inequality used that $p_k = O(\log (d/\epsilon)/\gamma)$ for all $k$. Combining this with Equation (32) and Equation (34), we obtain that

$$
\mathbb{E}_{t_{end}} \left[ \phi_{k, t_{end} + 1} \mathbb{I}(\mathcal{E}_{k,t_{end}}^{(1)}) \right] \leq (d/\epsilon)^{0.1} C_2 \log d \| \Sigma \|_{op}^{2p_k + 1}. 
$$

Combining this with Equation (31) and the observation that $(2p_k + 1)/2p_k + 1 \leq 2$ by definition of $p_k$ (Line 8 in Algorithm 4), we conclude that

$$
\mathbb{E}_{t_{end}} \left[ \phi_{k+1, t_{end}} \mathbb{I}(\mathcal{E}_{k,t_{end}}^{(1)}) \right] \leq (d/\epsilon) C_2 \log d
$$

Thus far, we have shown that $\mathbb{E}_{t_{end}} \left[ \phi_{k, t_{end}} \mathbb{I}(\mathcal{E}_{k,t_{end}}^{(1)}) \right] \leq (d/\epsilon) C_2 \log d$ for $k = 1, \ldots, k_{end}$.

It remains to analyze the last iteration $k = k_{end}$. For that, we again use Corollary 3.11 to obtain the following:

$$
\mathbb{E}_{t_{end}} \left[ \phi_{k_{end}, t_{end} + 1} \mathbb{I}(\mathcal{E}_{k_{end}, t_{end}}^{(1)}) \right] \leq (1 - \gamma) t_{end} + 1 \left( d/\epsilon \right) C_2 \log d \| \Sigma \|_{op}^{1+2p_{k_{end}}}
$$

(35)

where the last step uses that $t_{end} := C \log^2 (d/\epsilon)/\gamma$ for sufficiently large $C$ and recalling that the constant $C$ is large relative to the constant $C'$ (c.f. Line 2).

Recall the definition of the event $\mathcal{E}_{k,t}$ given in the statement of Lemma 3.9 as intersection of the two events:

1. $\mathcal{E}_{k,t}^{(1)}$: $(1 - \epsilon) \mathbb{E}_{X \sim G} [1 - w_{k', t'}(X)] + \epsilon \mathbb{E}_{X \sim B} [w_{k', t'}(X)] \leq 3 \epsilon$, for every iteration $(k', t')$ prior to (and including) $(k, t)$.
2. \( E^{(2)}_{k,t} : (\Sigma, M^{2}_{k,t}) < (1 - 250\gamma)(\Sigma_{k',t'}, M^{2}_{k',t'}) \) for every iteration \((k', t')\) from \((k, 1)\) up to (and including) \((k, t)\).

By Lemma A.9 we have that \( \Pr[E^{(1)}_{kend,tend}] > 0.2 \). Thus

\[
\mathbf{E}_{\forall k, tend} \left[ \phi_{kend,tend} + 1 I(E^{(2)}_{kend,tend}) \mid E^{(1)}_{kend,tend} \right] = \frac{\mathbf{E}_{\forall k, tend} \left[ \phi_{kend,tend} + 1 I(E^{(2)}_{kend,tend}) \right]}{\Pr[E^{(1)}_{kend,tend}]} \leq 5(d/\epsilon)^{-100C'} \|\Sigma\|_{op}^{1+2p_{kend}}
\]

Therefore, we have that

\[
\Pr[\phi_{kend,tend} + 1 I(E^{(2)}_{kend,tend}) > 500(d/\epsilon)^{-100C'} \|\Sigma\|_{op}^{1+2p_{kend}} \text{ or } I(E^{(1)}_{kend,tend}) = 0] \leq \Pr[I(E^{(1)}_{kend,tend}) = 0] + \Pr[I(E^{(1)}_{kend,tend}) = 0] \leq 0.2 + 1/100 ,
\]

(36)

where the last step uses Markov’s inequality.

Let \( t^* \) be the first time during the \( k_{end} \)-th outer loop iteration of Algorithm 4 such that \( \langle \Sigma, M^{2}_{kend,t} \rangle \geq (1 - 250\gamma)(\Sigma_{kend,t}, M^{2}_{kend,t}) \) (or equivalently \( I(E^{(2)}_{kend,t}) = 0 \)). In the following, we argue that \( t^* \leq t_{end} \). We prove this by contradiction. Assume that \( I(E^{(2)}_{kend,tend}) = 1 \).

In the event that \( \phi_{kend,tend} + 1 I(E^{(2)}_{kend,tend}) < 500(d/\epsilon)^{-100C'} \) and \( I(E^{(1)}_{kend,tend}) = 1 \) simultaneously (by Equation (36) the two events will hold simultaneously with probability at least 0.79), we have that

\[
\phi_{kend,tend} + 1 = \phi_{kend,tend} + 1 I(E^{(2)}_{kend,tend}) \leq 500(d/\epsilon)^{-100C'} \|\Sigma\|_{op}^{2p_{kend} + 1} < 0.5(1 - 2\gamma)^{2p_{kend}} \|\Sigma\|_{op}^{2p_{kend} + 1} \leq \frac{\mathbf{E}_{X \sim \mathcal{P}}[w_{k,t}(X)](1 - 2\gamma)^{2p_{kend}}}{1 - 250\gamma} \|\Sigma\|_{op}^{2p_{kend} + 1} ,
\]

(37)

where the first line uses the assumption \( I(E^{(2)}_{kend,tend}) = 1 \), the third line uses \( p_{kend} = C' \log(d/\epsilon)/\gamma \), and the fourth line uses that \( \mathbf{E}_{X \sim \mathcal{P}}[w_{k,t}(X)] \geq 1/2 \) because the event \( E^{(1)}_{kend,tend} \) holds.

Lemma 3.8 and Equation (37) imply that \( \langle \Sigma, M^{2}_{kend,t} \rangle \geq (1 - 250\gamma)(\Sigma_{kend,t}, M^{2}_{kend,t}) \), which yields a contradiction. Therefore, \( t^* \leq t_{end} \).

To complete the proof it remains to show that if the algorithm hasn’t terminated earlier with a good solution, then, during the \((k_{end}, t^*)\)-th iteration, the algorithm will return a good approximation of the top eigenvalue of \( \Sigma \) and terminate. We use Lemma 3.3 for that (the lemma is applicable since its requirement \( \mathbf{E}_{X \sim \mathcal{G}}[w_{k,t}(X)] \geq 1 - 3\epsilon \) follows by the fact that we have conditioned on the event \( E^{(1)}_{kend,tend} \), which as we saw earlier happens with constant probability). The conclusion of the lemma implies that there is probability 0.9 that when \( t = t^* \), the output of SAMPLETOPEIGENVECTOR satisfies \( u^{\top} \Sigma u > (1 - O(\gamma))u^{\top}(\Sigma_{k,t} u \) and \( \frac{u^{\top} \Sigma_{k,t} u}{\|u\|_{2}} \geq (1 - \gamma)\|\Sigma_{k,t}\|_{op} \). Conditioning on this event and by noting that the estimator \( \hat{r}_{t} \) of Line 3 will satisfy \( \hat{r}_{t} \geq (1 - \gamma)\|\Sigma\|_{op} \) with high probability over all the iterations, the check of Line 7 will be activated, and by Lemma A.7, the algorithm will return a vector \( u \) such that \( u^{\top} \Sigma u / \|u\|_{2} \geq (1 - O(\gamma))\|\Sigma\|_{op} \).
3.4.4 Runtime Analysis

Let the input distribution $P$ be the uniform distribution over $n$ points in $\mathbb{R}^d$. The outer loop is repeated $k_{\text{end}}$ times and the inner loop is repeated $t_{\text{end}}$ times, where $k_{\text{end}} = O(\log(1/\gamma))$, and $t_{\text{end}} = O(\log^2(d/\epsilon)/\gamma)$.

Inside each loop, the runtime is determined by the following: Calculating $v_{k,t}$ in 13 can be implemented in time $O(ndp_k)$ by starting with $z_{k,t}$ and repeatedly multiplying it by $B_{k,t}$ (multiplication of a vector $z$ with a second moment matrix $\sum_x xx^\top$ can be implemented in $O(nd)$ time by first calculating the inner product inside the parenthesis $\sum_x x(x^\top z)$, and then calculating the average of the resulting vectors). The power iteration estimator of Line 3 in Algorithm 2 runs in time $O(\frac{nd}{\gamma^2} \log(d \cdot k_{\text{end}} \cdot t_{\text{end}}/\gamma))$ (and is being called at most $k_{\text{end}} \cdot t_{\text{end}}$ many times). It remains to analyze the runtime of HARDTHRESHOLDINGFILTER (Algorithm 8): We have that the scores $\tau_{k,t}(x)$ that are not zeroed out by the thresholding satisfy the following upper and lower bound: $\text{poly}(\epsilon/d)\|\Sigma\|_\text{op} \|v_{k,t}\|_2^2 \leq 0.1(\hat{\sigma}_{\text{op}}/d)\|v_{k,t}\|_2^2 \leq \tau_{k,t}(x) \leq \|x\|_2^2\|v_{k,t}\|_2^2 \leq 2\hat{\sigma}_{\text{op}}(d^4/\epsilon)\|v_{k,t}\|_2^2 \leq \text{poly}(d/\epsilon)\|\Sigma\|_\text{op} \|v_{k,t}\|_2^2$, where we used the pruning of Line 5, the definition of Line 16, and the naïve estimator of Line 6. Since the HARDTHRESHOLDINGFILTER in expectation halves the maximum value of $\tau(x)$, the number of filtering steps it performs before terminating will be in expectation $O(\log(d/\epsilon))$, and thus by Markov’s inequality, the contribution to the runtime from all the $k_{\text{end}} \cdot t_{\text{end}}$ executions of HARDTHRESHOLDINGFILTER will be $O(nd \cdot k_{\text{end}} t_{\text{end}} \log(d/\epsilon))$ with high constant probability.

Therefore, the total runtime is

$$T = O\left(\frac{nd}{\gamma^2} k_{\text{end}} t_{\text{end}} \log(d \cdot k_{\text{end}} \cdot t_{\text{end}}/\gamma)\right) + O(nd \cdot k_{\text{end}} t_{\text{end}} \log(d/\epsilon)) + O\left(nd \cdot t_{\text{end}} \sum_{k=1}^{k_{\text{end}}} p_k\right)$$

$$= O\left(\frac{nd}{\gamma^2} \log(1/\gamma) \log^2(d/\epsilon) \log\left(\frac{d}{\gamma} \log(d/\epsilon)\right) + \frac{nd}{\gamma} \log(1/\gamma) \log^3(d/\epsilon) + \frac{nd}{\gamma^2} \log^2(d/\epsilon) \log(d/\gamma)\right)$$

$$= O\left(\frac{nd}{\gamma^2} \log^4(d/\epsilon)\right).$$

4 A Streaming Algorithm for Robust PCA

**Organization** In this section, we present the streaming version of our nearly-linear time robust PCA algorithm. We provide the statement in Theorem 4.1 and discuss the differences in comparison to the setting before and the adaptation needed to be made in our proof technique. This results in a set of additional deterministic conditions listed in Condition 4.2, which are established in Section 4.1 and Appendix B.

