Realizable Learning is All You Need

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Abstract

The equivalence of realizable and agnostic learnability is a fundamental phenomenon in learning theory. With variants ranging from classical settings like PAC learning and regression to recent trends such as adversarially robust and private learning, it’s surprising that we still lack a unified theory; traditional proofs of the equivalence tend to be disparate, and rely on strong model-specific assumptions like uniform convergence and sample compression.

In this work, we give the first model-independent framework explaining the equivalence of realizable and agnostic learnability: a three-line blackbox reduction that simplifies, unifies, and extends our understanding across a wide variety of settings. This includes models with no known characterization of learnability such as learning with arbitrary distributional assumptions or general loss, as well as a host of other popular settings such as robust learning, partial learning, fair learning, and the statistical query model.

More generally, we argue that the equivalence of realizable and agnostic learning is actually a special case of a broader phenomenon we call property generalization: any desirable property of a learning algorithm (e.g. noise tolerance, privacy, stability) that can be satisfied over finite hypothesis classes extends (possibly in some variation) to any learnable hypothesis class.

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1 Introduction

The equivalence of realizable and agnostic learnability in Valiant’s Probably Approximately Correct (PAC) model \[1\] is one of the best known results in learning theory, and numbers among its most surprising. Given a set \(X\) and a family of binary classifiers \(H\), the result states that the ability to learn a classifier \(h \in H\) from examples of the form \((x, h(x))\) is in fact sufficient for something much stronger: given samples from any distribution \(D\) over \(X \times \{0, 1\}\), it is possible to learn the best approximation to \(D\) in \(H\). This surprising equivalence stems from a classic result of Blumer, Ehrenfeucht, Haussler, and Warmuth (BEHW) \[2\] and Haussler \[3\], who equate both the former model (known as realizable learning) and the latter model (known as agnostic learning) to a strong property of pairs \((X, H)\) called uniform convergence \[1\].

BEHW and Haussler’s result was certainly a breakthrough in its own right, but its proof technique is too indirect to reveal any deeper connections between realizable and agnostic learning beyond the PAC setting. Further, recent years have seen both theory and practice shift away not only from this original formalization, but more generally from the “uniform convergence equals learnability” paradigm, often in favor of distributional or data-dependent assumptions like margin that are more applicable to the real world. The inability of BEHW and Haussler’s proof technique to generalize to such scenarios raises a fundamental question: is the equivalence of realizable and agnostic learning a fundamental property of learnability, or simply a happy coincidence derived from the original PAC framework?

In the 30 years since these works, a mountain of evidence has amassed in favor of the former: almost every reasonable variant of learning shares some sort of similar equivalence. This includes a long list of popular settings such as regression \[4\], distribution-dependent learning \[5\], multi-class learning \[6\], robust learning \[7\], private learning \[8, 9\], and partial learning \[10, 11\]. What’s more, the uniform convergence paradigm fails miserably in most of these models. In the distribution-dependent model, for instance, it is easy to build classes which are trivially learnable (even with one sample!) but completely fail to satisfy uniform convergence \[5\]. On the other hand, models such as private learning give well-known examples where uniform convergence fails to imply learnability \[12\]. In spite of this, we are really no closer today to a general understanding of this phenomenon than we were in the early 90s. Much like Blumer, Ehrenfeucht, Haussler, and Warmuth \[2\] and Haussler’s \[3\] proofs, the above works often use indirect methods and tend to rely on powerful model-dependent assumptions.

In this work, we aim to offer a generic, unifying theory by way of the first direct reduction from agnostic to realizable learning. Unlike any previous work, our reduction is blackbox, relies on no additional assumptions, and, perhaps most importantly, is incredibly simple. In fact, the basic algorithm can be stated in three lines.

**Algorithm 1:** Agnostic to Realizable Reduction

**Input:** Realizable PAC-Learner \(\mathcal{A}\), Unlabeled Sample Oracle \(\mathcal{O}_U\), Labeled Sample Oracle \(\mathcal{O}_L\)

**Algorithm:**

1. Draw an unlabeled sample \(S_U \sim \mathcal{O}_U\), and labeled sample \(S_L \sim \mathcal{O}_L\).

2. Run \(\mathcal{A}\) over all possible labelings of \(S_U\) to get:

\[
C(S_U) := \{\mathcal{A}(S_U, h(S_U)) \mid h \in H\}_{S_U}\.
\]

3. **Return** the hypothesis in \(C(S_U)\) with lowest empirical error over \(S_L\).

This basic reduction simplifies and unifies classic results such as BEHW \[2\] and Haussler’s \[3\] distribution-

\[1\] Uniform convergence promises that a large enough sample gives a good approximation for loss of every \(h \in H\) simultaneously.
free equivalence and Benedek and Itai’s\footnote{Formally Benedek and Itai only consider random classification noise, but it is clear that their analysis extends to the agnostic model.} analogous result in the distribution-dependent setting\footnote{In the distribution-free setting, the resulting sample complexity is optimal up to a log factor, though it was later discovered this could be removed through a more complicated chaining technique \cite{indyk2001learning}.} with no loss in sample-complexity\footnote{In some cases (e.g. privacy) the resulting learner has a \textit{weaker variant} of property $P$ (e.g. semi-privacy or private prediction).}. Moreover, because Algorithm\footnote{Over infinite label spaces, we will require some weak assumptions on the loss.} doesn’t rely on model-dependent properties like uniform convergence, it extends to learning regimes without known characterizations. One such example is the notoriously difficult distribution-family model, in which the adversary is given a restricted family of distributions $\mathcal{D}$ along with the pair $(X,H)$. While no characterization of learnability is known in this model, Algorithm\footnote{Mechanisms}

\begin{meta-theorem}{Property Generalization (Informal)}\label{meta-theorem}
Let $A$ be a (sample-efficient) base learner and $P$ a finitely-satisfiable property. Then $A$ can be used to build a (sample-efficient) learner satisfying property $P$.
\end{meta-theorem}

In fact, the idea behind Algorithm\footnote{Mechanisms} touches at a much more general phenomenon. Think of realizable learning as a “base learner,” and of agnostic learning as an additional property we’d like to satisfy (a sort of noise tolerance). Algorithm\footnote{Mechanisms} works in part because the agnostic property is easy to satisfy over \emph{finite classes}. More generally, call a property $P$ (e.g. privacy, noise-tolerance) \emph{finitely-satisfiable} if for every finite class, there exists a learner satisfying property $P$. We argue that any such property should in fact generalize to all “learnable” classes.

\section{The Basic Reduction}

Since all of our results are derived from variants of Algorithm\footnote{Mechanisms} it is instructive to start by considering its basic analysis in our simplest non-trivial setting: distribution-family classification. This framework captures learnability with arbitrary distributional assumptions, a well-studied relaxation of PAC learning in practice where worst-case distributional assumptions are often too strong, and captures both the distribution-free and distribution-dependent PAC settings. Unlike these models, however, the distribution-family setting has no...
known characterization of learnability: uniform convergence is not necessary as in the former, and finite coverability is not sufficient as in the latter. Indeed, it is plausible no combinatorial characterization of this model exists at all, as it shares characteristics with EMX learnability which was recently shown to be independent of the ZFC set theory axioms. As a result, we cannot hope to prove the equivalence of agnostic and realizable learning in this model by finding some common characterization.

With this in mind, let’s define distribution-family learning a bit more formally. Let $X$ be a set (called the instance space), $Y = \{0, 1\}$ the set of binary labels, $\mathcal{D}$ a family of distributions over $X$, and $H = \{h : X \to Y\}$ a family of binary classifiers. A tuple $(\mathcal{D}, X, H)$ is said to be realizable learnable if there exists an algorithm $\mathcal{A}$ and a function $n(\varepsilon, \delta)$ such that for every $\varepsilon, \delta > 0$, every choice of distribution $D \in \mathcal{D}$, and every hypothesis $h \in H$, $\mathcal{A}$ outputs a good classifier with high probability on samples of size $n(\varepsilon, \delta)$:

$$\Pr_{S \sim D^n(\varepsilon, \delta)}[err_{D \times h}(\mathcal{A}(S, h(S))) \leq \varepsilon] \geq 1 - \delta,$$

where $err_{D \times h}(\mathcal{A}(S, h(S)))$ is commonly called the error or risk of $h$:

$$err_{D \times h}(h') = \Pr_{x \sim D}[h'(x) \neq h(x)].$$

Likewise, a tuple $(\mathcal{D}, X, H)$ is said to be agnostically learnable if there exists an algorithm $\mathcal{A}$ which for every distribution $D$ over $X \times Y$ whose marginal $D_X \in \mathcal{D}$ outputs $h'$ close to the best hypothesis in $H$ with probability $1 - \delta$:

$$\Pr_{S \sim D^n(\varepsilon, \delta)}[err_D(\mathcal{A}(S)) \leq OPT + \varepsilon] \geq 1 - \delta,$$

where $OPT = \min_{h \in H}\{err_D(h)\}$ is the error of the best hypothesis in the class and the risk $err_D(\mathcal{A}(S))$ is similarly defined:

$$err_D(h') = \Pr_{(x, y) \sim D}[h'(x) \neq y].$$

With this in mind, we can now state the most basic application of Algorithm 1: the equivalence of agnostic and realizable learning for distribution-family classification.

**Theorem 2.1 (Agnostic → Realizable (Distribution-Family Classification)).** Let $\mathcal{A}$ be a realizable learner for $(\mathcal{D}, X, H)$ using $n(\varepsilon, \delta)$ samples. Then Algorithm 1 is an agnostic learner for $(\mathcal{D}, X, H)$ using:

$$m_U(\varepsilon, \delta) \leq n(\varepsilon/2, \delta/2)$$

unlabeled samples, and

$$m_L(\varepsilon, \delta) \leq O\left(\frac{n(\varepsilon/2, \delta/2) + \log(1/\delta)}{\varepsilon^2}\right)$$

labeled samples. Moreover if $(X, H)$ has finite VC dimension $d$, Algorithm 1 needs only

$$m_L(\varepsilon, \delta) \leq O\left(\frac{d \log (1/\varepsilon) + \log(1/\delta)}{\varepsilon^2}\right)$$

labeled samples.

In the distribution-free model, this returns the same complexity bound as standard analysis of uniform convergence (though a more complicated chaining argument can be used to remove the $\log(1/\varepsilon)$ factor). It’s also worth noting that while unlabeled sample complexity is not usually considered separately from labeled complexity in the PAC setting, this will become a useful distinction in semi-supervised extensions considered later in the work. As such, it is instructive to keep the complexities separate for the time being.