We work under the standard single-pass streaming model, where instead of having a fixed dataset, the algorithm draws samples from the distribution in an online manner. Let $G$ be an $(20\epsilon, \gamma)$-stable distribution, which will be the underlying distribution of inliers. Let the “contaminated” distribution be $P$, which is assumed to be an $\epsilon$-corrupted version of $G$ in total variation (c.f. Definition 1.4). Note that up to some small change in the constant in front of $\epsilon$, we can equivalently use a mixture representation $P = (1 - \epsilon)G + \epsilon B$ (see discussion below Definition 2.12). In contrast to the previous section, where the input dataset was stored in essentially “free” memory, now the data access model is that $n$ samples are drawn i.i.d. from $P$ and presented to the algorithm one by one (Definition 1.3).

The algorithm is still allowed to maintain a local memory, but anything stored in the local memory will be counted against its space complexity. The goal is again to find a unit vector $u$ such that $u^\top \Sigma u \geq (1 - O(\gamma))\|\Sigma\|_{\text{op}}$. 

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Theorem 4.1. Let an integer $d > 2$, and reals $0 < 20\epsilon < \gamma < \gamma_0$, for a sufficiently small $\gamma_0$. Let $G$ be $(20\epsilon, \gamma)$-stable distribution (Definition 2.12) with respect to a PSD matrix $\Sigma \in \mathbb{R}^{d \times d}$, and $r \geq 1$ be a radius such that $\Pr_{X \sim G}[\|X\|_2 > r\sqrt{d\|\Sigma\|_{op}}] \leq \epsilon$. Let $P$ be a distribution with $d_{TV}(P, G) \leq \epsilon$. There exists an algorithm that takes $\epsilon, \gamma$ as input, uses a stream of
\[ n \lesssim \left(\frac{r^2d^2}{\gamma^2} + \frac{1}{\epsilon^2}\right) \text{polylog}(d/\epsilon). \]
i.i.d. samples from $P$ (cf. Definition 1.3), uses additional memory of storing $O((d/\gamma + 1/\epsilon)\text{polylog}(d/\epsilon))$ many real numbers (or a bit complexity of $(d/\gamma^2)\text{polylog}(d/\epsilon)$ in the word RAM Model), runs for $O(\frac{d^2}{\gamma^2}\text{polylog}(d/\epsilon))$ time, and with probability at least 0.99 outputs a vector $u$ such that $u^\top \Sigma u \geq (1 - O(\gamma))\|\Sigma\|_{op}$.

The algorithm realizing the above theorem is outlined in Algorithms 6 and 7. The specialization of the above theorem for subgaussians, where $\gamma = O(\epsilon \log(1/\epsilon))$ and $r = O(\sqrt{\log(1/\epsilon)})$ yields sample complexity $O((d^2/\epsilon^5)\text{polylog}(d/\epsilon))$. Moreover, the memory usage stated in Theorem 4.1 is in terms of the number of real numbers that need to be stored in the algorithm’s memory. See Section 4.2.1 for how our algorithm can be implemented with bounded precision (i.e., in the standard word RAM model) using $(d/\gamma^2)\text{polylog}(d/\epsilon)$ registers of size $(1/\gamma)\text{polylog}(d/\epsilon)$ bits each (for a total bit complexity of $(d/\gamma^2)\text{polylog}(d/\epsilon)$).

Differences in Streaming Setting Drawing inspiration from techniques in [DKPP22], we see that the algorithm from the previous section is partly already amenable to the streaming setting: The filters used are of the form $\sum_{k=1}^{n} x_k$ and repeatedly multiplying by $M_{k,t}$. This was done by starting with $x$ and repeatedly multiplying by $B_{k,t}$. Instead of $B_{k,t}$ the new algorithm will now use an empirical moment matrix $\hat{B}_{k,t}$. As we have seen in the previous section, the multiplication of an empirical second moment matrix with a vector can be performed with $O(d)$ memory usage in a single pass over the samples and in linear time. However, since the algorithm cannot store
this matrix, it will repeatedly multiply $x$ by fresh estimators $\hat{B}_{k,t,1}, \ldots, \hat{B}_{k,t,p_k}$ each time. Thus, the result will be of the form $\hat{M}_{k,t}z$ where $\hat{M}_{k,t} = \prod_{\ell=1}^{p_k} \hat{B}_{k,t,\ell}$ (also see Algorithm 5 for more detail).

Algorithm 5 Estimator of $B_{k,t}^\ell$ from minibatches.

1: Draw a batch $S_0$ of $\tilde{n}$ samples from $P$ and let the estimate $\hat{W}_{k,t} = E_{X \sim \mathcal{U}(S_0)}[w_{k,t}(X)]$.
2: Draw $p$ batches $S_1, \ldots, S_p$ of $\tilde{n}$ samples, each from $P_{k,t}$.
3: for $\ell \in [p]$ do
4: Let $\hat{\Sigma}_{k,t,\ell} = \frac{1}{\tilde{n}} \sum_{x \in S_\ell} xx^T$.
5: Let $\hat{B}_{k,t,\ell} = \hat{W}_{k,t}^2 \hat{\Sigma}_{k,t,\ell}$.
6: end for
7: return $\hat{M}_{k,t} = \prod_{\ell=1}^{p} \hat{B}_{k,t,\ell}$.

We will additionally need two new memory-efficient estimators that work in the streaming model. First, we need an estimator for the quantiles of the underlying distribution to replace Line 15 of the old algorithm. These will be computed by empirical quantiles. Second, we also need a memory-efficient way to evaluate the stopping condition inside the HARDTHRESHOLDINGFILTER (Algorithm 8) because the stopping condition, $E_{X \sim P}[w(x)\tau(x)]$, is a population-level quantity; a similar estimator is also needed to adapt the Line 18 of our old algorithm. This completes the short informal overview.

Formally, we aggregate all the guarantees needed regarding the estimators in Condition 4.2. As a small technical note, the conditions in Condition 4.2 are only needed to hold whenever $E_{X \sim P}[w_{k,t}(X)] \geq 1 - 3 \epsilon$ and $\|B_{k,t}\|_\text{op} \geq 0.5\|\Sigma\|_\text{op}$. We can assume the first condition holds throughout the course of the algorithm because we filter out mostly outliers. The second condition holds because we also know that if $\|B_{k,t}\|_\text{op} \geq 0.5\|\Sigma\|_\text{op}$ is violated, then the potential function is small enough so that the algorithm must have already terminated (c.f. Lemma 3.8) holds with high probability thought the algorithm and dedicate Section 4.1 to establishing it.

**Condition 4.2 (Conditions for Algorithm 6).** In the context of Algorithm 6, assume that the following are true. For every $k \in [k_{\text{end}}]$ and $t \in [t_{\text{end}}]$, if $E_{X \sim P}[w_{k,t}(X)] \geq 1 - 3 \epsilon$ and $\|B_{k,t}\|_\text{op} \geq 0.5\|\Sigma\|_\text{op}$, then:

1. $\hat{g}_{k,t}(x) \geq 0.5g_{k,t}(x) - 0.01(\gamma/\epsilon)\|M_{k,t}\|_F^2\|\Sigma\|_\text{op}$.
2. $\|\hat{M}_{k,t}\|_F - \|M_{k,t}\|_F^2 \leq 0.01\|M_{k,t}\|_F$.
3. The estimator $\hat{L}_{k,t}$ of Line 16 satisfies $\Pr_{X \sim P_{k,t}}[\hat{g}_{k,t}(X) > \hat{L}_{k,t}] \in (3.99\epsilon, 4.01\epsilon)$.
4. Recall the parameter $r$ as the radius such that $\Pr_{X \sim G}[\|X\|_2 > r \sqrt{d\|\Sigma\|_\text{op}}] \leq \epsilon$. For any weight function $w : \mathbb{R}^d \rightarrow [0, 1]$, the algorithm has access to an estimator $\hat{F}$ for the quantity $F_{k,t} := E_{X \sim P}[w(X)\hat{f}_{k,t}(X)]$ that has accuracy $|\hat{F} - F_{k,t}| \leq 0.01\gamma F_{k,t} + \frac{0.1\gamma}{d}\|\hat{\Sigma}_{k,t}\|_2\|\Sigma_{k,t}\|_\text{op}$ across a total of $t_{\text{end}}k_{\text{end}} \cdot \log^5(d/\epsilon)$ calls.
5. The estimator $\hat{r}_t$ of Line 7 of Algorithm 7 satisfies $\hat{r}_t \geq (1 - \gamma)\|\Sigma_{k,t}\|_\text{op}$.
6. Every time Line 10 of of Algorithm 7 is executed, there is probability 0.9 that $u^\top\Sigma_{k,t}u/\|u\|_2^2 \geq (1 - \gamma)\|\Sigma_{k,t}\|_\text{op}$.

In particular, Item 1 of Condition 4.2 shows that Assumption 3.7 that we used in proving the main theorem of the previous section is satisfied for Algorithm 6 in our current setting. Item 2 is
Algorithm 6 ROBUSTPCA in streaming model