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6Note that $\mathcal{A}$ could be deterministic or randomized. This distinction has no effect on any of the arguments in this work.
With this out of the way, let’s prove Theorem 2.1. The analysis breaks naturally into two parts, corresponding respectively to Step 2 and Step 3 of Algorithm 1. In the first part, we’ll show that \( C(S_U) \), the set of outputs corresponding to running the realizable learner \( \mathcal{A} \) across all possible labelings of the unlabeled sample \( S_U \), is in some sense a “good approximation” of the class \( H \). More formally, the crucial observation is that for any choice of the adversary’s distribution, \( C(S_U) \) will (almost) always contain a hypothesis close to the optimal solution.

**Claim 2.2.** For any distribution \( D \) over \( X \times Y \) whose marginal \( D_X \in \mathcal{D} \), with probability \( 1 - \delta/2 \), there exists \( h' \in C(S_U) \) which is within \( \epsilon/2 \) of the optimal risk:

\[
err_D(h') \leq OPT + \epsilon/2.
\]

Once we have this claim, the second step is to show that Step 3, an empirical risk minimization process on \( C(S_U) \), gives the desired agnostic learner. This actually follows from standard arguments. In particular, given a hypothesis \( h \in C(S_U) \), let

\[
err_{S_L}(h) = \Pr_{(x,y) \sim S_L} [h(x) \neq y]
\]

denote its empirical risk with respect to \( S_L \). Since \( C(S_U) \) is finite, a standard Chernoff+Union bound gives that with probability at least \( 1 - \delta/2 \), the empirical risk of every hypothesis in \( C(S_U) \) with respect to \( S_L \) is close to its true risk. Then as long as \( S_L \) is sufficiently large, empirical risk minimization returns a solution with at most \( OPT + \epsilon \) error with high probability (we’ll formalize this in a moment).

It remains to prove Claim 2.2. The key observation lies in an equivalence between realizable PAC-learning and a weak type of covering: for any fixed \( S \), denote its empirical risk \( \err_S(h) \). For any distribution \( D \) over \( X \times Y \) whose marginal \( D_X \in \mathcal{D} \) and any \( h \in H \), with probability \( 1 - \delta/2 \), there exists \( h' \in C(S_U) \) which is within \( \epsilon/2 \) of \( h \) in classification distance:

\[
Pr_{x \sim D_X} [h'(x) \neq h(x)] \leq \epsilon/2.
\]

**Proof.** The proof is essentially immediate from the definition of realizable PAC-learning. \( \mathcal{A} \) promises that for any \( h \in H \) and \( D \in \mathcal{D} \), a \( 1 - \delta/2 \) fraction of labeled samples \( (S, h(S)) \sim D^{n(\epsilon/2,\delta/2)} \) satisfy

\[
\err_D[X \times h][\mathcal{A}(S, h(S))] = \Pr_D [h'(x) \neq h(x)] \leq \epsilon/2,
\]

where \( h' = \mathcal{A}(S, h(S)) \). Since \( C(S_U) \) contains \( \mathcal{A}(S_U, h(S_U)) \) for every \( h \in H \) by definition, the result follows. \( \square \)

We call \( C(S_U) \) a non-uniform cover. Note that Lemma 2.3 does not imply that \( C(S_U) \) contains hypotheses close to every \( h \in H \) simultaneously. This stronger object is called a uniform cover and takes provably more samples to construct (see Appendix C). In our case, a non-uniform cover is sufficient. Since the guarantee holds for every fixed \( h \in H \), it must hold in particular for the optimal hypothesis \( h_{OPT} \), so \( C(S_U) \) contains some \( h' \) within \( \epsilon/2 \) of optimal. Let’s now formalize these ideas and put everything together to prove Theorem 2.1.

**Proof of Theorem 2.1.** Let \( D \) be the adversary’s distribution over \( X \times Y \), and let \( h_{OPT} \in H \) be a hypothesis achieving the optimal error. By Lemma 2.3, with probability \( 1 - \delta/2 \), \( C(S_U) \) contains a hypothesis \( h' \) such that:

\[
Pr_{x \sim D_X} [h'(x) \neq h_{OPT}(x)] \leq \epsilon/2.
\]
This implies Claim \ref{claim:approximation} (that $C(S_U)$ contains a hypothesis with error at most $OPT + \varepsilon/2$) since

$$err_D(h') = \Pr_{(x,y) \sim D}[h'(x) \neq y]$$

$$\leq \Pr_{(x,y) \sim D}[h_{OPT}(x) \neq y] + \Pr_{(x,y) \sim D}[h_{OPT}(x) \neq h'(x)]$$

$$\leq OPT + \varepsilon/2.$$
this is not just an issue with our algorithm: it is an inherent barrier. Realizable and agnostic learning simply
aren’t equivalent for most reasonable losses over infinite label classes.

**Proposition 3.1** (Proposition 8.1). Let $\ell$ be any loss function over $\mathbb{R}$ satisfying the identity of indiscernibles
that is continuous in the first variable. Then there exists a class $(\mathcal{D}, X, H, \ell)$ which is realizable learnable
but not agnostically learnable.

The construction in Proposition 3.1 uses the infinite label space to exactly encode each hypothesis (i.e. a single labeled example always uniquely determines the corresponding hypothesis). On the other hand, a small amount of noise can completely erase this, so the class clearly won’t be agnostically learnable.

On the positive side, we can modify Algorithm 1 to show this is essentially the only barrier to agnostic learnability. Somewhat more formally, we call a class *discretely learnable* if for every $\varepsilon > 0$, there exists an $\varepsilon$-discretization\(^7\) of $(\mathcal{D}, X, H, \ell)$ that is learnable up to $O(\varepsilon)$ error. Discrete learnability can informally be thought of as a very weak type of noise tolerance that essentially acts only to rule out the above construction.

We prove that discrete learnability is equivalent to agnostic learnability over two broad classes of loss functions. The first is a basic generalization of loss functions over finite label classes we call doubly bounded loss.

**Theorem 3.2** (Informal Theorem A.2: Agnostic → Realizable (Doubly Bounded Loss)). Let $\ell : Y \times Y \to \mathbb{R}_{\geq 0}$ be a loss function such that $\forall y_1 \neq y_2 \in Y$:

$$\ell(y_1, y_2) \in [\alpha, \beta]$$

for some $\beta > \alpha > 0$. Then for any class $(\mathcal{D}, X, H, \ell)$, the following are equivalent:

1. $(X, H, \mathcal{D}, \ell)$ is (properly) discretely-learnable.
2. $(X, H, \mathcal{D}, \ell)$ is (properly) agnostically-learnable.

This result also implies that realizable and agnostic learning are equivalent over any loss function satisfying the identity of indiscernibles over finite $Y$, since realizable and discrete learnability are equivalent in this case and every such loss function is doubly bounded.

While many reasonable loss functions on infinite label classes (e.g. $\ell_p$-loss) aren’t bounded away from 0, they do tend to come with other structure we can utilize. We’ll prove a similar result to the above under the weak assumption that our loss satisfies an approximate triangle inequality. Such loss functions, which we call approximate pseudometrics, can informally be thought of as generalizing any sort of distance-based loss.

**Theorem 3.3** (Informal Theorem 8.5: Agnostic → Realizable (Approximate Pseudometrics)). Let $\ell : Y \times Y \to \mathbb{R}_{\geq 0}$ be an (upper) bounded\(^8\) loss function satisfying a $c$-approximate triangle inequality:

$$\forall y_1, y_2, y_3 \in Y : \ell(y_1, y_3) \leq c(\ell(y_1, y_2) + \ell(y_2, y_3)).$$

Then for any class $(\mathcal{D}, X, H, \ell)$, the following are equivalent:

1. $(\mathcal{D}, X, H, \ell)$ is (properly) discretely-learnable.
2. $(\mathcal{D}, X, H, \ell)$ is (properly) $c$-agnostically learnable.

where “$c$-agnostic” learnability only requires error $c \cdot \text{OPT} + \varepsilon$ rather than $\text{OPT} + \varepsilon$.

While $c$-agnostic learnability is a weaker guarantee than we get for doubly bounded loss, it is actually necessary for approximate pseudometrics. In particular, there exist simple discretely-learnable classes over $c$-approximate pseudometrics which are not $c'$-agnostically learnable for any $c' < c$ (see Proposition 8.6).

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\(^7\)A discretization is a class $H'$ over a finite (or probably finite) label space such that every $h \in H$ is close to some $h' \in H'$. See Section 8.1 for details.

\(^8\)By this we mean the range of $\ell$ is $[0, B]$ for some $B \in \mathbb{R}_{\geq 0}$. We note that this condition can be replaced with the weaker assumption that $(\mathcal{D}, X, C, \ell)$ is agnostically learnable for every finite subset $C \subseteq H$. 

8
3.2 Beyond the PAC Setting

While allowing distributional assumptions through the distribution-family model is a good step towards practice, recent trends have starting branching even further away from the PAC setting. In this section, we’ll discuss a prototypical example of applying Algorithm 1 to an extended model: adversarial robustness. In the appendix, we cover similar extensions to Partial Learning (Appendix C), SQ-learning (Appendix E), and Fair Learning (Appendix F).

Robust learning is an extension of the PAC model introduced to handle adversarial perturbations at test time. Practically, this is meant to ensure that an integrated prediction system (e.g. in a self-driving car) cannot be tricked by possibly imperceptible adversarial changes to the outside world. This can be formalized by modifying the way we compute error. Given a perturbation function mapping $X$ to its power set, $U : X \rightarrow P(X)$ (think of $U$ as specifying a neighborhood set for each $x \in X$), the robust risk of a labeling $c : X \rightarrow Y$ with respect to a distribution $D$ over $X \times Y$ is:

$$R_{err,U,D}(c) = \mathbb{E}_{(x,y) \sim D} \left[ \max_{x' \in U(x)} (\ell(c(x'), y)) \right].$$

Realizable and agnostic robust learning are then defined analogously to the PAC-model where the standard error is replaced with robust error (though the distribution-family model does require a slight twist, see Appendix B for details). We show that a basic modification to Algorithm 1 again implies the two models are equivalent.

**Theorem 3.4** (Informal Theorem B.5: Agnostic $\rightarrow$ Realizable (Robust Classification)). If $(\mathcal{D}, X, H)$ is robustly PAC-learnable in the realizable setting with sample complexity $n(\varepsilon, \delta)$, then (a modification of) Algorithm 1 robustly learns $(\mathcal{D}, X, H)$ in the agnostic setting in:

$$m_U(\varepsilon, \delta) \leq O \left( \max_{\mu \in [0, 1 - \varepsilon]} \left\{ \frac{n(\varepsilon/(2(1 - \mu)), \delta/3)}{1 - \mu} \right\} \right)$$

unlabeled samples and

$$m_L(\varepsilon, \delta) \leq O \left( \frac{m_U(\varepsilon, \delta) + \log(1/\delta)}{\varepsilon^2} \right)$$

labeled samples.