1: **Input:** \( P, \epsilon, \gamma. \)
2: Let \( C, C' \) be sufficiently large absolute constants with their ratio \( C/C' \) being sufficiently large.
3: Let an estimation \( R \) be such that \( \left| \Pr_{X \sim C} \left[ \|X\|_2 \geq R \right] \right| - \epsilon \leq 2\epsilon \) (note that \( R \leq r \sqrt{d\|\Sigma\|_{\text{op}}} \)).
4: Initialize \( w_{1,1}(x) = 1 \left( \|x\|_2 \leq R \right). \)
5: Let \( k_{\text{end}} := \log\left( (\log(d/\gamma)/\log(d))/\gamma \right) \), and \( t_{\text{end}} := \frac{C\log^2(d/\epsilon)}{\gamma} \).
6: Find estimator \( \hat{\sigma}_{\text{op}} \in (0.8\|\Sigma\|_{\text{op}}, 2d\|\Sigma\|_{\text{op}}). \) \( \triangleright \) c.f. Item (iv) of Lemma 2.14 with \( U = I. \)
7: for \( k = 1, \ldots, k_{\text{end}} \) do
8: \hspace{1em} Let \( p_k = 2^{k-1}p, \) where \( p = C'\log(d). \) \( \triangleright \) \( p_k \) ranges from \( C'\log(d) \) to \( C'\log(d)/\gamma. \)
9: for \( t = 1, \ldots, t_{\text{end}} \) do
10: \hspace{2em} Let \( P_{k,t} \) be the distribution of \( P \) weighted by \( w_{k,t}: P(x) w_{k,t}(x) / \mathbb{E}_{X \sim P}[w_{k,t}(X)]. \)
11: \hspace{2em} Let \( B_{k,t} := \mathbb{E}_{X \sim P}[w_{k,t}(X)XX^\top] \) and \( M_{k,t} := \hat{B}_{k,t}. \)
12: \hspace{2em} Let \( g_{k,t}(x) := \|M_{k,t}x\|_2^2. \)
13: \hspace{2em} Let \( \hat{M}_{k,t} \) be a sample-based version of \( M_{k,t} \) as defined in Algorithm 5.
14: \hspace{2em} \( \hat{v}_{k,t} \leftarrow \hat{M}_{k,t}z_{k,t}, \) where \( z_{k,t} \sim N(0, I). \)
15: \hspace{2em} Let \( \hat{f}_{k,t}(x) = (\hat{v}_{k,t}^\top x)^2. \)
16: \hspace{2em} Let \( \hat{L}_{k,t} \) be estimator for the \( 3\epsilon \)-quantile of \( \hat{f}_{k,t}(\cdot) \) under \( P_{k,t} \) such that \( \Pr_{X \sim P_{k,t}}[\hat{f}_{k,t}(X) > \hat{L}_{k,t}] - 3\epsilon \leq 0.01\epsilon, \) and then do \( \hat{L}_{k,t} \leftarrow \max \{ \hat{L}_{k,t}, \frac{0.1}{d\epsilon} \hat{\sigma}_{\text{op}} \|v_{k,t}\|_2^2 \}. \)
17: \hspace{2em} Let \( \hat{\sigma}_{k,t} = \hat{f}_{k,t}(x) \mathbb{I}(\hat{f}_{k,t}(x) > \hat{L}_{k,t}). \)
18: \hspace{2em} Call SAMPLETOPEIGENVECTOR \( (P, w_{k,t}, \epsilon, \gamma, 1/(k_{\text{end}} \cdot t_{\text{end}})). \) \( \triangleright \) c.f. Algorithm 7.
19: \hspace{2em} Let \( \hat{\sigma}_{k,t} \) such that \( \|\hat{\sigma}_{k,t} - \hat{v}_{k,t}^\top \hat{\Sigma} \hat{v}_{k,t}\| \leq 4\gamma \hat{v}_{k,t}^\top \hat{\Sigma} \hat{v}_{k,t}. \)
20: \hspace{2em} (e.g., \( \hat{\sigma}_{k,t} := \mathbb{E}_{X \sim P}[w_{k,t}(X)\hat{f}_{k,t}(X)\mathbb{I}(\hat{f}_{k,t}(X) \leq \hat{L}_{k,t})]). \) \( \triangleright \) c.f. Item (iv) of Lemma 2.14.
21: \hspace{2em} Find an estimator \( \hat{\sigma}'_{k,t} \) such that \( \|\hat{\sigma}'_{k,t} - \hat{\sigma}_{k,t}\| \leq 0.01\hat{\sigma}_{k,t} + \frac{0.1\gamma}{r\epsilon d}\|\hat{v}_{k,t}\|_2^2\|\hat{\Sigma}_{k,t}\|_{\text{op}}. \)
22: \hspace{2em} Let \( \hat{T}_{k,t} = 2.35\gamma \hat{\sigma}'_{k,t}. \)
23: \hspace{2em} \( w_{k,t+1} \leftarrow \text{HardThresholdingFilter}(P, w_{k,t}, \tau_{k,t}, \hat{T}_{k,t}, R, \frac{0.1\gamma}{r\epsilon d}\|\hat{v}_{k,t}\|_2^2\|\hat{\Sigma}_{k,t}\|_{\text{op}}), \) where \( \hat{\Sigma}_{k,t} \) is a sample-based version of \( \Sigma_{k,t}. \) \( \triangleright \) c.f. Algorithm 8.
24: end for
25: Set \( w_{k+1,0} = w_{k,t+1}. \)
26: end for
27: return FAIL.

relevant to equation Equation (16) in our previous analysis. To adapt that in our current setting, \( w_{k,t} \) will be replaced by \( \hat{v}_{k,t} \) and after we take expectation over \( \hat{v}_{k,t} \) it’s norm will become \( \|\hat{M}_{k,t}\|_F^2 \) that we can relate to \( \|M_{k,t}\|_F^2. \) Item 3 takes care of the quantile estimation. Item 4 is the estimator that we will use to evaluate the stopping condition inside HARDTHRESHOLDINGFILTER as well as adapting line 18 of the old algorithm. Items 5 and 6 are the adaptation of the power iteration guarantee when we do not have access to the target matrix \( M_{k,t} \) but instead we start with a random Gaussian vector and multiply it repetitively with fresh estimates \( \hat{\Sigma}_{k,t}. \)

4.1 Establishing Condition 4.2

In this section, we focus on showing that Condition 4.2 holds with high probability given enough samples. The proofs of Items 1, 2 and 4 closely follow [DKPP22] and are thus deferred to Appendix B. Item 3 is a basic application of Chernoff-Hoeffding bounds and is also provided in Appendix. Thus, in this section, we focus on establishing Items 5 and 6, both of which are related to the power
Algorithm 7 SampleTopEigenvector 2

1: **Input**: Distribution $P$, weights $w$, parameters $\epsilon, \gamma, \delta$.
2: Let $M := (E_{X \sim P}[w(X)XX^\top])^p$ for $p = C\log(d/\gamma) / \gamma$.
3: $\tilde{r} \leftarrow 0$.
4: **for** $j \in [\log(1/\delta)]$ **do**
5: Let $\hat{M}$ be a sample based estimator for $\Sigma_{P^w}^p$ calculated using Algorithm 5.
6: $y \leftarrow \hat{M} \cdot y$ for $g \sim \mathcal{N}(0, I)$.
7: $\tilde{r} \leftarrow \max(\tilde{r} \leftarrow, y^\top \hat{\Sigma}_{P^w} y/\|y\|^2_{2})$.
8: **end for**
9: Let $\hat{M}$ be a sample based estimator for $\Sigma_{P^w}^p$ calculated using Algorithm 5.
10: $u \leftarrow \hat{M}z$ for $z \sim \mathcal{N}(0, I)$.
11: Let $\hat{\sigma}_{u}$ be any approximation such that $|\hat{\sigma}_{u} - u^\top \Sigma u| \leq 4\gamma u^\top \Sigma u$, for example $\hat{\sigma}_{u} := E_{X \sim P}[w(X)(u^\top X)^2I((u^\top X)^2 \leq Q)]$, where $Q$ is within 0.01ε from the $3\epsilon$-quantile of $(u^\top X)^2$ under $P_w$.
12: Find estimator $\hat{\sigma}_{u}$ such that $|\hat{\sigma}_{u} - \sigma_{u}| \leq 0.01\gamma \sigma_{u} + 0.01\gamma \|u\|^2_{2}\|\Sigma\|_{op}$.
13: Let $\hat{\Sigma}_{P^w}$ be a sample-based version of $\Sigma_{P^w}$.
14: **if** $\hat{\sigma}_{u} \geq (1 - C\gamma) u^\top \hat{\Sigma}_{P^w} u$ and $\|u^\top \hat{\Sigma}_{P^w} u\|_{2} \geq (1 - \gamma)\hat{r}$ **then**
15: \textbf{return} $u/\|u\|_{2}$. **end if**

iteration method, as explained below.

For the standard power iteration method, we know that for a $d \times d$ PSD matrix $A$, $y^\top Ay/\|y\|^2_{2}$ is close to $\|A\|_{op}$ with high probability if $y = Apz$ for a large $p$ and $z \sim \mathcal{N}(0, I)$. Formally, we have that $y^\top Ay/\|y\|^2_{2} \geq (1 - \gamma)\|A\|_{op}$ for $p = C\log(d/\gamma)/\gamma$; cf. Fact 2.11. However, in our analysis, we will be able to compute only $y = A z$, where $\|A - Ap\|_{op} \leq \delta \|A\|_{op}$ for a small $\delta$; Here, $\delta$ accounts for sampling error. Despite having access to only $A$, we still want that $y^\top Ay/\|y\|^2_{2}$ is at least $(1 - O(\gamma))\|A\|_{op}$. This is obviously true for $\delta = 0$ by power iteration and we want to understand how large can $\delta$ be so that it is still true, since larger $\delta$ will lead to smaller sample complexity. We state our result in Lemma 4.3 below.

Once we have Lemma 4.3, then Items 5 and 6 will follow as described next. For Item 6 we use the lemma below with $A = B_{k,t}$ and $\hat{A} = \hat{M}_{k,t}$, for which it is guaranteed by Lemma B.1 that $\|\hat{A} - Ap\|_{op} \leq \delta \|Ap\|_{F}$ for a small $\delta$; Here, $\delta$ accounts for sampling error. Denote $\Delta := \hat{A} - Ap$, which has operator norm at most $\delta \|Ap\|_{F}$. We first analyze the random variable $U = z^\top (Ap\Delta + \Delta Ap) z$. Then

$$E[U] = \text{tr} (Ap\Delta + \Delta Ap) = 2\langle Ap, \Delta \rangle \leq 2\|Ap\|_{F}\|\Delta\|_{F} \leq 2\sqrt{d}\|Ap\|_{F}\|\Delta\|_{op} \leq 2\delta\sqrt{d}\|Ap\|_{F}^2,$$

since $\|\Delta\|_{op} \leq \delta \|Ap\|_{F}$. Since $Ap\Delta + \Delta Ap$ is symmetric, by Fact 2.9, the variance of $U$ satisfies $\text{Var}(U) \leq \|Ap\|_{F}^2 \|Ap\|_{F}^2 \|\Delta\|_{op}^2 \leq \delta^2 \|Ap\|_{F}^4$. By applying Chebyshev’s inequality, we
have that with probability at least 0.999
\[ |U| = |z^\top (A^p \Delta + \Delta A^p) z| \lesssim \delta \sqrt{d} ||A^p||_F^2. \] (38)

By a similar argument, we obtain that with probability at least 0.999,
\[ |z^\top (A^{p+1} \Delta + \Delta A^{p+1}) z| \lesssim \delta \sqrt{d} ||A||_{op} ||A^p||_F^2. \] (39)

Now let \( U' = z^\top A^{2p} z \). Then \( E[U'] = \text{tr}(A^{2p}) = ||A^p||_F^2 \). By Gaussian anticoncentration (Fact 2.9), the standard deviation of \( U' \) is at most \( O(||A^p||_F) \), which gives an upper bound on \( U' \) with high probability. Combining this with Fact 2.9, we have that with probability 0.999:
\[ z^\top A^{2p} z \asymp ||A^p||_F^2. \] (40)

Similarly, we obtain that with high probability, \( ||y||_2^2 \lesssim d \). Combining this with Equations (38) to (40), we have that with probability at least 0.99:
\[ ||y||_2^2 = z^\top \hat{A}^2 z = z^\top (A^p + \Delta)^2 z \]
\[ = z^\top A^{2p} z + z^\top (A^p \Delta + \Delta A^p) z + z^\top \Delta^2 z \]
\[ \leq z^\top A^{2p} z + C \delta \sqrt{d} ||A^p||_F^2 + C d ||\Delta||_{op}^2 \]
\[ \leq z^\top A^{2p} z + C ||A^p||_F^2 \left( \delta \sqrt{d} + d \delta^2 \right) \]
\[ \leq z^\top A^{2p} z + 2C \delta \sqrt{d} ||A^p||_F^2. \] (41)

Similarly, we obtain the following lower bound that with probability at least 0.99:
\[ ||y||_2^2 \geq z^\top A^{2p} z - C \delta \sqrt{d} ||A^p||_F^2. \] (42)

By Fact 2.11, the standard power iteration implies that with probability 0.999 we have that
\[ \frac{z^\top A^{2p+1} z}{z^\top A^{2p} z} \geq (1 - \gamma)||A||_{op}. \] (43)

At the intersection of all these events, which itself holds with probability at least 0.9 by union bound, we obtain the following lower bound on \( y^\top Ay \):
\[ y^\top Ay = z^\top \hat{A} \hat{A} \hat{A} z = z^\top (A^p + \Delta) A (A^p + \Delta) z \]
\[ = z^\top A^{2p+1} z + z^\top (A^{p+1} \Delta + \Delta A^{p+1}) z + z^\top (\Delta A \Delta) z \]
\[ \geq z^\top A^{2p+1} z - C \delta \sqrt{d} ||A||_{op} ||A^p||_F^2 \]
\[ \geq (1 - \gamma)||A||_{op} z^\top A^{2p} z - C \delta \sqrt{d} ||A||_{op} ||A^p||_F^2 \] (using Equation (39) and \( z^\top \Delta A \Delta z \geq 0 \))
\[ \geq (1 - \gamma)||A||_{op} \left( z^\top A^{2p} z - 3C \delta \sqrt{d} ||A^p||_F^2 \right) \]
\[ \geq (1 - \gamma)||A||_{op} \left( ||y||^2 - 5C \delta \sqrt{d} ||A^p||_F^2 \right). \] (using Equation (41))

Finally, dividing by \( ||y||_2^2 \) both sides yields
\[ \frac{y^\top Ay}{||y||_2^2} \geq (1 - \gamma)||A||_{op} \left( 1 - 5C \delta \sqrt{d} ||A^p||_F^2 \frac{1}{||y||_2^2} \right). \]
We will now show that $5C\delta \sqrt{d} \|A^p\|_F^2 \frac{1}{\|y\|_2^2} \lesssim \gamma$ on the intersection of the events mentioned above, which will complete the result:

$$\delta \sqrt{d} \|A^p\|_F^2 \frac{1}{\|y\|_2^2} \leq \frac{\delta \sqrt{d} \|A^p\|_F^2}{z^T A^{2p} z - 2C\delta \sqrt{d} \|A^p\|_F^2}$$

(using Equation (42))

$$\leq \frac{\|A^p\|_F^2}{\|A^p\|_F^2 - C' \delta \sqrt{d} \|A^p\|_F^2}$$

(using Equation (40))

$$= \frac{\delta \sqrt{d}}{1 - C' \delta \sqrt{d}} \lesssim \delta \sqrt{d} \lesssim \gamma.$$  

(using $\delta < c\gamma/\sqrt{d}$)

\[\square\]

4.2 Proof Sketch of Theorem 4.1

The proof of correctness of Algorithm 6 is the same as in Section 3 modulo small changes in the constants of the bounds to account for the usage of our sample-based estimators. Thus, here we only derive the sample complexity, memory, and runtime bounds.

**Sample complexity:** Consider the $(k, t)$-th iteration of Algorithm 6 (that is, the $k$-th iteration of the outer loop and $t$-th iteration of the inner loop). In order to ensure all parts of Condition 4.2 we use Lemma B.1 with $\delta = \frac{0.01}{\sqrt{d}} \min \left(\frac{\sqrt{\gamma/\epsilon}}{\tau}, \gamma\right)$ and probability of failure, $\tau$, equal to a sufficiently large polynomial of $\epsilon/d$ so that with high probability the estimators are accurate across all iterations simultaneously. Thus, each estimator $\hat{B}_{k,t,\ell}$ uses $(r^2 p_k^3 d/\delta^2) \polylog(d/\epsilon)$ samples. We use $p_k$ of these estimators in Line 13 resulting in $(r^2 p_k^3 d/\delta^2) \polylog(d/\epsilon)$ samples. Moreover, for the estimator of Line 16 we use Lemma B.4 (again with $\tau = \poly(\epsilon/d)$) which contributes another $(1/\epsilon) \polylog(d/\epsilon)$ to the sample complexity. We also call the estimator of Lemma B.3 which uses $O((r^4 d^2/\gamma^2) \polylog(d/\epsilon))$ many samples. Thus, the number of samples used during the $(k,t)$-th iteration of Algorithm 6 is $n_{k,t} := O\left(r^2 p_k^3 d/\delta^2 \polylog(d/\epsilon) + (1/\epsilon) \polylog(d/\epsilon) + (r^4 d^2/\gamma^2) \polylog(d/\epsilon)\right)$, and the total sample complexity is

$$n = \sum_{k=1}^{t_{end}} \sum_{t=1}^{t_{end}} n_{k,t} \leq t_{end} \left(\sum_{k=1}^{t_{end}} p_k^3\right) \frac{r^2 d}{\delta^2} \polylog(d/\epsilon) + \frac{t_{end} k_{end} d}{\epsilon} \polylog(d/\epsilon) + t_{end} k_{end} \frac{r^4 d^2}{\gamma^2} \polylog(d/\epsilon)$$

$$\lesssim \frac{r^2 d^2 \max(r^2, 1)}{\gamma^3} \polylog(d/\epsilon) + \frac{1}{\epsilon \gamma} \polylog(d/\epsilon) + \frac{r^4 d^2}{\gamma^3} \polylog(d/\epsilon).$$

**Memory Usage:** The estimator of Line 16 in Algorithm 6 uses memory $O((1/\epsilon) \polylog(d/\epsilon))$, and that memory can be freed and reused the next time Line 16 is executed. The filters created by HARDTHRESHOLDINGFILTER are of the form of a vector and a one dimensional threshold, meaning that they can be stored in $O(d)$ memory. The number of the filters created in each one of the $t_{end} \cdot k_{end}$ calls of HARDTHRESHOLDINGFILTER is at most $O(\log(d/\epsilon))$. Thus, the memory used to store all of them is $O(t_{end} \cdot k_{end} \cdot \log(d/\epsilon)) = O((d/\gamma) \polylog(d/\epsilon))$. Every other operation done in Algorithm 6 is either a multiplication of an empirical second moment matrix with a vector (that as we have noted earlier can be implemented in $O(d)$ memory) or even more basic operation.

**Runtime:** The same runtime analysis from Section 3.4.4 is applicable here as well.

4.2.1 Bit Complexity

We briefly discuss how our algorithm can be implemented with bounded precision (i.e., in the standard word RAM model) using $(d/\gamma) \polylog(d/\epsilon)$ registers of size $(1/\gamma) \polylog(d/\epsilon)$ bits each. We assume
that all but \( \epsilon \) mass of the inlier distribution is supported in a ball of radius \( R = (d/\epsilon)^{\text{polylog}(d/\epsilon)} \) and we also assume that \( \Sigma \succeq (\epsilon/d)^{\text{polylog}(d/\epsilon)} I \). Our algorithm ignores all points \( x \) outside of the ball and rounds the remaining ones to \( x' \) such that \( \|x - x'\|_2 \leq \eta \). We want \( \eta \) to be small enough so that it does not affect the correctness of our algorithm. Picking \( \eta = (\epsilon/d)^{\text{polylog}(d/\epsilon)} \) means that the rounded distribution has covariance matrix close enough to the original one so that it satisfies the same stability condition (modulo a difference in the constants), thus our algorithm will be correct. So far, by the aforementioned rounding, every point can be stored in \( d \) words of size \( O(\log(Rd/\eta)) = \text{polylog}(d/\epsilon) \) bits. Regarding the bit complexity of intermediate calculations of our algorithm, the most expensive one is the computation of \( \text{Mz} \) for \( M \) being the empirical covariance raised to the power \( p = \gamma^{-1}\text{polylog}(d/\epsilon) \) and \( z \) a random Gaussian vector. Since the power \( p \) includes a \( 1/\gamma \) factor, the final result may have bit complexity increased by \( 1/\gamma \) factor, thus we need \( d \) registers of size \( (1/\gamma)^{\text{polylog}(d/\epsilon)} \) to store the resulting vector \( \text{Mz} \). The filters that we create are \( (1/\gamma)^{\text{polylog}(d/\epsilon)} \) in number and the representation of each is a vector of the previous form (and a one-dimensional threshold). Thus, overall, we have to use \( (d/\gamma)^{\text{polylog}(d/\epsilon)} \) many registers of size \( (1/\gamma)^{\text{polylog}(d/\epsilon)} \) bits each.

5 Conclusion

We gave the first nearly-linear time algorithm for robust PCA that attains near-optimal error guarantees, without eigenvalue gap assumptions in the spectrum of \( \Sigma \). Additionally, we presented the first sub-quadratic space streaming algorithm for robust PCA.

In terms of future improvements, one potential avenue for optimization is the calculation of \( B^p \) for a PSD matrix \( B \) and \( p = \tilde{\Theta}(1/\epsilon) \). It is likely that these matrix-dot products can be approximated faster through the use of Chebyshev approximation [SV14, Chapter 10]. Additionally, it remains to be determined if the runtime of our algorithm improves in the presence of a gap among the large eigenvalues of the true covariance matrix. Lastly, as highlighted by [DKPP22], designing streaming algorithms with optimal sample complexities and robustness to strong contamination is an open problem.
References


Appendix

Organization  The Appendix is organized as follows: Appendix A includes the additional standard results (and their proofs) that were omitted from the main paper. Finally, Appendix B focuses on the streaming algorithm achieving Theorem 1.5.

A Preliminaries: Omitted Facts

A.1 Information-theoretic Error

Let us briefly outline why the best approximate guarantee for subgaussian distributions is of the order $(1 - \tilde{\Theta}(\epsilon))$. We will do so by focusing on Gaussian distributions and establishing a lower bound of $(1 - \Omega(\epsilon))$. Let $P$ be the standard Gaussian distribution in $\mathbb{R}^d$. For a unit vector $v$ and $\epsilon \in (0, 0.1)$, let $Q$ be the distribution $\mathcal{N}(0, I + \epsilon vv^\top)$. Using simple calculations given below, it can be seen that $d_{TV}(P, Q) \leq \epsilon/2$: By Pinsker’s inequality and the expression for KL divergence between two Gaussians, we obtain the following series of inequalities

$$d_{TV}(Q, P) \leq \sqrt{d_{KL}(Q||P)} \quad \text{(Using Pinsker’s inequality)}$$
$$= \sqrt{\frac{1}{2} (-\log |\Sigma_Q| + \text{tr}(\Sigma_Q) - d)} \quad \text{(KL divergence between two Gaussians)}$$
$$= \sqrt{\frac{1}{2} (-\log(1 + \epsilon) + \epsilon)} \quad \text{(Using $\Sigma_Q = I + \epsilon vv^\top$)}$$
$$\leq \epsilon/2. \quad \text{(since $\log(1 + x) \geq x - x^2/2$ for $x \geq 0$)}$$

Let $\hat{v} \in \mathbb{R}^d$ be the output of any robust PCA algorithm, when given i.i.d. samples from $P$ as input. The true variance along the direction $\hat{v}$ under the distribution $Q$ is $1 + \epsilon(\hat{v}^\top v)^2$. In particular, the approximation factor (on the distribution $Q$) obtained by the algorithm outputting $\hat{v}$ is

$$\frac{\hat{v}^\top \Sigma_Q \hat{v}}{\|\Sigma_Q\|_{op}} = \frac{1 + \epsilon(\hat{v}^\top v)^2}{1 + \epsilon} \leq (1 - 0.5\epsilon)(1 + \epsilon(\hat{v}^\top v)^2).$$

Since $P$ and $Q$ have total variation distance at most $\epsilon/2$, an adversary, that is allowed to corrupt an $\epsilon$-fraction of samples, can simulate i.i.d. samples of $P$ given i.i.d. samples from $Q$, and vice versa. Now, suppose the true distribution was $Q$, but the adversary gives i.i.d. samples from $P$ (which is a valid adversary as mentioned above). Since $P$ contains no information about $v$ whatsoever, no algorithm can identify $v$ from the set of $\epsilon$-corrupted samples of $Q$. As $v$ could be an arbitrary unit vector, no algorithm can output a $\hat{v}$ such that $|v^\top \hat{v}| \leq 0.01$ for $d$ larger than a constant. As a result, it is not possible to obtain an approximation better than $(1 - 0.1\epsilon)$ for Gaussians because $(1 - 0.5\epsilon)(1 + 0.1\epsilon) \leq (1 - 0.1\epsilon)$.