We note that this result can also be combined with our analysis for more general loss functions, albeit with a slightly worse $c$-agnostic parameter.

3.3 Beyond Agnostic Learning

So far we have only considered using Algorithm 1 to reduce from agnostic to realizable learning, albeit in a number of extended settings beyond the PAC model. On the other hand, we claimed in our meta-theorem that Algorithm 1 can be used to build a learner satisfying any “finitely-satisfiable” property. In this section, we’ll discuss two such examples: privacy, and malicious noise. In Appendix D we cover a similar application to uniform stability. Note that since we are only modifying the learning property in this section, the base (realizable) learner remains the same and does not require any additional constraints.

We’ll start with Kearns and Li’s malicious noise [14]. In this model, the learner has access to a faulty sample oracle $O_M(\cdot)$ which returns a labeled sample from the adversary’s true distribution with probability $1 - \eta$, and otherwise receives an adversarially chosen pair $(x, y)$. A realizable or agnostic learner is said to be tolerant to malicious noise if it achieves the standard PAC guarantees while drawing from the malicious oracle instead of the standard sample oracle. Like agnostic learning, tolerance to malicious noise is easy to achieve on finite hypothesis classes. As a result, a basic modification of Algorithm 1 gives a blackbox reduction from agnostic learning with malicious noise to realizable learning.
Theorem 3.5 (Informal Theorem 8.8: Malicious → Realizable). If \((\mathcal{D}, X, H)\) is realizably PAC-learnable with sample complexity \(n(\varepsilon, \delta)\), then for any \(\eta < \frac{\varepsilon}{1+\varepsilon}\), (a modification of) Algorithm 7 gives an agnostic learner for \((\mathcal{D}, X, H)\) tolerant to \(\eta\) malicious noise. Furthermore letting \(\Delta = \frac{\varepsilon}{1+\varepsilon} - \eta\), the sample complexity is at most \(\text{poly}(\Delta^{-1}, \log(1/\delta), n(\Delta, \delta))\).

This result is tight in the sense that \(\frac{\varepsilon}{1+\varepsilon}\) is the best possible error tolerance in the malicious model [14]. As before, the result can be combined with our discretization techniques to give a similar equivalence for approximate pseudometric loss as well.

While malicious noise is certainly distinct from the agnostic model, both are examples of noise-tolerance properties. We’ll finish this informal discussion of our results with a learning property of a different flavor: privacy. Informally, an algorithm is said to be \(\alpha\)-differentially private if its output is not susceptible to small changes in the underlying sample (see Section 8.3 for exact definitions). Privacy is a very strong condition, even relaxed notions such as \((\alpha, \delta)\)-differential privacy (which essentially allows for a \(\delta\) probability of privacy failure) require finite Littlestone dimension in the distribution-free setting [12], which rules out any sort of reduction directly from realizability.

On the other hand, our reduction actually is able to recover a different relaxation known as semi-private learning [16]. In this setting, the learning algorithm has access not only to a private database of labeled samples, but also a public database of unlabeled data. This models situations in practice where there may be a portion of “opt-in” users who are willing to share their participation. By replacing the empirical risk minimization (ERM) process in Step 3 of Algorithm 1 with a popular private algorithm known as the exponential mechanism [20], we give a direct reduction from semi-private to realizable learning in the distribution-family setting.

Theorem 3.6 (Informal Theorem 8.15: Semi-Private → Realizable). If \((\mathcal{D}, X, H)\) is realizably learnable with sample complexity \(n(\varepsilon, \delta)\), then (a modification of) Algorithm 4 gives an \(\alpha\)-semi-private, agnostic learner using
\[
m_{\text{pub}}(\alpha, \varepsilon, \delta) \leq n(\varepsilon/2, \delta/2)
\]
unlabeled (public) samples, and
\[
m_{\text{pri}}(\alpha, \varepsilon, \delta) \leq O\left(\frac{n(\varepsilon/2, \delta/2) + \log(1/\delta)}{\varepsilon \cdot \min\{\varepsilon, \alpha\}}\right)
\]
labeled (private) samples.

As before, this result also extends to the general loss functions covered in Section 3.1. Let’s take a look at a concrete application of this result to the well-studied case of distribution-free classification. In this setting we can improve the sample complexity bounds using standard VC dimension arguments.

Corollary 3.7 (Informal Corollary 8.16: Semi-Private → Realizable (Distribution-Free Classification)). Let \((X, H)\) be a class with VC-dimension \(d\). Then \((X, H)\) is \(\alpha\)-semi-private, agnostically learnable in
\[
m_{\text{pub}}(\varepsilon, \delta, \alpha) \leq O\left(\frac{d + \log(1/\delta)}{\varepsilon}\right)
\]
unlabeled samples and
\[
m_{\text{pri}}(\varepsilon, \delta, \alpha) \leq O\left(\frac{d \log(1/\varepsilon) + \log(1/\delta)}{\varepsilon \cdot \min\{\varepsilon, \alpha\}}\right)
\]
labeled samples.

\(^9\)We note that this algorithm is improper. A proper algorithm may need an additional \(\log(1/\varepsilon)\) unlabeled samples for some classes.
For fixed $d$ and $\delta$, Corollary 3.7 resolves the unlabeled sample complexity of semi-private learning, as Alon, Bassily, and Moran [17] recently showed that any class which is not privately learnable requires at least $\Omega(\frac{1}{\epsilon})$ public samples to (semi)-privately learn. The private sample complexity, on the other hand, remains off by a logarithmic factor from known lower bounds [21]. Resolving the latter requires improving the standard application of the exponential mechanism, and remains an interesting open problem.

It is also worth noting that Theorem 3.6 and Corollary 3.7 are robust to some amount of shift in distribution between the public and private databases. This problem, often called covariate shift in other contexts, is a commonly observed issue in machine learning practice, and is especially of concern in privacy where a distribution over “opt-in” public users could easily differ from the overall distribution of private data. We discuss covariate shift in the semi-private setting in more depth in Section 8.4.

4 Proof Overview: Modification Archetypes

We’ve already seen how Algorithm 1 works in the basic case of binary classification, but covering the extended regimes above require some modifications. In this section, we’ll overview the four generic types of modification we use to extend Algorithm 1 across the aforementioned settings.

Discretization. We’ll start by discussing our main technique to extend Algorithm 1 to infinite label spaces. The basic idea is simple: since we cannot afford to run our learner over all possible labelings of $S_U$, we instead run the learner over labelings coming from some discretization of the class. As long as we have access to a learner for the discretization, we can then use the same arguments covered in Section 2 to prove various occurrences of property generalization. We formalize these notions more generally in Section 8.1, where we use the technique to prove Theorem 3.3 (Theorem 3.2 is proved in Appendix A). Discretization can also be used to handle learning models such as the statistical query setting which output real-valued query responses (see Appendix E).

Subsampling. Another core limitation to Algorithm 1 is access to clean unlabeled data. Algorithm 1 works by running a realizable learner over a representative set of unlabeled data, but, in practice, such data may often be corrupted, and data-dependent assumptions such as margin might mean that the optimal hypothesis isn’t even well-defined on this set. We handle cases like these by a simple sub-sampling procedure: instead of running our realizable learner over labelings of $S_U$, we run the learner over all labelings of all subsets of $S_U$. As long as $S_U$ contains some amount of uncorrupted data, this subsampling procedure will find it and we can maintain the guarantees discussed in Section 2. We use this technique to prove property generalization for models such as robust learning (see Theorem 3.4 and Appendix B), partial learning (see Appendix C), and malicious noise (see Theorem 3.5 and Section 8.2).

Replacing the Finite Learner. In our meta-theorem, we proposed a general paradigm of property generalization: that a variant of any learning property which holds for finite classes should in fact hold for any “learnable” class in the base model. The main idea relies on replacing Step 3 of Algorithm 1 (which, as stated, is an empirical risk minimization process) with a generic learner for finite classes with the desired property. For noise-tolerance properties such as agnostic and malicious noise, empirical risk minimization works. Properties such as privacy or stability, however, require a different finite learner. To prove Theorem 3.6, for example, we replace the ERM process in Algorithm 1 with the exponential mechanism [20]. We use a similar strategy in Appendix D to prove an analogous result for uniform stability.

Replacing the Base Learner. Finally we note a very basic modification of Algorithm 1 that allows us to extend property generalization beyond the PAC setting: simply replace the input realizable PAC learner
with a realizable learner in the desired model. This is usually combined with one of the techniques above depending on the specific application, e.g. to prove property generalization for robust learning and the statistical query model. The same idea can also be used to analyze semi-private learning with covariate shift (see Section 8.4) and property generalization for fair learning (see Appendix F).

5 Related Work

Agnostic learning is a very widely studied model across learning theory, and works across many different sub-areas have noted model specific equivalences with realizable learning. Here we’ll survey a few representative examples, and discuss how they differ from our approach.

5.1 Beyond Binary Classification

Uniform Convergence and Multiclass Classification. It is well known that the uniform convergence equals learnability paradigm continues to hold for 0/1-valued loss functions over constant-size label spaces [2, 22–24], and that agnostic and realizable learning are equivalent as a result. On the other hand, Daniely, Sabato, Ben-Devid, and Shalev-Shwartz [25] showed this is no longer the case as the number of labels grows large. In this regime, even basic multi-class learning is no longer equivalent to uniform convergence, so the connection between realizable and agnostic learning becomes non-trivial. Not long after, David, Moran, and Yehuyadoff (DMY) [6] were able to prove the equivalence of these models for 0/1-valued loss functions over arbitrary label spaces through the weaker sample compression equals learnability paradigm. While this holds more generally than the uniform convergence paradigm, it remains a model-specific technique and fails in many of the settings we consider, e.g. partial learning [11].

Discretization and General Loss Functions. Basic forms of discretization were also considered back in the mid 90s in work on characterizing the learnability of real-valued functions. In a seminal work, Bartlett, Long, and Williamson (BLW) [4] proved that a scale-sensitive measure introduced by Kearns and Schapire [26] called fat-shattering dimension characterizes learnability under bounded Lipschitz loss functions. BLW use a basic form of discrete learning (called quantization) to prove that fat-shattering dimension is a necessary condition, and use uniform convergence to prove sufficiency. We give a similar argument as BLW in the necessary direction, but show that uniform convergence is not necessary for the equivalence to hold, and instead use Algorithm 1 to appeal directly to discrete learnability. This allows us to extend BLW’s result across a much more general set of loss functions and scenarios without strong model-specific assumptions.

5.2 Private and Semi-Private Classification

Starting with the seminal work of Beimel, Nissim, and Stemmer (BNS) [28], agnostic learning has seen significant study in the privacy literature [17, 9]. Similar to our approach, both BNS and later work on the semi-private model [17] use unlabeled samples to construct a small set of hypothesis that “cover” the hypothesis space and then apply the exponential mechanism. More recent work by Alon, Beimel, Moran, and Stemmer [9] gives a direct reduction from agnostic to realizable private learning by relabling the dataset using a carefully chosen hypothesis \(h\) and then using the private realizable learner to learn the relabelled dataset. However, these previous techniques all relied on the fact that uniform convergence is necessary for private learnability, and therefore break down in more general settings. Moreover, by using a non-uniform cover instead of the uniform covers employed by [17], our reduction resolves the public sample complexity

Though the original work only considers \(\ell_1\) loss, their techniques generalize to Lipschitz loss, see for instance [27].
of semi-private learning. In fact, one can show uniform covers cannot be used to build sample-optimal semi-private learners (at least in the distribution-family model), as they require provably more unlabeled samples to construct (see Appendix G).

5.3 Semi-supervised and Active Learning

Our reduction hinges on combining a realizable learner with unlabeled data to cut down the number of potential hypotheses in our class. The use of unlabeled samples to this effect is one of the core ideas in the field of *semi-supervised learning* [29, 30]. Here, it is usually additionally assumed that the function to be learnt has some additional relation to the underlying data distribution, for example it might have large margin on unlabelled data as in Transductive SVM [31] or redundant sufficient information as in Co-training [32]. Since this use of unlabeled data is a core tenet of semi-supervised learning, it is not surprising that strategies similar in flavor to our reduction have appeared in this literature (see e.g. [29]). To our knowledge, however, these works (e.g. [33, 29, 34]) focus more on giving algorithms themselves (either realizable or agnostic) than a reduction between the two, and rely on model-dependent properties such as uniform convergence or fixed covers which break down in the more general settings we consider. Moreover, as in the privacy literature, the works that do use a similar covering strategy (e.g. [29]) use uniform rather than non-uniform covers. Finally, we note that a similar strategy has also seen use in the related *active learning* literature. Here, Hanneke [35] used uniform covers built from unlabeled data (again based on uniform convergence) to improve the query complexity of learning.

**Paper Organization**

The remainder of this paper is split into two portions, the main body and the appendix. The main body covers our base reduction for finite label classes along with four archetypes of modification, and is meant to be read as written. On the other hand, the Appendix covers various applications of these archetypes to a variety of learning models and properties. These sections are all self-contained, and are meant more as a reference text in the sense that the reader interested in some particular model or property should simply skip to the section covering that application.

The main body is organized as follows: we cover preliminary definitions in Section 6, our base reduction from agnostic to realizable learning for finite label classes in Section 7, and discuss the four modification archetypes along with a representative application in Sections 8.1, 8.2, 8.3 and 8.4. In more detail, these sections respectively cover: extensions to infinite label classes via discretization, malicious noise via sub-sampling, agnostic semi-private learning via replacing ERM, and covariate shift via replacing the base learner.

In the Appendix we cover applications to doubly-bounded loss (Appendix A), robust learning (Appendix B), partial learning (Appendix C), uniformly-stable learning (Appendix D), the statistical query model (Appendix E), and fair learning (Appendix F), and discuss further connections of non-uniform covers to previous notions of covering (Appendix G).

6 Preliminaries

Before moving to a more formal discussion of our results, we’ll cover the most basic learning models discussed in this work: standard (distribution-free) PAC-learning and distribution-family PAC-learning. Extended models we consider beyond these (e.g. malicious noise, robust learning, partial learning, etc.) will instead be introduced in their respective sections.
6.1 PAC-Learning

We’ll start by reviewing the seminal PAC-learning model of Valiant [1] and Vapnik and Chervonenkis [36]. We start with a few core definitions for the setting of general loss. Let $X$ be an arbitrary set called the instance space (e.g. $\mathbb{R}^d$), $Y$ a set called the label space (e.g. $\{0, 1\}$), and $H$ a family of labelings of $X$ by $Y$ (that is a family of functions of the form $h : X \rightarrow Y$). Given a class $(X, H)$, it will often be useful to consider its growth function $\Pi_H(n)$ which measures the maximum size of $H$ when restricted to a sample of size $n$:

$$\Pi_H(n) = \max_{h \in H, S \subseteq X^n}(|\{h|S\}|).$$

We note that the growth function is trivially bounded by $|Y|^n$, but one can often give stronger bounds when $(X, H)$ satisfies some finite combinatorial dimension (e.g. VC-dimension for the binary case).

While PAC-learning is sometimes used to refer only to classification, we will study the model under general loss functions. With that in mind, we call a function $\ell : Y \times Y \rightarrow \mathbb{R}_{\geq 0}$ a loss function if $\ell(y, y) = 0$ for all $y \in Y$. We say a loss $\ell$ satisfies the identity of indiscernibles if $\ell(y_1, y_2) = 0$ iff $y_1 = y_2$. Given any distribution $D$ over $X \times Y$ and loss $\ell$, the risk of a labeling $h : X \rightarrow Y$ with respect to $D$ and $\ell$ is its expected loss:

$$\text{err}_{D, \ell}(h) = \mathbb{E}_{(x, y) \sim D}[\ell(h(x), y)].$$

The goal of learning is generally to find a classifier $h \in H$ that minimizes risk. More formally, there are two commonly studied variants of this problem. The original formulation, now called realizable learning, assumes the existence of a hypothesis in $H$ with no loss.

Definition 6.1 ((Realizable) PAC-learning). We say $(X, H, \ell)$ is realizable PAC-learnable if there exists an algorithm $A$ and function $n(\varepsilon, \delta)$ such that for all $\varepsilon, \delta > 0$ and distributions $D$ over $X \times Y$ such that $\min_{h \in H} \text{err}_{D, \ell}(h) = 0$:

$$\Pr_{S \sim D^n(\varepsilon, \delta)}[\text{err}_{D, \ell}(A(S)) > \varepsilon] \leq \delta.$$

$A$ is called proper if it outputs only labels in $H$.

Perhaps a more realistic variant of PAC-learning is to drop this restriction on the adversary, and let them choose an arbitrary distribution over $X \times Y$. This model, introduced by Haussler [3] and Kearns, Schapire, and Sellie [14], is known as agnostic learning.

Definition 6.2 ((Agnostic) PAC-learning). We say $(X, H, \ell)$ is agnostic PAC-learnable if there exists an algorithm $A$ and function $n(\varepsilon, \delta)$ such that for all $\varepsilon, \delta > 0$ and distributions $D$ over $X \times Y$:

$$\Pr_{S \sim D^n(\varepsilon, \delta)}[\text{err}_{D, \ell}(A(S)) > \text{OPT} + \varepsilon] \leq \delta,$$

where $\text{OPT} = \min_{h \in H} \{\text{err}_{D, \ell}(h)\}$.

For some settings covered in this work, it will turn out that reaching $\text{OPT} + \varepsilon$ error is too stringent of a condition. However, we will show in these cases that it is sometimes possible to maintain a weaker guarantee and learn up to $c \cdot \text{OPT} + \varepsilon$ error for some constant $c > 1$. We call such classes $c$-agnostic learnable.

Finally, we note that for simplicity when $\ell$ is the standard “classification error”:

$$\ell(y_1, y_2) = \begin{cases} 0 & \text{if } y_1 = y_2 \\ 1 & \text{else,} \end{cases}$$

we’ll simply write $(X, H)$ to mean $(X, H, \ell)$. Realizable and Agnostic Learning are well studied under many basic loss functions including binary classification, where both models are known to be characterized by a combinatorial parameter called VC-dimension.
6.2 Learning Under Distribution Families

The standard PAC-models described above are often called distribution-free due to the fact that no assumptions are made on the marginal distribution over $X$. In practice, however, this is usually too worst-case an assumption. We often expect distributions in nature to be “nice” in some way, or at least somewhat restricted. This is reflected in the fact that popular machine learning algorithms usually significantly outperform the PAC-model’s worst-case generalization bounds. Indeed such niceness assumptions have long been popular in learning theory as well, where conditions such as tail bounds or anti-concentration are frequently used to build efficient algorithms.

These ideas are captured more generally by a simple (but notoriously difficult) extension to the PAC framework originally proposed by Benedek and Itai [5], where the adversary is restricted to picking from a fixed, known set of distributions.

**Definition 6.3 ((Realizable) Distribution-Family PAC-learning).** Let $X$ be an instance space and $D$ a family of distributions over $X$. We say $(D, X, H, \ell)$ is realizable PAC-learnable if there exists an algorithm $A$ and function $n(\varepsilon, \delta)$ such that for all $\varepsilon, \delta > 0$ and distributions $D$ over $X \times Y$ satisfying:

1. The marginal $D_X \in D$,
2. $\min_{h \in H} err_{D,\ell}(h) = 0$,

we have

$$\Pr_{S \sim D^n(\varepsilon, \delta)} [err_{D,\ell}(A(S)) \leq \varepsilon] \leq \delta.$$  

Agnostic learning is defined similarly. The adversary must still choose a marginal distribution in $D$, but the conditional labeling can be arbitrary.

**Definition 6.4 (Agnostic Distribution-Family PAC-learning).** We say $(D, X, H, \ell)$ is agnostic PAC-learnable if there exists an algorithm $A$ and function $n(\varepsilon, \delta)$ such that for all $\varepsilon, \delta > 0$ and distributions $D$ over $X \times Y$ satisfying:

1. The marginal $D_X \in D$,

$A$ outputs a good hypothesis with high probability:

$$\Pr_{S \sim D^n(\varepsilon, \delta)} [err_{D,\ell}(A(S)) > OPT + \varepsilon] \leq \delta$$

where $OPT = \min_{h \in H} \{err_{D,\ell}(h)\}$.

The weaker $c$-agnostic learning is defined analogously with $OPT$ replaced by $c \cdot OPT$. Unlike the standard model, very little is known about distribution-family learnability. While a number of works have made some progress on this front [5, 18, 37, 38], a characterization of learnability remains elusive despite some 30 years of effort.

7 The Core Reduction: Agnostic to Realizable Learning

In this section, we give a more detailed exposition of our main reduction as covered in Section 2, including the more general setting of arbitrary loss on constant size label spaces (in the distribution-family model), matching lower bounds, and additional discussion of non-uniform covers. As mentioned previously, since there is no known combinatorial characterization of learnability in the distribution-family model, standard
techniques \cite{2, 22, 24} cannot be used, and it is plausible that no combinatorial characterization of learnability exists for this model at all \cite{19}.