A.2 Comparison Between the Two Stopping Conditions

Consider the setting in Section 3.1. Let $w : \mathbb{R}^d \to \{0, 1\}$ be weights such that probability of inliers is at least $1 - 3\epsilon$. Let $r = E_{X \sim P}[w_t(X)]$. Recall that $M = (E_{X \sim P}[w_t(X)XX^\top]) = (E_{X \sim P}[w_t(X)]\Sigma_t)^p = r^p\Sigma_t^p$.

Our algorithm stops when $\langle \Sigma_t, M^2 \rangle \leq (1 + C\gamma)\langle \Sigma, M^2 \rangle$. We will now show that this stopping condition implies the stopping condition that depends solely on the Schatten norms for large $p$. Using the definition of the Schatten norm, we begin as follows:

$$\|\Sigma_t\|_{2p+1}^{2p+1} = \langle \Sigma_t, \Sigma_t^{2p} \rangle = r^{-2p}\langle \Sigma_t, M^2 \rangle$$
where we use that 
\[ p \gg \log(d)/\gamma \] and thus \[ d^{1/(2p+1)} \leq (1 + C\gamma) \]. Rearranging it, we obtain that

\[ \|\Sigma_t\|_{2p+1} \leq (1 + 2C\gamma)\|\Sigma\|_{\text{op}}, \]

and thus \[ \|\Sigma_t\|_{2p+1}^{2p+1} \leq (1 + 2C\gamma)^{2p+1}\|\Sigma\|_{\text{op}}^{2p+1} \], which is the corresponding stopping condition that depends solely on the Schatten norms of \( \Sigma_t \) and \( \Sigma \). Thus, the stopping condition of Lemma 3.3 is stronger.

### A.3 Omitted Proofs Regarding Stability: Proof of Lemma 2.14

In this section we restate and prove Lemma 2.14. We have not optimized the constants (up to the choice of constants \( C \)).

**Lemma 2.14.** In Setting 2.13, define the following: (i) \( g(x) := \|UX\|_2^2 \) for some \( U \in \mathbb{R}^{m \times d} \), (ii) \( L \) be the top 3-def. quantile of \( g(x) \) under \( P_w \) (\( P \) re-weighted by \( w \)) and \( S := \{ x : x \leq L \} \), (iii) \( \tilde{\sigma} := E_{X \sim \rho}[w(X)g(X)1\{g(X) \leq L\}] \). Then,

(i) \[ \left| E_{X \sim G}[w(X)g(X)] - \langle UTU, \Sigma \rangle \right| \leq 2\gamma \langle UTU, \Sigma \rangle, \]

(ii) \[ E_{X \sim G}[w(X)g(X)1\{X \in S\}] \leq 2.35\gamma \langle UTU, \Sigma \rangle, \]

(iii) \[ L \leq 1.65(\gamma/\epsilon)\langle UTU, \Sigma \rangle, \]

(iv) \[ |\tilde{\sigma} - \langle UTU, \Sigma \rangle| \leq 4\gamma \langle UTU, \Sigma \rangle, \]

(v) \[ \Pr_{X \sim G}[\|X\|_2^2 > 2(d/\epsilon)\|\Sigma\|_{\text{op}}] \leq \epsilon. \]

We prove each part of the lemma separately. For the first one, we have the following, which holds under the slightly stronger condition \( E_{X \sim G}[w(X)] \geq 1 - 7\epsilon \) (this is because this version will be needed later on).

**Lemma A.1.** Let \( 0 < 20\epsilon \leq \gamma < 1 \). Let \( G \) be a \((20\epsilon, \gamma)\)-stable distribution with respect to \( \Sigma \). Let a matrix \( U \in \mathbb{R}^{m \times d} \) and a function \( w : \mathbb{R}^d \rightarrow [0, 1] \) with \( E_{X \sim G}[w(X)] \geq 1 - 7\epsilon \). For the function \( g(x) = \|UX\|_2^2 \), we have that

\[ (1 - 1.35\gamma)\langle UTU, \Sigma \rangle \leq E_{X \sim G}[w(X)g(X)] \leq (1 + \gamma)\langle UTU, \Sigma \rangle. \]

**Proof.** For the upper bound,

\[ E_{X \sim G}[w(X)g(X)] = \langle UTU, E_{X \sim G}[w(X)XX^\top] \rangle \leq (1 + \gamma)E_{X \sim G}[w(X)] \langle UTU, \Sigma \rangle \leq (1 + \gamma)\langle UTU, \Sigma \rangle, \]

where the first step rewrites it in the notation of trace inner product, the second step uses the stability condition and Fact 2.2 with \( A = UTU, B = E_{X \sim G}[w(X)XX^\top], C = E_{X \sim G}[w(X)](1 + \gamma)\Sigma, \) and
the last step uses $w(x) \leq 1$. The other direction is similar, with the only difference being that we lower bound $\mathbb{E}_{X \sim G}[w(X)] \geq 1 - 7\epsilon$:

$$\mathbb{E}_{X \sim G}[w(X)g(X)] \geq (1 - \gamma)(1 - 7\epsilon)\left< U^\top U, \Sigma \right> \geq (1 - \gamma - 7\epsilon)\left< U^\top U, \Sigma \right> \geq (1 - 1.35\gamma)\left< U^\top U, \Sigma \right>,$$

where we also used that $\epsilon \leq \gamma/20$.

We will need the following intermediate result:

**Lemma A.2.** Let $0 < 20\epsilon \leq \gamma < 1$. Let $G$ be a $(20\epsilon, \gamma)$-stable distribution with respect to $\Sigma$. Let a matrix $U \in \mathbb{R}^{m \times d}$, a function $w : \mathbb{R}^d \rightarrow [0, 1]$ with $\mathbb{E}_{X \sim G}[w(X)] \geq 1 - 3\epsilon$, and a set $S$ such that $\mathbb{E}_{X \sim G}[w(X)1\{X \in S\}] \leq 4\epsilon$. Then, for the function $g(x) := \|Ux\|_2^2$, we have the following:

$$\mathbb{E}_{X \sim G}[w(X)g(X)1\{X \in S\}] \leq 2.35\gamma\left< U^\top U, \Sigma \right>.$$

**Proof.** Let $w'(x) := w(x)1\{X \not\in S\} = w(x) - w(x)1\{X \in S\}$, and note that $\mathbb{E}_{X \sim G}[w'(X)] = \mathbb{E}_{X \sim G}[w(X)] - \mathbb{E}_{X \sim G}[w(X)1\{X \in S\}] \geq 1 - 7\epsilon$ by our assumptions. Thus, we have that

$$\mathbb{E}_{X \sim G}[w(X)g(X)1\{X \in S\}] = \mathbb{E}_{X \sim G}[w(X)g(X)] - \mathbb{E}_{X \sim G}[w'(X)g(X)]
\leq (1 + \gamma)\left< U^\top U, \Sigma \right> - (1 - 1.35\gamma)\left< U^\top U, \Sigma \right>
= 2.35\gamma\left< U^\top U, \Sigma \right>,$$

where the second line is the application of **Lemma A.1** for $w'(x)$.

Now, Item (ii) of **Lemma 2.14** follows directly as shown below.

**Corollary A.3.** In the setting of **Lemma A.2**, let a mixture distribution $P = (1 - \epsilon)G + \epsilon B$ and $S = \{x : x > L\}$, where $L$ is defined to be the top $3\epsilon$-quantile of $g(x)$ under $P_w$ (the weighted version of $P$). Then,

$$\mathbb{E}_{X \sim G}[w(X)g(X)1\{X \in S\}] \leq 2.35\gamma\left< U^\top U, \Sigma \right>.$$

**Proof.** In order to apply **Lemma A.2**, we only have to verify that $\mathbb{E}_{X \sim G}[w(X)1\{X \in S\}] \leq 4\epsilon$. Since $P = (1 - \epsilon)G + \epsilon B$ and $L$ is by definition the $3\epsilon$-quantile of $g(x)$ under $P_w$, and $\epsilon < 1/10$, we have that

$$\mathbb{E}_{X \sim G}[w(X)1\{g(X) > L\}] \leq \frac{1}{1 - \epsilon}\mathbb{E}_{X \sim P}[w(X)1\{g(X) > L\}]
\leq \frac{1}{1 - \epsilon}\mathbb{P}_{X \sim P_w}[g(X) > L]
\leq \frac{3\epsilon}{1 - \epsilon} < 4\epsilon. \quad (44)$$

An application of **Lemma A.2** completes the proof.

We show Item (iii) of **Lemma 2.14** below.