Before jumping into our reduction proper, it is worth discussing why we can’t simply take the approach of prior works and rely on uniform convergence, a strong condition which promises that on a large enough sample, the empirical error of every hypothesis will be close to its true error. While uniform convergence was a very popular technique in the early years of learning, practitioners have since moved away from the paradigm which fails to capture learning rates seen in practice \cite{39, 40}. Indeed it soon became clear that the technique failed to capture even basic theoretical models such as the distribution-dependent setting \cite{39}.

Proposition 7.1 (Benedek and Itai \cite{5}). There exists a PAC-learnable class \((D, X, H)\) over binary labels and classification loss without the uniform convergence property.

Proof. Let \(X = [0, 1]\), \(D\) be the uniform distribution over \(X\), \(Y = \{0, 1\}\), and \(H\) consist of all indicator functions for finite sets \(S \subset X\), as well as for \(X\) itself. It is not hard to see that \((D, X, H)\) is realizably PAC-learnable by the following scheme in only a single sample: if the learner draws a sample labeled 1, output the all 1’s function. Otherwise, output all 0s. When the adversary has chosen a finite set, with probability 1 the learner draws a sample labeled 0, and outputs a hypothesis with 0 error (since the finite set has measure 0). If the adversary chooses the all 1’s function, the learner will always output the all 1’s function.

On the other hand, it is clear that when the adversary chooses the all 1’s function, no matter how many samples the learner draws, there will exist a hypothesis in the class that is poorly approximated by the sample. Namely the hypothesis whose support is given by the support of the sample itself has empirical measure 1, but true measure 0. As a result, this class fails to have the uniform convergence property despite its learnability.

In later sections, we will even see distribution-free models where uniform convergence fails, such as the Partial PAC model \cite{10, 11} which captures realistic scenarios such as learning with margin. Since even the most basic modifications of PAC-learning fail to satisfy uniform convergence, it is clear we need to move beyond the condition to gain a more general understanding of the common phenomenon of equivalence between learning models.

Instead of relying on uniform convergence, our core observation is an equivalence between learning and sample access to a combinatorial object we call a non-uniform cover.

Definition 7.2 (Non-uniform Cover). Let \((X, H)\) be a class over label space \(Y\) and \(L_{X,Y}\) denote the family of all labelings from \(X\) to \(Y\). If \(C\) is a random variable over the power set \(P(L_{X,Y})\) and \(d : L_{X,Y} \times L_{X,Y} \to \mathbb{R}_{\geq 0}\) is a “distance” function between labelings, we call \(C\) a non-uniform \((\varepsilon, \delta)\)-cover of \(H\) with respect to \(d\) if for all \(h \in H\):

\[
\Pr_{T \sim C} \left[ \exists h' \in T : d(h', h) \leq \varepsilon \right] \geq 1 - \delta.
\]

We call \(C\) bounded if its support lies entirely on subsets of size at most some \(k \in \mathbb{N}\), and we call the smallest such \(k\) its size.

Non-uniform covers share a close connection to several notions of covering used throughout the learning literature such as uniform covers \cite{17} and fractional covers \cite{41}. We discuss these connections in more detail in Appendix C. For the moment, we note only that previous works using the strictly stronger notion of uniform covering necessarily lose factors in the sample complexity as a result. We discuss this further in Section 8.3 as well.

In Section 2 we argued (at least implicitly) that once we have sampling access to a bounded non-uniform cover, agnostic learnability follows from standard arguments. Namely since a sample \(T\) has bounded size

\[\text{In this setting only a single marginal distribution } D \text{ is allowed.}\]
and is guaranteed to contain a concept “close” to optimal, it suffices to run empirical risk minimization over about $\log(|T|/\delta)/\varepsilon^2$ samples. The key to our reduction therefore boils down to turning blackbox access to a realizable PAC-learner into sampling access to some relevant non-uniform cover. This is given by Step 2 of Algorithm 1 which we rewrite here as a subroutine called LEARNINGTOCOVER.

**Algorithm 2: LEARNINGTOCOVER($H$, $A$, $S_U$)**

**Input:** Hypothesis Class $H$, Realizable PAC-Learner $A$, Unlabeled Sample size $S_U$.

**Algorithm:**

1. Run $A$ over all possible labelings of $S_U$ by $H$.
2. Return the set of responses $C(S_U)$:
   \[
   C(S_U) := \{A(S_U, h(S_U)) \mid h \in H\}_{S_U}.
   \]

In fact, we already argued in Section 2 that LEARNINGTOCOVER gives sampling access to a non-uniform cover, but we will re-write the result here in this formulation for convenience.

**Lemma 7.3** (Core Lemma: Realizable Learning Implies Non-Uniform Covering). Let $A$ be an algorithm that $(\varepsilon, \delta)$-PAC learns a class $(\mathcal{D}, X, H, \ell)$ in $n = n(\varepsilon, \delta)$ samples. Then for any $D \in \mathcal{D}$, running LEARNINGTOCOVER on $S_U \sim D^n$ returns a sample from a size $\Pi_H(n)$, non-uniform $(\varepsilon, \delta)$-cover with respect to the standard distance between hypotheses:

\[
d(h, h') = E_D[\ell(h'(x), h(x))].
\]

**Proof.** The proof is essentially immediate from the definition of realizable PAC-learning. $A$ promises that for any $h \in H$ and $D \in \mathcal{D}$, a $1 - \delta$ fraction of labeled samples $(S_U, h(S_U)) \sim D^n$ satisfy

\[
\text{err}_{D \times h, \ell}(A(S_U, h(S_U))) = E_D[\ell(h'(x), h(x))] \leq \varepsilon,
\]

where $h' = A(S_U, h(S_U))$. Since $C(S_U)$ contains $A(S_U, h(S_U))$ for every $h$ by definition, the result follows.

This means that as long as we have blackbox access to a realizable PAC-learner and unlabeled samples from the adversary’s distribution, we can simulate access to a non-uniform cover. Let’s now formalize our previous intuition that this is sufficient to turn a realizable learner into an agnostic one for any finite label class. We will generalize this result to doubly-bounded loss in Appendix A but it is instructive to consider the setting of finite $Y$ first.

**Theorem 7.4** (Agnostic $\rightarrow$ Realizable (Finite Label Classes)). Let $(\mathcal{D}, X, H, \ell)$ be any class on a finite label space $Y$ with loss function $\ell: Y \rightarrow Y$ satisfying the identity of indiscernibles. Then Algorithm 1 is an agnostic learner with sample complexity:

\[
m(\varepsilon, \delta) \leq n(\eta_{\ell} \varepsilon, \delta/2) + O\left(\log \left(\frac{\Pi_H(n(\eta_{\ell} \varepsilon, \delta/2))}{\delta \varepsilon^2}\right)\right)
\]

where $\eta_{\ell} \geq \Omega\left(\min_{a \neq b}(\ell(a,b)) / \max_{a \neq b}(\ell(a,b))\right)$ is a constant depending only on $\ell$. 17
Proof. Let \( A \) be the promised realizable learner for \((X, H, \mathcal{D}, \ell)\) with sample complexity \( n(\varepsilon, \delta) \). Run LEARNINGTOCOVER with parameters \( \varepsilon' = \eta \varepsilon \) and \( \delta' = \delta/2 \). We argue that the output contains some \( h' \) such that \( \text{err}_{D, \ell}(h') \leq OPT + \varepsilon/2 \). Since \( C(S_U) \) is finite and \( \ell \) is upper bounded, a standard Chernoff bound gives that choosing an empirical risk minimizer from \( C(S_U) \) based on \( O \left( \frac{\log \left( \frac{|C(S_U)|}{\varepsilon} \right)}{\varepsilon^2} \right) \) additional samples gives the desired learner. The sample complexity then follows immediately from the definition of \( C(S_U) \).

To see why \( C(S_U) \) has this property, recall that for any \( h \in H \), Lemma 7.3 states that \( C(S_U) \) contains some \( h' \) such that:

\[
\mathbb{E}_{x \sim D_X} [\ell(h'(x), h(x))] \leq \eta \varepsilon.
\]

Because we assume that \( \ell(a, b) = 0 \) iff \( a = b \), this actually implies a stronger relation—\( h \) and \( h' \) must be close in classification error:

\[
\Pr_{x \sim D_X} [h(x) \neq h'(x)] \leq \frac{\eta \varepsilon}{\min_{a \neq b}(\ell(a, b))}.
\]

Let \( h_{OPT} \in H \) be an optimal hypothesis, and let \( h'_{OPT} \) denote the corresponding output of LEARNINGTOCOVER. Then by the above, we have that:

\[
\text{err}_{D, \ell}(h'_{OPT}) &= \mathbb{E}_{(x, y) \sim D} [\ell(h'_{OPT}(x), y)] \\
&= \mathbb{E}_{(x, y) \sim D} [\ell(h'_{OPT}(x), y) - \ell(h_{OPT}(x), y) + \ell(h_{OPT}(x), y)] \\
&= OPT + \mathbb{E}_{(x, y) \sim D} [\ell(h'_{OPT}(x), y) - \ell(h_{OPT}(x), y)] \\
&\leq OPT + \Pr_D [h_{OPT}(x) \neq h'_{OPT}(x)] \max_{a \neq b}(\ell(a, b)) \\
&\leq OPT + \varepsilon/2
\]

where we have used the assumption that we set \( \eta \ell = c \min_{a \neq b}(\ell(a, b)) / \max_{a \neq b}(\ell(a, b)) \) for some universal \( c > 0 \).

It’s worth spending a moment discussing our only assumption on the loss function \( \ell \), that it satisfies the identity of indiscernibles. This is not only a natural assumption for most cases in practice (that mislabeling has non-zero error), it is theoretically justified as well: realizable and agnostic learning aren’t necessarily equivalent for \( \ell \) without this property.

**Proposition 7.5 (Identity of Indiscernibles Lower Bound).** There exists a realizable class \((X, H, \ell)\) over a finite label space \( Y \) which is not agnostically learnable.