**Lemma A.4.** Let $0 < 20\epsilon \leq \gamma < 1$. Let a mixture distribution $P = (1 - \epsilon)G + \epsilon B$ on $\mathbb{R}^d$, where $G$ is $(20\epsilon, \gamma)$-stable distribution with respect to a matrix $\Sigma$. Let $U \in \mathbb{R}^{m \times d}$ and a function $w : \mathbb{R}^d \rightarrow [0, 1]$ with $\mathbb{E}_{X \sim G}[w(X)] \geq 1 - 3\epsilon$. Define $g(x) := \|Ux\|_2^2$ and let $L$ be the top $3\epsilon$-quantile of $g(x)$ under $P_w$. Then, $L \leq 1.65(\gamma/\epsilon)(U^\top U, \Sigma)$.
Proof. Let $w'(x) := w(x)I(g(x) > L)$. Note that

$$E_{X \sim G}[w'(X)] = \frac{E_{X \sim P}[w(X)I(g(X) > L)] - \epsilon E_{X \sim B}[w(X)I(g(X) > L)]}{1 - \epsilon}$$

$$\geq \frac{E_{X \sim P}[w(X)I(g(X) > L)] - \epsilon}{1 - \epsilon}$$

$$= \frac{E_{X \sim P}[w(X)] Pr_{X \sim P}(g(X) > L) - \epsilon}{1 - \epsilon}$$

$$\geq \frac{(1 - \epsilon)(1 - 3\epsilon)3\epsilon - \epsilon}{1 - \epsilon} \geq 1.43\epsilon,$$

where the second step uses that $w(x) \leq 1$, the fourth step uses that $E_{X \sim P}[w(X)] \geq (1 - \epsilon)E_{X \sim P}[w(X)] \geq (1 - \epsilon)(1 - 3\epsilon)$ and that $L$ is the $3\epsilon$-quantile of $g$ under $P_w$, and the last step used that $\epsilon < 1/20$. Using this lower bound on $E_{X \sim G}[w'(X)]$ with Corollary A.3, we have that

$$E_{X \sim G_w'}[g(X)] = \frac{E_{X \sim G}[w'(X)g(X)]}{E_{X \sim G}[w'(X)]} \leq \frac{1 - \epsilon}{1.43\epsilon} E_{X \sim G}[w(X)g(X)I(g(X) > L)] \leq \frac{1.65\gamma}{\epsilon} \left\langle U^T U, \Sigma \right\rangle.$$

Since the minimum is less than the average, the result above implies that there exists a point in the support of $G_w'$ which is smaller than $1.65(\gamma/\epsilon)\left\langle U^T U, \Sigma \right\rangle$. Since $G_w'$ is supported on points bigger than $L$, it means that $L \leq 1.65(\gamma/\epsilon)\left\langle U^T U, \Sigma \right\rangle$. \hfill \Box

We now provide the proof of Item (iv) of Lemma 2.14 below.

Lemma A.5 (Variance estimator). Let $0 < 20\epsilon \leq \gamma < 1$. Let a mixture distribution $P = (1-\epsilon)G + \epsilon B$ on $\mathbb{R}^d$, where $G$ is $(20\epsilon, \gamma)$-stable distribution with respect to a matrix $\Sigma$. Let $U \in \mathbb{R}^{m \times d}$ and a function $w : \mathbb{R}^d \to [0, 1]$ with $E_{X \sim G}[w(X)] \geq 1 - 3\epsilon$. Define $g(x) := \|Ux\|^2_2$ and let $L$ be the top $3\epsilon$-quantile of $g(x)$ under $P_w$. For the estimator $\hat{\sigma} := E_{X \sim P}[w(X)g(X)1(g(X) \leq L)]$, we have that

$$\left| \hat{\sigma} - \left\langle U^T U, \Sigma \right\rangle \right| \leq 4\gamma \left\langle U^T U, \Sigma \right\rangle.$$

Proof. We consider the contribution to $\hat{\sigma}$ from the two components of the distribution $P$ separately. For the component $G$, we first note that $E_{X \sim G}[w(X)1(g(X) \leq L)] \geq E_{X \sim G}[w(X)] - E_{X \sim G}[w(X)1(g(X) > L)] \geq 1 - 3\epsilon - 4\epsilon = 1 - 7\epsilon$ (the last step can be shown as in Equation (44)). Thus, by Lemma A.1, we have that

$$E_{X \sim G}[w(X)g(X)1(g(X) \leq L)] - \left\langle U^T U, \Sigma \right\rangle \leq 1.35\gamma \left\langle U^T U, \Sigma \right\rangle. \quad (45)$$

We now consider the contribution to $\hat{\sigma}$ from $B$. We first note that $L \leq 1.65(\gamma/\epsilon)\left\langle U^T U, \Sigma \right\rangle$ by Lemma A.4. Therefore,

$$\left| E_{X \sim B}[w(X)g(X)1(g(X) \leq L)] - \left\langle U^T U, \Sigma \right\rangle \right| \leq L + \left\langle U^T U, \Sigma \right\rangle \leq \left( \frac{1.65\gamma}{\epsilon} + 1 \right) \left\langle U^T U, \Sigma \right\rangle. \quad (46)$$

Finally, combining Equations (45) and (46) and using that $\epsilon \leq \gamma$ yields:

$$\left| E_{X \sim G}[w(X)g(X)1(g(X) \leq L)] - \left\langle U^T U, \Sigma \right\rangle \right| \leq (1 - \epsilon) \left| E_{X \sim G}[w(X)g(X)1(g(X) \leq L)] - \left\langle U^T U, \Sigma \right\rangle \right| + \epsilon \left| E_{X \sim B}[w(X)g(X)1(g(X) \leq L)] - \left\langle U^T U, \Sigma \right\rangle \right|$$
Then, \( \Pr \left[ \| X \|^2 > 2(d/\epsilon) \| \Sigma \|_{\text{op}} \right] \leq \epsilon. \)

We conclude with the last part of Lemma 2.14.

Claim A.6. Let \( 0 < 20\epsilon \leq \gamma < 1. \) Let \( G \) be a \((20\epsilon, \gamma)\)-stable distribution with respect to a matrix \( \Sigma. \) Then, \( \Pr_{X \sim G} \left[ \| X \|^2 > 2(d/\epsilon) \| \Sigma \|_{\text{op}} \right] \leq \epsilon. \)

Proof. Using Markov’s inequality and Lemma A.1 with \( U = I \)

\[
\Pr_{X \sim G} \left[ \| X \|^2 > 2(d/\epsilon) \| \Sigma \|_{\text{op}} \right] \leq \frac{\mathbb{E}_{X \sim G} \| X \|^2 \Sigma_{\text{op}}}{2(d/\epsilon) \| \Sigma \|_{\text{op}}} \leq \frac{(1 + \gamma) \text{tr}(\Sigma)}{2(d/\epsilon) \| \Sigma \|_{\text{op}}} \leq \epsilon.
\]

We finally restate and prove Lemma 2.15 from Section 2.2.

Lemma A.7. Let \( 0 < 20\epsilon \leq \gamma < \gamma_0 \) for a sufficiently small absolute constant \( \gamma_0. \) Let \( P = (1 - \epsilon)G + \epsilon B, \) where \( G \) is a \((20\epsilon, \gamma)\)-stable distribution with respect to \( \Sigma. \) Let \( w : \mathbb{R}^d \to [0, 1] \) with \( \mathbb{E}_{X \sim G}[w(X)] \geq 1 - 3\epsilon, \) and recall that in our notation \( \Sigma_{P_w} := \mathbb{E}_{X \sim P_w}[X X^\top] = \mathbb{E}_{X \sim P}[w(X) X X^\top] / \mathbb{E}_{X \sim P}[w(X)]. \) If \( u \) is a vector such that \( u^\top \Sigma_{P_w} u / \| u \|^2 \geq (1 - \gamma) \| \Sigma_{P_w} \|_{\text{op}} \) and \( u^\top \Sigma u \geq (1 - O(\gamma)) u^\top \Sigma_{P_w} u, \) then \( u^\top \Sigma u / \| u \|^2 \geq (1 - O(\gamma)) \| \Sigma \|_{\text{op}}. \)

Proof. Since \( \mathbb{E}_{X \sim G}[w(X)] \geq 1 - 3\epsilon, \) we can use the stability condition as follows: Let \( a \) be the (normalized) top eigenvector of \( \Sigma_{P_w} \) and \( b \) be the (normalized) top eigenvector of \( \Sigma. \) Using the definition of the operator norm and stability, we first obtain the following lower bound on \( \| \Sigma_{P_w} \|_{\text{op}} \) in terms of \( \| \Sigma \|_{\text{op}}: \)

\[
\| \Sigma_{P_w} \|_{\text{op}} = a^\top \Sigma_{P_w} a \geq b^\top \Sigma_{P_w} b = b^\top \left( \frac{\mathbb{E}_{X \sim P}[w(X) X X^\top]}{\mathbb{E}_{X \sim P}[w(X)]} \right) b \geq (1 - \epsilon) b^\top \left( \frac{\mathbb{E}_{X \sim G}[w(X) X X^\top]}{\mathbb{E}_{X \sim P}[w(X)]} \right) b \geq (1 - \gamma) \| \Sigma \|_{\text{op}}, \tag{47}
\]

where the penultimate step used that \( \epsilon < \gamma/20. \) Combining this with the assumption that \( u^\top \Sigma u \) is large compared to \( u^\top \Sigma_{P_w} u, \) we obtain the following:

\[
u^\top \Sigma u / \| u \|^2 \geq (1 - O(\gamma)) u^\top \Sigma_{P_w} u / \| u \|^2 \geq (1 - O(\gamma))(1 - \gamma) \| \Sigma_{P_w} \|_{\text{op}} \geq (1 - O(\gamma))(1 - \gamma)(1 - 2\gamma) \| \Sigma \|_{\text{op}} = (1 - O(\gamma)) \| \Sigma \|_{\text{op}}. \tag{48}\]

A.4 Filtering

In this subsection, we prove the filter guarantees that were described in Section 2.3. We use a slightly more general version of the filter so that it can be employed later on when we will develop a streaming algorithm. The difference is the introduction of an additional error parameter \( \delta \) in the filter. This will be useful in Appendix B, where the quantity \( \mathbb{E}_{X \sim P}[w(x) \tau(x)] \) as well as \( \hat{T} \) will have to be estimated by averaging samples, thus \( \delta \) will then account for a small additive error in that approximation. For the non-streaming algorithm, we will use \( \delta = 0 \)
Algorithm 8 HARDTHRESHOLDINGFILTER

1: **Input:** Distribution \( P = (1 - \epsilon)G + \epsilon B \), weights \( w \), scores \( \tau \), parameters \( \tilde{T}, R, \delta \).
2: \( w_0(x) \leftarrow w(x) \), and \( r_0 \leftarrow R, \ell \leftarrow 1 \).
3: while \( E_{X \sim P}[w(X)\tau(X)] > \frac{5}{2}(\tilde{T} + \delta) \)
   4: Draw \( r_\ell \sim U([0, r_{\ell-1}]) \).
5: \( w_{\ell+1}(x) \leftarrow w_\ell(x) \cdot I(\tau(x) > r_\ell) \).
6: \( \ell \leftarrow \ell + 1 \).
7: **return** \( w_\ell(x) \).

Lemma A.8 (Guarantees of Filtering). Consider one call of HARDTHRESHOLDINGFILTER\((P, w, \tau, \tilde{T}, R, \delta)\). Suppose there exists \( T \) such that \((1 - \epsilon)E_{X \sim G}[w(X)\tau(X)] < T \) and \( \tilde{T} \) such that \( |\tilde{T} - T| < T/5 + \delta \). Denote by \( F \) the randomness of the filter (i.e., the collection of the random thresholds \( r_1, r_2 \ldots \) used). Let \( w' \) be the weight function returned. Then,

1. \( E_{X \sim P}[w'(X)\tau(X)] \leq 3T + 5d \) with probability 1 over the randomness of the filter \( F \).

2. \( E_F[\epsilon E_{X \sim B}[w(X) - w'(X)] > (1 - \epsilon)E_{X \sim G}[w(X) - w'(X)] \).

Proof. The first part of the lemma follows by the termination condition \( E_{X \sim P}[w'(X)\tau(X)] \leq (5/2)(\tilde{T} + \delta) \) and the assumption \(|\tilde{T} - T| < T/5 + \delta \).