**Proof.** Let the instance space \( X = \mathbb{N} \) be the set of natural numbers, the label space \( Y = \{0, 1\}^2 \). We consider the hypothesis class \( H \) with all functions which output the first bit as 0, that is:

\[
H = \{ h : h(x) = (0, \cdot) \ \forall x \in X \}.
\]

Furthermore, we define the loss function \( \ell : Y \times Y \to \{0, 1, c\} \) as

\[
\ell((b_1, r_1), (b_2, r_2)) = \begin{cases} 
0 & b_1 = b_2 \\
1 & b_1 \neq b_2 \text{ and } r_1 = r_2 \\
c & \text{otherwise.}
\end{cases}
\]

Note that \((X, H, \ell)\) is trivially learnable in the realizable setting simply by returning any \( h \in H \). On the other hand, we will show it is only \( O(c) \)-agnostically learnable. First, notice that for any labelling \( f : X \to Y \),
there exists a hypothesis $h \in H$ which matches $f$ on the second bit, and therefore for any marginal $D$ over $X$:

$$OPT \leq \text{err}_{D,f}(h) \leq 1.$$ 

As a result, it suffices to show that for every $m \in \mathbb{N}$ and (randomized) algorithm $A$ using $m$ samples there exists a labeling $f : X \rightarrow Y$ and marginal distribution $D_X$ such that

$$E_{S \sim D^S_X}E_{x \sim D_X}[\ell(A(S,f(S))(x),f(x))] \geq c/12.$$ \hspace{1cm} (1)

As long as this holds Markov’s inequality gives that every algorithm must have error at least $\Omega(c)$ with constant probability.

For simplicity, we will restrict our attention in the rest of the proof to the marginal distribution $D_X$ which is uniform over the set $[k]$ for some natural number $k$ we will fix later. To prove Equation (1), by Yao’s minimax principle it is enough to prove there is a distribution $\mu$ over functions $f : [k] \rightarrow Y$ such that any deterministic algorithm $A$ has expected loss at least $c/12$ over $\mu$:

$$E_{f \sim \mu}E_{S \sim D^S_X}E_{x \sim D_X}[\ell(A(S,f(S))(x),f(x))] > c/12.$$ 

We now show that the above holds for $\mu$ being uniform over all functions from $[k]$ to $Y$ for any $k > 2m$. Here, we have that

$$E_{x \sim D_X}E_{f \sim \mu}E_{S \sim D^S_X}[\ell(A(S,f(S))(x),f(x))] \geq E_{x \sim D_X} \left[ \Pr_{S \sim D^S_X}[x \notin S] \cdot c/4 \right],$$

where the last step follows from noting that for any value $(a,b)$ that $A(S,f(S))$ assigns to $x \notin S$, $f(x)$ will be $(1-a,1-b)$ with probability $1/4$ incurring a loss of $c$. The result then follows by noting that for every $x \in [k]$:

$$\Pr_{S \sim D^S_X}[x \notin S] = \left(1 - \frac{1}{k}\right)^m \geq 1/e$$

since $1 - x \geq \exp\{-x/(1 - x)\}$ and $k > 2m$. Therefore, we get that

$$E_{x \sim D_X}E_{f \sim \mu}E_{S \sim D^S_X}[\ell(A(S,f(S))(x),f(x))] \geq c/4e,$$

which completes the proof.  

Note that this bound holds even if $A$ is allowed to be improper. It is worth noting that if we are willing to increase the size of $Y$, the learner’s error in this bound can actually be increased all the way to $c$, the maximum possible (see Proposition 8.1). This is in fact tight, as we will show that any loss function like the above satisfying a $c$-approximate triangle inequality can be $c$-agnostically learned (that is learned to within $c \cdot OPT + \varepsilon$ error).

8 Four Modification Archetypes

8.1 Discretization: Infinite Label Classes

In the previous section, we showed that our base reduction characterizes the equivalence of realizable and agnostic learning for loss functions satisfying the identity of indiscernibles for all finite label classes. In this section, we discuss a technique called discretization that allows our reduction to extend this result to infinite label classes. Normally, it’s clear that when $Y$ is infinite our standard reduction will fail: since the total number of possible labelings of a finite sample may be infinite, LEARNINGTOCOVER may output an infinite set. In fact, this is more than a technical barrier: realizable and agnostic learning simply aren’t equivalent for infinite label classes.
Proposition 8.1. Let $\ell$ be any continuous loss function (in the first variable) over $\mathbb{R}$ satisfying the identity of indiscernibles. Then there exists a class $(\mathcal{D}, X, H, \ell)$ which is realizably learnable but not agnostically learnable.

Proof. Let $X = \mathbb{N}$ and $Y = [0, 2]$. To construct our class, we first consider the set of all boolean functions over $X$ with finite support. Each function $f$ in this class may equivalently be thought of as a binary string in $\{0, 1\}^n$. Denote the corresponding decimal value of this string in $[0, 1]$ by $s_f$. To construct $H$, for every boolean function $f : \mathbb{N} \rightarrow \{0, 1\}$ with finite support, include in $H$ the function $h_f(x) = f(x) + s_f$. Note that $(X, H)$ is clearly realizably learnable under any distribution family $\mathcal{D}$ and any loss function, since a single sample always uniquely determines $h_f$. On the other hand, adding even the smallest amount of noise erases this unique identification, making the class impossible to learn.

More formally, let $\mathcal{D}$ be the family of all distributions. Since $\ell$ is continuous in the first variable and $\ell(0, 0) = \ell(1, 1) = 0$, there exists some $\gamma = \gamma(\varepsilon)$ such that $\max_{0 \leq \gamma' \leq \gamma} \{\ell(\gamma', 0), \ell(1 + \gamma', 1)\} < O(\varepsilon)$. Let $n_\gamma \in \mathbb{N}$ be the index of the first non-zero digit in the binary representation of $\gamma$. The key is to notice that any boolean function $f$ which is 0 on $[n_\gamma]$ satisfies $\text{OPT}_H(f) \leq O(\varepsilon)$. The argument now essentially boils down to standard VC-lower bounds. By Yao’s minimax principle it is enough to show that for any potential sample complexity $m(\varepsilon, \delta)$, there exists a randomized strategy for the adversary such that the learner cannot achieve $1 - \varepsilon$ accuracy with some constant probability.

To this end, consider the following strategy: the adversary chooses the uniform distribution over $[n_\gamma, n_\gamma + 2m(\varepsilon, \delta)]$, and a binary function on that interval uniformly at random. Since the learner can only see 1/2 of the mass, any strategy must be incorrect on half of the remaining points in expectation. The only subtlety here is that $\ell$ is not binary classification error. However it is still the case that any $y \in \mathbb{R}$ chosen by the learner on these points incurs $\max_{y \in \mathbb{R}} \{\ell(y, 0), \ell(y, 1)\}$ error with constant probability. Let $\ell_{\min-	ext{err}} = \min_{y \in \mathbb{R}} \max_{y \in \mathbb{R}} \{\ell(y, 0), \ell(y, 1)\}$. This quantity exists and is bounded away from 0 since $\ell$ is continuous and satisfies the identity of indiscernibles. The learner then incurs at least $\ell_{\min-	ext{err}}$ on each point outside of its sample with probability 1/2. Since $\ell_{\min-	ext{err}}$ is just some constant, setting $\varepsilon \leq O(\ell_{\min-	ext{err}})$ gives the desired result. \qed

Proposition 8.1 relies crucially on the fact that the adversary can erase a significant amount of information with a very small label perturbation. In the rest of this section, we’ll discuss a technique for modifying our reduction that shows this is essentially the only barrier between realizable and agnostic learning (at least for a broad class of loss functions). The key is to require a slightly stronger notion of learnability based upon discretization.

Definition 8.2 (Discretization). We say $(\mathcal{D}, X, H', \ell)$ is an $\varepsilon$-discretization of $(\mathcal{D}, X, H, \ell)$ if the following three conditions hold:

1. $H'$ is probably bounded. That is for all $n \in \mathbb{N}$, $\delta > 0$, and $D \in \mathcal{D}$ there exists a bound $m(n, \delta) \in \mathbb{N}$ such that:

\[
\Pr_{S \sim D^n} [\text{Im}(H'|_S) \leq m(n, \delta)] \geq 1 - \delta.
\]

2. $H'$ point-wise $\varepsilon$-covers $H$ with respect to $\ell$. That is for all for all $h \in H$, there exists $h' \in H'$ satisfying:

\[
\forall x \in X : \ell(h'(x), h(x)) \leq \varepsilon.
\]

3. $H'$ is always useful. That is for all $h' \in H'$, there exists $h \in H$ such that:

\[
\forall x \in X : \ell(h'(x), h(x)) \leq \varepsilon.
\]

We note that in most cases this condition can generally be weakened to the expected guarantee $\mathbb{E}_{x \sim D_X} [\ell(h'(x), h(x))] \leq \varepsilon$, but the stronger notion is needed for some applications such as adversarial robustness.
Note that most realistic settings have reasonable discretizations (e.g. it is enough to have some Lipschitz-like condition and a weak tail-bound on the loss). We now define a basic notion of learnability based on discretization which essentially serves to rule out adversarial constructions in the vein of Proposition 8.1.

**Definition 8.3 (Discretely-learnable).** We say \((\mathcal{D}, X, H, \ell)\) is discretely-learnable with sample complexity \(n(\varepsilon, \delta)\) if there is some constant \(c_1 > 0\) such that for all \(\varepsilon, \delta > 0\) there exists an \(\varepsilon\)-discretization \(H_\varepsilon\) which is \((c_1 \varepsilon, \delta)\)-PAC-learnable in at most \(n(\varepsilon, \delta)\) samples. We call the learner proper if it outputs hypotheses in \(H\).

We’ll prove that discrete and agnostic learnability are equivalent as long as the loss satisfies an approximate triangle inequality.

**Definition 8.4 (Approximate pseudometric).** We call a loss function \(\ell : Y \rightarrow Y\) a \(c\)-approximate pseudometric if for all triples \(y_1, y_2, y_3 \in Y\):

\[
\ell(y_1, y_3) \leq c(\ell(y_1, y_2) + \ell(y_2, y_3)).
\]

Approximate pseudometrics are natural choices for loss functions in practice and capture a broad set of scenarios including finite-range losses and standard setups such as \(\ell_p\)-regression, and have seen some previous study in the literature [42]. By modifying the first step of our reduction to take discretization into account and leveraging the approximate triangle inequality in the second, we prove that discrete learnability and \(c\)-agnostic learnability are equivalent under \(c\)-approximate pseudometrics.

**Theorem 8.5.** Let \(\ell : Y \times Y \rightarrow \mathbb{R}_{\geq 0}\) be a bounded \(c\)-approximate pseudometric. Then the following are equivalent for all \((\mathcal{D}, X, H, \ell)\):

1. \((\mathcal{D}, X, H, \ell)\) is discretely-learnable.
2. \((\mathcal{D}, X, H, \ell)\) is \(c\)-agnostically learnable.

**Proof.** The proof is similar to Theorem 7.4. We first show the forward direction. Assume \((\mathcal{D}, X, H, \ell)\) is discretely-learnable. Fix \(\varepsilon' = \frac{\varepsilon}{4c^2}c_1\) (where \(c_1\) is the constant from Definition 8.3), and let \(H_{\varepsilon'}\) be a learnable \(\varepsilon'\)-discretization of \(H\). We argue that running LEARNINGTOCOVER on \(H_{\varepsilon'}\) gives the desired agnostic learner. Since \(\ell\) is bounded, it is sufficient to prove that \(C(S_U)\) contains a hypothesis \(h'\) such that:

\[
\text{err}_{D,\ell}(h') \leq c \cdot OPT + \varepsilon/2.
\]

Empirical risk minimization then works as in the finite case.

Let \(h_{OPT} \in H\) be an optimal hypothesis. Since \(H_{\varepsilon'}\) is a discretization of \(H\), there exists \(h_{OPT}^{\varepsilon'} \in H_{\varepsilon'}\) such that:

\[
\forall x \in X : \ell(h_{OPT}(x), h_{OPT}^{\varepsilon'}(x)) < \varepsilon'.
\]

Further, by the guarantees of discrete learnability, with probability at least \(1 - \delta/2\) there exists \(h' \in C(S_U)\) such that close to \(h_{OPT}^{\varepsilon'}\) in the following sense:

\[
\mathbb{E}_{x \sim D_X} [\ell(h'(x), h_{OPT}^{\varepsilon'}(x))] \leq c_1 \varepsilon' = \frac{\varepsilon}{4c^2}.
\]

Plugging in the previous observation and applying our approximate triangle inequality, we get that \(h'\) is close to \(h_{OPT}\) in the following sense:

\[
\mathbb{E}_{x \sim D_X} [\ell(h'(x), h_{OPT}(x))] \leq c \left( \mathbb{E}_{x \sim D_X} [\ell(h'(x), h_{OPT}(x))] + \mathbb{E}_{x \sim D_X} [\ell(h_{OPT}^{\varepsilon'}(x), h_{OPT}(x))] \right) \leq \frac{\varepsilon}{2c}.
\]
The final step is to transfer from the marginal $D_X$ to the full joint distribution of the adversary, which follows immediately from a similar application of the approximate triangle inequality. This is the only step that loses a factor in the $OPT$ term:

$$
err_{D,\ell}(h') = \mathbb{E}_{(x,y) \sim D}[\ell(h'(x), y)]
$$

$$
\leq c \left( \mathbb{E}_{(x,y) \sim D}[\ell(h'(x), h_{OPT}(x))] + \mathbb{E}_{(x,y) \sim D}[\ell(h_{OPT}(x), y)] \right)
$$

$$
\leq c \cdot OPT + \varepsilon/2
$$
as desired.

We now prove the reverse direction, which is essentially immediate. Assume the existence of a $c$-agnostic learner for $(\mathcal{D}, X, H, \ell)$. Given a discretization $H_\varepsilon$, we want to show $(\mathcal{D}, X, H_\varepsilon, \ell)$ is learnable to within $c_1 \varepsilon$ error for some $c_1 > 0$. This is achieved simply by running the agnostic learner on $(\mathcal{D}, X, H_\varepsilon)$. Since $H_\varepsilon$ is “always useful”, every $h \in H_\varepsilon$ is $\varepsilon$-close to some $h' \in H$ in the sense that:

$$
\forall x \in X : \ell(h'(x), h(x)) \leq \varepsilon.
$$

In particular, this means that for any choice of $h$ by the adversary there exists $h' \in H$ with low error:

$$
err_{D,\ell}(h') = \mathbb{E}[\ell(h'(x), h(x))] \leq \varepsilon.
$$

As a result, running the $c$-agnostic learner for $(\mathcal{D}, X, H, \ell)$ returns a hypothesis of at most $(c + 1)\varepsilon$ error with high probability. 

It is worth noting that bounded loss is not really necessary for Theorem 8.5. More generally we can require that $(\mathcal{D}, X, H, \ell)$ is “finitely learnable” in the sense that for all finite subsets $H' \subset H$, $(\mathcal{D}, X, H', \ell)$ is agnostically learnable. When $\ell$ is bounded, this is true for any finite class by empirical risk minimization.

It is also worth noting that various modifications to the definition of loss (e.g. defining loss between hypotheses rather than on $Y$ directly) will continue to work with the above. Similarly, there are various cases when one can get better than $c \cdot OPT$ accuracy for a $c$-approximate pseudometric, generally by instead optimizing over some surrogate loss function. For instance, if a simple transformation of the loss gives a $c'$-approximate pseudometric for $c' < c$, then one can generally learn up to $c' \cdot OPT$.

As an example, note that while square loss $\ell_2(x, y) = (x - y)^2$ is a 2-approximate pseudometric, taking $\sqrt{err_{D,\ell_2}}$ gives a true metric between hypotheses. As a result, as long as $OPT$ is bounded, we can get truly agnostic learning by optimizing $\sqrt{err_{D,\ell_2}}$ instead. This strategy works for any polynomial loss, such as $\ell_p(x, y) = |x - y|^p$.

On the other hand, outside of these special cases, Theorem 8.5 is tight: there exist $c$-approximate pseudometric loss functions which cannot be $c'$-agnostically learned for any $c' < c$. The argument is similar to Proposition 8.1, but requires a bit more care.

**Proposition 8.6.** There exists a discretely-learnable class over a $c$-approximate pseudometric that is not $c'$-agnostically learnable for any $c' < c$.

**Proof.** The proof is similar to Proposition 7.5. We consider the same instance space $X = \mathbb{N}$ and hypothesis class $H$:

$$
H = \{ h : h(x) = (0, \cdot) \forall x \in X \}.
$$

---

13This does require that $OPT$ is bounded.
The loss function $\ell: Y \times Y \rightarrow \{0, 1, c\}$ is also the same, but extended to the larger domain $Y = \mathbb{N}^2$:

$$\ell((b_1, r_1), (b_2, r_2)) = \begin{cases} 0 & b_1 = b_2 \\ 1 & b_1 \neq b_2 \text{ and } r_1 = r_2 \\ c & \text{otherwise.} \end{cases}$$

As before, note that $(X, H, \ell)$ is trivially realizable learnable by always returning any $h \in H$, $\ell$ is a c-psuedometric by definition, and for any labeling $f: X \rightarrow Y$ there exists $h \in H$ such that for all distributions $D$:

$$OPT \leq err_{D,f}(h) \leq 1.$$ 

We now show that the class $(X, H, \ell)$ is only c-agnostically learnable. Since $OPT \leq 1$, it suffices to show that for every $m \in \mathbb{N}$, large enough $n \in \mathbb{N}$, and randomized algorithm $A$ on $m$ samples, there exists a labeling $f: X \rightarrow Y$ and a marginal distribution $D_X$ such that:

$$\mathbb{E}_{S \sim D_X^m} \mathbb{E}_{x \sim D_X} [\ell(A(S, f(S))(x), f(x))] \geq \left(1 - \frac{1}{n}\right)^3 c.$$  

(2)

For $n \geq \frac{1}{1-(1-(c-c')/c)2^m}$, applying Markov’s inequality to Equation (2) implies that $A$ has error at least $c'$ with constant probability.

For simplicity, we now restrict our attention to the marginal $D_X$ which is uniform over the set $[k]$ for some $k \in \mathbb{N}$ to be fixed. By Yao’s minimax principle, its enough to prove that there exists a distribution $\mu$ over functions $f: [k] \rightarrow [n]^2$ such that any deterministic algorithm $A$ the following holds

$$\mathbb{E}_{f \sim \mu} \mathbb{E}_{S \sim D_X^m} \mathbb{E}_{x \sim D_X} [\ell(A(S, f(S))(x), f(x))] \geq \left(1 - \frac{1}{n}\right)^3 c.$$ 

We now show that the above holds for $\mu$ being uniform over all functions from $[k]$ to $[n]^2$ when $k > \frac{2m}{\ln(n/(n-1))}$. Similar to Proposition 7.5 we have that

$$\mathbb{E}_{x \sim D_X} \mathbb{E}_{f \sim \mu} \mathbb{E}_{S \sim D_X^m} [\ell(A(S, f(S))(x), f(x))] \geq \mathbb{E}_{x \sim D_X} \left[ \Pr_{S \sim D_X^m} [x \notin S] \cdot \left(1 - \frac{1}{n}\right)^2 \cdot c \right],$$

since no matter the assignment $A$ gives to $x \notin S$, it will be wrong on both coordinates with probability $(1 - 1/n)^2$ over the randomness of $\mu$. The result follows by noting that for every $x \in [k]$

$$\Pr_{S \sim D_X^m \times f(X)} [(x, \cdot) \notin S] = \left(1 - \frac{1}{k}\right)^m \geq 1 - \frac{1}{n}$$

since $1 - x \geq \exp\{-x/(1-x)\}$, and we have assumed $k > \frac{2m}{\ln(n/(n-1))}$ and $n > 1$. Therefore, we get that

$$\mathbb{E}_{x \sim D_X} \mathbb{E}_{f \sim \mu} \mathbb{E}_{S \sim D_X^m} [\ell(A(S, f(S))(x), f(x))] \geq \left(1 - \frac{1}{n}\right)^3 c$$

which completes the proof.

\(\square\)

### 8.2 Sub-sampling: Malicious Noise

Now that we’ve seen how to handle practical problems like regression over infinite label spaces, we’ll discuss a technique that helps handle data corruption and data-dependent assumptions: sub-sampling. The main idea is as follows. Say that the original unlabeled sample we draw is, in some sense, partially corrupted: perhaps an adversary has changed some fraction of examples (malicious noise), or some portion of the sample is

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un-realizable for a concept in the class (robust and partial learning). In either case, there generally exists a core subset of “clean” samples that we can use to recover the guarantees of LEARNINGToCover. Since we cannot necessarily identify these, the idea is to run LEARNINGToCover over enough subsets of the unlabeled sample that we find a clean subsample with high probability. In this section we’ll discuss the application of this technique in detail to Kearns and Li’s [14] well-studied malicious noise model. In the appendix, we discuss applications to recently popular adversarially robust (Appendix B) and partial learning (Appendix C) models.

To start, let’s recall the standard malicious noise model. In this variant of PAC learning, instead of having access to the standard sample oracle from the adversary’s distribution $D$ over $X \times Y$, we have access to a malicious oracle $O_M(\cdot)$ which, with probability $\eta$, outputs an adversarially chosen pair $(x, y)$, and otherwise samples from $D$ as usual.

**Definition 8.7 (PAC-learning with Malicious Noise).** We call $(X, H, D, \ell)$ (agnostically) $(\varepsilon, \delta)$-learnable with malicious noise at rate $\eta = \eta(\varepsilon)$ if there exists an algorithm $A$ and function $m = m_{\text{mal}}(\varepsilon, \delta)$ such that for all $\varepsilon, \delta > 0$, and distributions $D$ over $X \times Y$ satisfying:

1. The marginal $D_X \in D$,

$A$ outputs a good hypothesis with high probability over samples drawn from the malicious oracle of size $n(\varepsilon, \delta)$:

$$\Pr_{S \sim O_M(\cdot)^m}[err_{D, \ell}(A(S)) > OPT + \varepsilon] \leq \delta.$$  

where $OPT = \min_{h \in H} \{err_{D, \ell}(h)\}$.