We show that the second desideratum holds for every filtering round, conditioned on the past rounds. Consider the \( \ell \)-th round of filtering. The probability that a point gets removed (over the random selection of \( r_\ell \)) is \( \tau(x)/r_{\ell-1} \). As a result, for any point \( x \) that has not been filtered, the probability that a point that gets removed is equal to \( E[((w_\ell(x) - w_{\ell+1}(x))] = \tau(x)/r_{\ell-1} \). Also, note that since the weights are binary decreasing functions, \( w_\ell(x) - w_{\ell+1}(x) = w_\ell(x)(w_\ell(x) - w_{\ell+1}(x)) \).

For the inliers, we have that

\[
E_{r_\ell}[(1 - \epsilon)E_{X \sim G}[w_\ell(X) - w_{\ell+1}(X)]] = E_{r_\ell}[(1 - \epsilon)E_{X \sim G}[w_\ell(X)(w_\ell(X) - w_{\ell+1}(X))]]
\]

\[
= (1 - \epsilon)E_{X \sim G}[w_\ell(X)E_{r_\ell}[w_\ell(X) - w_{\ell+1}(X)]]
\]

\[
= \frac{1 - \epsilon}{r_{\ell-1}} E_{X \sim G}[w_\ell(X)\tau(X)] < \frac{T}{r_{\ell-1}}.
\]

We now turn to the outliers. If the filtering hasn’t stopped, it means that the stopping condition of the while loop is still violated and thus \( E_{X \sim P}[w_\ell(X)\tau(X)] > (5/2)(\tilde{T} + \delta) > 2T \), (since we have assumed \(|\tilde{T} - T| < T/5 + \delta \)). Thus,

\[
E_{r_\ell}[\epsilon E_{X \sim B}[w_\ell(X) - w_{\ell+1}(X)]] = E_{r_\ell}[\epsilon E_{X \sim B}[w_\ell(X)(w_\ell(X) - w_{\ell+1}(X))]]
\]

\[
= E_{X \sim P}[w_\ell(X)E_{r_\ell}[w_\ell(X) - w_{\ell+1}(X)]]
\]

\[
- (1 - \epsilon)E_{X \sim G}[w_\ell(X)E_{r_\ell}[w_\ell(X) - w_{\ell+1}(X)]]
\]

\[
= \frac{1}{r_{\ell-1}} E_{X \sim P}[w_\ell(X)\tau(X)] - \frac{1 - \epsilon}{r_{\ell-1}} E_{X \sim G}[w_\ell(X)\tau(X)]
\]

\[
> \frac{2T}{r_{\ell-1}} - \frac{T}{r_{\ell-1}} = \frac{T}{r_{\ell-1}}.
\]
We now provide the guarantees of the filtering algorithm over the course of the entire execution of RobustPCA.

Lemma A.9. Let $0 < 20\epsilon \leq \gamma < 1/20$. Let $P = (1 - \epsilon)G + \epsilon B$, where $G$ is a $(20\epsilon, \gamma)$-stable distribution with respect to $\Sigma$. Consider an execution of RobustPCA. With probability at least 0.2 over the randomness of the algorithm, we have that $(1 - \epsilon)E_{X \sim G}[1 - w_{k,t}(X)] + \epsilon E_{X \sim B}[w_{k,t}(X)] \leq 2.9\epsilon$.

Proof. For notational simplicity, first define an ordering over the multi-indices \{(k, t) : k \in \{1, \ldots, k_{\text{end}}\}, t \in \{1, \ldots, t_{\text{end}}\}\} in the natural way: $(k', t') \leq (k, t)$ is defined to mean that either $k' \leq k$ or $k' = k$ and $t' \leq t$. Also define a successor operator $s(\cdot)$ that returns the next multi-index according to the way the two loops are structured in RobustPCA, that is, $s(k, t) = (k + 1, t)$ if $t < t_{\text{end}}$ and $s(k, t) = (k + 1, 0)$ otherwise.

Let $(k^*, t^*)$ be the first time that $(1 - \epsilon)E_{X \sim G}[1 - w_{k,t}(X)] + \epsilon E_{X \sim B}[w_{k,t}(X)] \geq 2.9\epsilon$ (if there exists one, otherwise set $(k^*, t^*) = (k_{\text{end}}, t_{\text{end}})$). Define the sequence

$$\Delta_{(k, t)} = (1 - \epsilon)E_{X \sim G}[1 - w_{\min_{(k,t)}(k^*, t^*)}(X)] + \epsilon E_{X \sim B}[w_{\min_{(k,t)}(k^*, t^*)}(X)]$$.

We note that $\Delta_{(k, t)}$ is a supermartingale with respect to the randomness of the filter: If $(k, t) < (k^*, t^*)$, then $E[\Delta_{s(k, t)} | F_{(k, t)}] \leq \Delta_{(k, t)}$, where $F_{(k, t)}$ denotes the randomness of the filters applied so far. This follows by Lemma A.8, as long as the lemma is applicable: We apply that lemma with $T = 2.35\gamma v_{k,t}^\top \Sigma v_{k,t}$ and $\bar{T} = 2.35\gamma \tilde{\sigma}_{k,t}$. Its first requirement that $(1 - \epsilon)E_{X \sim G}[w_{k,t}(x)] < T$ is satisfied by Item (ii) of Lemma 2.14 (specifically, the special case that uses $\gamma_{\text{iter}} = 2$). Its second requirement $|\bar{T} - T| < T/5$ we have that $|\bar{T} - T| = 2.35\gamma \tilde{\sigma}_{k,t} - v_{k,t}^\top \Sigma v_{k,t} \leq 2.35\gamma \cdot 4\gamma \cdot v_{k,t}^\top \Sigma v_{k,t} \leq (1/5) \cdot 2.35\gamma v_{k,t}^\top \Sigma v_{k,t} = T/5$, where the second step uses Item (iv) of Lemma 2.14 (with $U = v_{k,t}$) and the last inequality uses that $\gamma < 1/20$. All these intermediate lemmata that we just referred to are applicable when $(1 - \epsilon)E_{X \sim G}[1 - w_{k,t}(X)] + \epsilon E_{X \sim B}[w_{k,t}(X)] \leq 2.9\epsilon$.

Otherwise, if $(k, t) \geq (k^*, t^*)$ we still have $E[\Delta_{s(k, t)} | F_{(k, t)}] \leq \Delta_{(k, t)}$, since by definition the sequence $\Delta_{(k, t)}$ remains unchanged.

In the beginning, we have that $\Delta_{(1, 1)} \leq 2\epsilon$. This is because we start with $\epsilon$-fraction of outliers and the naïve pruning of line 5 of the algorithm cannot remove more than $\epsilon$ mass from the distribution $G$, as shown in Item (v) of Lemma 2.14.

By Doob’s inequality for martingales, we have that

$$\Pr \left[ \max_{(1,1) \leq (k,t) \leq (k_{\text{end}}, t_{\text{end}})} \Delta_{(k, t)} > 2.5\epsilon \right] \leq \frac{\Delta_{(1,1)}}{2.5\epsilon} \leq 0.8 \ . \ (51)$$

This implies that with probability 0.2, it holds $(1 - \epsilon)E_{X \sim G}[1 - w_{k,t}(X)] + \epsilon E_{X \sim B}[w_{k,t}(X)] \leq 2.5\epsilon$ throughout the algorithm’s iterations (which is already a bit stronger than the conclusion of the lemma that wanted to show). To see that, assume the contrary, and define $(\tilde{k}, \tilde{t})$ to be the first time that $(1 - \epsilon)E_{X \sim G}[1 - w_{k,t}(X)] + \epsilon E_{X \sim B}[w_{k,t}(X)] > 2.5\epsilon$. Combining that with the trivial observation that $\Delta_{(\tilde{k}, \tilde{t})} \leq (k^*, t^*)$ yields that $\Delta_{(\tilde{k}, \tilde{t})} > 2.5\epsilon$, which by Equation (51) cannot happen with probability more than 0.2.

\[\square\]

**B Omitted Proofs from Section 4**

We now provide details that were omitted from Section 4. In particular, we will show that the deterministic conditions in Condition 4.2 hold with high probability. Our focus in this section will
be on Items 1 to 4 of Condition 4.2 since we have already shown Items 5 and 6 in Section 4.1. The proofs of Items 1, 2 and 4 will closely follow [DKPP22], while Item 3 is a standard implication of Chernoff-Hoeffding bounds.

B.1 Items 1 and 2

Items 1 and 2 of Condition 4.2 rely on the fact that the empirical covariances used in the estimator \( \prod_{\ell=1}^{\ell_p} \hat{B}_{k,t,\ell} \) are close enough to \( B_{k,t} \), as in Lemma B.1 below. We first prove the lemma and then show how the two conditions follow.

Lemma B.1. Let \( \delta < 1 \). Consider Algorithm 5 and assume that \( P_{k,t} \) is supported on an \( \ell_2 \)-ball of radius \( r \sqrt{d} \|\Sigma \|_{op} \). If \( \hat{W}_{k,t} \) and every \( \hat{B}_{k,t,\ell} \) in Algorithm 5 are calculated using \( \tilde{n} > C \frac{r^2 d p^2}{\delta^2} \log \left( \frac{d}{\tau} \right) \) samples, where \( C \) is a sufficiently large constant, then with probability at least \( 1 - \tau \), we have that

\[
\left\| \hat{M}_{k,t} - M_{k,t} \right\|_2 \leq \delta \left\| M_{k,t} \right\|_{op},
\]

Proof. Let \( \| \hat{B}_{k,t,\ell} - B_{k,t,\ell} \|_{op} \leq \delta_1 \| B_{k,t,\ell} \|_{op} \) and \( \| \hat{W}_{k,t} - E_{X \sim P}[w_{k,t}(X)] \| \leq \delta_2 \), for \( \delta_1 \), \( \delta_2 \) to be determined. The accuracy of the estimators \( \hat{B}_{k,t,\ell} \) and that of \( \hat{M}_{k,t} \) are related as follows:

Lemma B.2 (Lemma 4.14 in [DKPP22]). Let \( A, B, B_1, \ldots, B_p \) be symmetric \( d \times d \) matrices and define \( M = B_p, M_S = \prod_{i=1}^{p} B_i \). If \( \| B_i - B \|_2 \leq \delta_1 \| B \|_2 \), then \( \| M_S - B_p \|_{op} \leq \rho \delta_1 (1 + \delta_1) \rho \| B \|_{op} \).

By Lemma B.2, it suffices to ensure \( \rho \delta_1 e^{\rho \delta_1} < \delta \). For that, it suffices to use \( \delta_1 := \delta / (3p) \). In the reminder we focus on ensuring \( \| \hat{B}_{k,t,\ell} - B_{k,t,\ell} \|_{op} \leq \delta_1 \| B_{k,t,\ell} \|_{op} \) for that choice of \( \delta_1 \).

Let \( \| \hat{\Sigma}_{k,t,\ell} - \Sigma_{k,t,\ell} \|_{op} \leq \delta_3 \| \Sigma_{k,t,\ell} \|_{op} \) for \( \delta_3 \) to be specified. We have that

\[
\| \hat{B}_{k,t,\ell} - B_{k,t,\ell} \|_{op} = \| \hat{W} \hat{\Sigma}_{k,t,\ell} - E_{X \sim P}[w_{k,t}(X)] \Sigma_{k,t} \|_{op} \leq \| \hat{W} \hat{\Sigma}_{k,t,\ell} - \Sigma_{k,t,\ell} \|_{op} + \| \hat{W}_{k,t} - E_{X \sim P}[w_{k,t}(X)] \| \| \hat{\Sigma}_{k,t,\ell} \|_{op} \leq (1 + \delta_2) \| \hat{\Sigma}_{k,t,\ell} - \Sigma_{k,t,\ell} \|_{op} + \delta_2 \| \Sigma_{k,t,\ell} \|_{op} \leq 2 \delta_3 \| \Sigma_{k,t,\ell} \|_{op} + \delta_2 \| \Sigma_{k,t,\ell} \|_{op}.
\]

To make the right hand side above bounded by \( \delta_1 \| \Sigma_{k,t,\ell} \|_{op} \), we choose \( \delta_3 = 0.1 \delta_1 \) and \( \delta_2 = 0.1 \delta_1 \).

The sample complexity of achieving \( \| \hat{W}_{k,t} - E_{X \sim P}[w_{k,t}(X)] \| \leq \delta_2 = \delta / (30p) \) is \( O \left( \frac{p^2 \log(1/\tau)}{\tau} \right) \) by Hoeffding inequality. The sample complexity for achieving \( \| \hat{\Sigma}_{k,t} - \Sigma_{k,t} \|_{op} \leq \delta_3 \| \Sigma_{k,t} \|_{op} \) can be obtained by Fact 2.10 as follows:

\[
\tilde{n} \leq \frac{r^2 d \| \Sigma \|_{op} \log \left( \frac{d}{\tau} \right)}{\delta^2 \| \Sigma_{k,t} \|_{op}} \leq \frac{r^2 d p^2 \| \Sigma \|_{op} \log \left( \frac{d}{\tau} \right)}{\delta^2 \| \Sigma_{k,t} \|_{op}} \leq \frac{r^2 d p^2}{\delta^2} \log \left( \frac{d}{\tau} \right),
\]

where the last inequality uses that because of our assumptions \( E_{X \sim P}[w_{k,t}(X)] \geq 1 - O(\epsilon) \) and \( \| B_{k,t,\ell} \|_{op} \geq 0.5 \| \Sigma \|_{op} \), we have that \( \| \Sigma_{k,t} \|_{op} = \| B_{k,t} \|_{op} / E_{X \sim P}[w_{k,t}(X)] \geq \| \Sigma \|_{op} \).
Proof of Items 1 and 2 of Condition 4.2. The two conditions follow directly from Lemma B.1 using \( \delta = \frac{0.01}{\sqrt{d}} \min \left( \frac{\gamma}{\epsilon}, 1 \right) \). For the first one: Since \( \| \tilde{M}_{k,t} - M_{k,t} \|_{\text{op}} \leq \frac{0.1}{r \sqrt{d}} \sqrt{\gamma/\epsilon} \| M_{k,t} \|_{\text{op}} \) and we are interested only for points with \( ||x||_2 \leq r \sqrt{d/\Sigma} \| ||\text{op}, \) we have that

\[
g(x) \leq \| M_{k,t}x \|^2_2 \leq 2 \left( \| \tilde{M}_{k,t}x \|^2_2 + 2 \left( \| M_{k,t} - \tilde{M}_{k,t} \|_2 \right)^2 \right) \\
\leq 2 \left( \| \tilde{M}_{k,t}x \|^2_2 + 2 ||x||_2^2 \right) \| M_{k,t} - \tilde{M}_{k,t} \|_2^2 \leq 2 \tilde{g}(x) + 0.02 (\gamma/\epsilon) \| M_{k,t} \|_{\text{op}}^2 \| \Sigma \|_{\text{op}}.
\]

Equivalently, \( \tilde{g}(x) \geq 0.5g(x) - 0.01 (\gamma/\epsilon) \| M_{k,t} \|_{\text{op}}^2 \| \Sigma \|_{\text{op}} \). The proof of Item 2 can be found in [DKPP22].

\( \Box \)

### B.2 Item 4

We prove Item 4 in the lemma below, which is a simple adaptation of Lemma 4.16 from [DKPP22].

For the sake of completeness we include a proof.

**Lemma B.3.** In the setting of Algorithm 6, let \( \tau > r \) be a radius such that \( \mathsf{Pr}_{X \sim \mathcal{G}}[\|X\|_2 > r \sqrt{d/\Sigma \|_{\text{op}}} \leq \epsilon \) and assume that \( \| B_{k,t} \|_{\text{op}} \geq 0.5 \| \Sigma \|_{\text{op}} \). For any \( \tau \) and any weight function \( w : \mathbb{R}^d \rightarrow [0,1] \) and any vector \( u \), there is an estimator \( \tilde{F} \) for the quantity \( F_{k,t} := \mathbb{E}_{X \sim P}[w(X)(u^\top X)^2] \) that uses \( n = O \left( \frac{r^4 \tau^2}{\tau \gamma} \log(1/\tau) \right) \) samples from \( P \), runs in \( O(nd) \) time, uses memory \( O(\log(1/\tau)) \), and, with probability \( 1 - \tau \) it satisfies \( |\tilde{F} - F_{k,t}| \leq 0.01 \gamma F_{k,t} + \frac{0.01 \gamma^2}{r^4 \tau^2 \gamma} \| u \|_2^2 \| \Sigma_{k,t} \|_{\text{op}} \).

**Proof.** We show the lemma for constant probability of success as then one can boost that to \( 1 - \tau \) by repeating \( \log(1/\tau) \) times and keeping the median. The estimator is just sample average. By Chebysev’s inequality, with constant probability, we have that

\[
\frac{1}{n} \sum_{i=1}^{n} w(x_i) (u^\top x_i)^2 - \mathbb{E}_{X \sim P}[w(X)(u^\top X)^2] \leq \frac{\sqrt{\text{Var}_{X \sim P}[w(X)(u^\top X)^2]}}{n}.
\]

We want to upper bound the RHS by \( 0.01 \gamma \mathbb{E}_{X \sim P}[w(X)(u^\top X)^2] + \frac{0.01 \gamma^2}{r^4 \tau^2} \| u \|_2^2 \| \Sigma_{k,t} \|_{\text{op}} \). A sufficient number of samples for that is big enough multiple of the following:

\[
\frac{\text{Var}_{X \sim P}[w(X)(u^\top X)^2]}{\left( \gamma \mathbb{E}_{X \sim P}[w(X)(u^\top X)^2] + \frac{0.01 \gamma^2}{r^4 \tau^2} \| u \|_2^2 \| \Sigma_{k,t} \|_{\text{op}} \right)^2} \leq \frac{r^2 d \text{Var}_{X \sim P}[w(X)(u^\top X)^2]}{\gamma^2 \mathbb{E}_{X \sim P}[w(X)(u^\top X)^2] \| u \|_2^2 \| \Sigma_{k,t} \|_{\text{op}}} \leq \frac{r^2 d \mathbb{E}_{X \sim P}[w^2(X)(u^\top X)^4]}{\gamma^2 \mathbb{E}_{X \sim P}[w(X)(u^\top X)^2] \| u \|_2^2 \| \Sigma_{k,t} \|_{\text{op}}} \leq \frac{r^2 d \| u \|_2^2 \| \Sigma \|_{\text{op}} \mathbb{E}_{X \sim P}[w(X)(u^\top X)^2]}{\gamma^2 \mathbb{E}_{X \sim P}[w(X)(u^\top X)^2] \| u \|_2^2 \| \Sigma_{k,t} \|_{\text{op}}}.
\]
where the third line from the end used \( w^2(x) \leq w(x), \quad (u^\top x)^2 \leq \|u\|^2\|x\|^2, \) and \( \|x\|^2 \leq r^2d\|\Sigma\|_{\text{op}} \) (by the pruning of Lines 3 and 4 in Algorithm 6) and the fourth line uses our assumption \( \|B_{k,t}\|_{\text{op}} \geq 0.5\|\Sigma\|_{\text{op}} \).

### B.3 Item 3

**Lemma B.4.** For any distribution \( P \) over \( \mathbb{R} \) and any \( \epsilon, \delta \in (0, 1) \), there is an estimator \( \hat{L} \) that uses \( O\left(\frac{1}{\epsilon} \log \left(\frac{1}{\tau}\right)\right) \) samples from \( D \) and satisfies

\[
\left| \Pr_{X \sim P} [X > \hat{L}] - \epsilon \right| \leq \frac{\epsilon}{100},
\]

with probability at least \( 1 - \tau \). The memory usage and runtime of the estimator are also \( O\left(\frac{1}{\epsilon} \log \left(\frac{1}{\tau}\right)\right) \).

**Proof.** The estimator \( \hat{L}(X_1, \ldots, X_m) \) from \( m \) samples \( X_1, \ldots, X_m \sim P \) is defined to be the \( (m \cdot \epsilon) \)-greatest sample. Let \( L \) denote the target threshold, that is, the real for which \( \Pr_{X \sim P} [X > L] = \epsilon \).

Additionally, let \( L_-, L_+ \) such that \( \Pr_{X \sim P} [L_- \leq X \leq L] = \Pr_{X \sim P} [L \leq X \leq L_+] = \epsilon/100 \). Then, the probability that \( \hat{L} \) is not accurate is

\[
\Pr_{X_1, \ldots, X_m \sim P} \left[ \left| \Pr_{X \sim P} [X > \hat{L}] - \epsilon \right| > \frac{\epsilon}{100} \right] \leq \Pr[\hat{L} > L_+] + \Pr[\hat{L} < L_-].
\]

Each term is bounded by an application of the multiplicative version of Chernoff bounds. Regarding first one, \( \hat{L} > L_+ \) implies that we had at least \( m \cdot \epsilon \) samples in the interval \( (L_+, +\infty) \). But this is a low probability event because each sample belongs in this region with probability only \( \epsilon(1 - 1/100) \):

\[
\Pr[\hat{L} > L_+] \leq \Pr_{X_1, \ldots, X_m \sim P} \left[ \frac{1}{m} \sum_{j=1}^{m} \mathbb{1}(X_j > L_+) - \Pr_{X \sim P} [X > L_+] > \frac{\epsilon}{100} \right] \leq e^{-\Omega(em)}.
\]

Choosing \( m \) to be a sufficiently large multiple of \( \epsilon^{-1} \log \left(\frac{1}{\tau}\right) \) makes that probability of failure less than \( \tau/2 \). The other term \( \Pr[\hat{L} < L_-] < \tau/2 \) can be shown similarly.

This algorithm stores \( O(\epsilon^{-1} \log(1/\tau)) \) one dimensional samples. If the algorithm selects the top \( \epsilon \cdot m \) element by sorting the data first, its runtime is \( O(\epsilon^{-1} \log(1/\tau) \log(\epsilon^{-1} \log(1/\tau))) \). However, \( O(\epsilon^{-1} \log(1/\tau)) \) is also possible for selection using a probabilistic divide and conquer algorithm. \( \square \)