In other words, malicious noise essentially gives a worst-case formalization of the idea that an $\eta$-fraction of the learner’s data is (adversarial) garbage.

Let’s now formalize the argument above: modifying LEARNINGToCover to run over subsamples gives a sample-efficient algorithm for learning with malicious noise. For readability, we’ll (somewhat informally) restate the algorithm with this change.

**Algorithm 3: Malicious to Realizable Reduction**

**Input:** Realizable PAC-Learner $A$, Accuracy Parameter $\varepsilon < 1/2$, Noise Parameter $\eta < \frac{\varepsilon}{1+\varepsilon}$, Unlabeled Sample Oracle $O_U$, Labeled Sample Oracle $O_L$

**Algorithm:**

1. Draw an unlabeled sample $S_U \sim O_U$, and labeled sample $S_L \sim O_L$.

2. Run LEARNINGToCover over all $S \in (S_U)_{\eta'} := \{S \subseteq S_U : |S| = \lfloor (1 - \eta') |S_U| \rfloor\}$, where:

$$\eta' = \frac{3\eta + \varepsilon/(1-\varepsilon)}{4}.$$  

3. Return the hypothesis in $C(S_U) := \bigcup_{S \in (S_U)_{\eta'}} C(S)$ with lowest empirical error over $S_L$.

We now prove that Algorithm 3 gives an (agnostic) learner that is tolerant to malicious noise.

**Theorem 8.8.** Let $(D, X, H)$ be realizably PAC-learnable with sample complexity $n(\varepsilon, \delta)$. Then for any $\eta < \frac{\varepsilon}{1+\varepsilon}$, Algorithm 3 is an agnostic learner for $(D, X, H)$ tolerant to $\eta$ malicious noise. Furthermore
letting $\Delta = \frac{\varepsilon}{1+\varepsilon} - \eta$ and $\beta = (1 + \frac{\eta}{\Delta}) \log \left( \frac{1}{\eta} \right)$, its sample complexity is at most:

$$m_{\text{mal}}(\varepsilon, \delta) \leq O \left( \beta^{n(\Delta/4, \delta/4)} + \log \left( \Pi_{\text{H}} \left( O \left( n(\Delta/2, \delta/2) + \frac{n^2 \log(1/\delta)}{\Delta^2} \right) \right) \right) + \log(1/\delta) + \frac{\beta \eta^2 \log(1/\delta)}{\Delta^3} \right)$$

where we’ve assumed $\varepsilon < 1/2$ for simplicity.

Proof. To start, we’ll review for completeness a fairly standard analysis of empirical risk minimization under malicious noise. Assume for the moment that the output of \textsc{LearningToCover}, $C(S_U)$, contains a hypothesis $h'$ satisfying $err(h') \leq OPT + \beta_1$. Say we draw $M$ labeled samples for the ERM step, and an $\eta' = \eta + \beta_2$ fraction are corrupted by the adversary. For large enough $M$, we can assume by a Chernoff and union bound that the empirical loss of every hypothesis returned by \textsc{LearningToCover} is at most some $\beta_3$ away from its true loss on the un-corrupted portion of $M$ (we will make all these assumptions formal in a moment). Given these facts, notice that the empirical loss of $h'$ on $M$ is at most:

$$err_M(h') \leq (1 - \eta')(OPT + \beta_1 + \beta_3) + \eta'.$$

On the other hand, the empirical error of any $h_\varepsilon \in H$ whose true error is greater than $OPT + \varepsilon$ is at least:

$$err_M(h) > (1 - \eta')(OPT + \varepsilon - \beta_3).$$

To ensure that our ERM works, it is enough to show that for any such $h$, $err_M(h) > err_M(h')$. A simple calculation shows that this is satisfied as long as $\beta_1 + 2\beta_3 \leq \varepsilon$ and $\beta_1 + \beta_2 + 2\beta_3 \leq \Delta$. Setting $\beta_1 = \beta_2 = \beta_3 = \Delta/4$ gives the desired result.

It is left to argue that our assumptions above hold with high probability. First, note that by a Chernoff bound, the probability that $\eta' \geq \eta + \Delta/4$ on a set of $M$ samples is at most $e^{-c(\frac{\Delta}{\eta})^2 M}$ for some $c > 0$, so this occurs with high probability as long as $M \geq \Omega(\log(1/\delta)\eta^2/\Delta^2)$. Similarly, the empirical error of every hypothesis in $C(S_U)$ on the remaining clean samples will be within $\Delta/4$ of its true error as long as $M \geq \Omega(\log(|C(S_U)|/\delta)\eta^2/\Delta^2)$.

It is then left to show that $C(S_U)$ contains a hypothesis of error at most $OPT + \Delta/4$. To show this, it is enough to ensure that we run \textsc{LearningToCover} over a clean subsample of size at least $n(\Delta/4, \delta/4)$ with high probability. If we draw $|S_U| = O \left( \frac{n(\Delta/4, \delta/4)}{1+\eta'} \log(1/\delta) \frac{n^2 \eta}{\Delta^2} \right)$ unlabeled samples, a similar Chernoff bound to the above promises that at most an $\eta'$ fraction are corrupted with high probability, and therefore that at least $n(\Delta/4, \delta/4)$ remain un-corrupted. Running \textsc{LearningToCover} over all subsets of size $(1 - \eta')|S_u|$ then gives the desired result. The sample complexity bound follows from choosing $M$ large enough to satisfy the above conditions along with the fact that $|C(S_U)| \leq (\frac{n}{1-\eta'}|n| \Pi_{\text{H}}(n))$. \hfill \Box

It’s worth noting that the error tolerance of Theorem 8.8 is tight. In their original introduction of malicious noise, Kearns and Li [14] proved that for most non-trivial concept classes, no PAC-learner can be tolerate $\frac{\varepsilon}{1+\varepsilon}$ malicious noise. Theorem 8.8 also extends to other scenarios we’ve seen so far such as arbitrary loss over finite label classes and approximate pseudometrics. The proof remains mostly the same, though the optimal error tolerance may differ.

Since our agnostic model restricts the adversary to choosing a distribution whose marginal lies in the original family, Theorem 8.8 provides the first insight on robustness against an adversary who can corrupt the underlying data as well as the labels. One might wonder whether this result can be pushed further: is it possible to be robust against an adversary who can corrupt the marginal over $X$ in some stronger sense? Unfortunately, the answer is no: malicious noise is necessarily the most distributional corruption we can handle. Let’s look at two basic lower bounds to see why. First, we’ll consider an adversary who can remove a portion of the learner’s sample.

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Proposition 8.9. For any $\delta > 0$, there exists a class $(\mathcal{D}, X, H)$ which is realizable PAC-learnable, but not learnable under an adversary who can remove a $\delta$ fraction of the learner's sample.

Proof. This follows from a result of Dudley, Kulkarni, Richardson, and Zeitouni [18] that there exists an unlearnable class $(\mathcal{D}, X, H)$ such that for some $n(\varepsilon, \delta)$, $(D, X, H)$ is learnable in $n(\varepsilon, \delta)$ samples for every $D \in \mathcal{D}$. The lower bound then follows simply from adding an extra unique identifying point $x_D$ to $X$ for every distribution $D$, and modifying each $D \in \mathcal{D}$ to have $\Theta(\delta)$ support on $x_D$. This modified class is clearly learnable, since after drawing $O(1/\delta)$ samples, the learner will draw $x_D$ and identify the distribution $D$ with good probability. However, the class is not learnable under an adversary who removes points, since with high probability the adversary can completely remove any mention of $x_D$ from the learner’s sample, reducing to the original unlearnable class $(\mathcal{D}, X, H)$. \hfill \square

An adversary who can add samples is similarly powerful. In the realizable setting, if the adversary is allowed to add an arbitrary number of correctly labeled points to the learner’s sample, basic classifiers such as halfspaces become unlearnable [43]. On the other hand if the adversary is limited to adding only a few additional samples, realizable learning may remain possible, but even trivial concept classes cannot be agnostically learnable.

Proposition 8.10. There exists a class $(X, H)$ which for any $\gamma > 0$ is realizable but not agnostically learnable under an adversary who can add a $\gamma$ fraction of correctly labeled points to the learner’s sample.

Proof. Let $X = \{x, x_1, x_2\}$ and $H = \{h_1, h_2\}$ be any class such that $h_1(x) = h_2(x)$, but $h_1(x_1) \neq h_2(x_1)$ for $i = 1, 2$. In the realizable setting, note that a single labeled example on $x_1$ or $x_2$ exactly determines the hypothesis. As long as there is less than $1 - \varepsilon$ mass on $x$, the learner will draw such a sample after $O(1/\varepsilon)$ samples with good probability. Further, if the mass on $x$ is at least $1 - \varepsilon$, then $h_1$ and $h_2$ are both valid outputs. As a result, any ERM is a valid PAC-learner. Since adding correctly labeled examples can only help this learner, the class remains realizable under an adversary who can add an arbitrary number of clean samples.

In the agnostic setting, consider an adversary who chooses a labeling $f$ such that $f(x) = h_1(x) = h_2(x)$, $f(x_1) = h_1(x_1)$, and $f(x_2) = h_2(x_2)$. The optimal hypothesis $h_{OPT}$ is then decided by the amount of mass on $x_1$ and $x_2$ in the marginal distribution. Namely if the adversary chooses a distribution $D$ over $\{x, x_1, x_2\}$, the optimal error is $\min\{D(x_1), D(x_2)\}$. The idea is then to note that for $\gamma' \leq \gamma/4$, the learner cannot distinguish between the two following distributions:

$$D_1(x_1) = c_1 \gamma', D_1(x_2) = (1 - c_1) \gamma'$$

and

$$D_2(x_1) = (1 - c_1) \gamma', D_2(x_2) = c_1 \gamma'$$

where $1/2 > c_1 > 0$ is some small constant. Informally, if the two distributions are indistinguishable, any learner will always incur error of around $\frac{1 - c_1}{2} \gamma'$, whereas OPT is $c_1 \gamma'$ for both distributions.

Let’s now give the formal argument. By Yao’s Minimax Principle it is enough to prove there is a strategy over distributions such that any deterministic learner has high error. In particular, if we can prove that the expected error is at least $3 \cdot OPT$, then $\Pr[\text{error} \geq 2 \cdot OPT] \geq OPT$. Since OPT is just some constant $c_1 \gamma'$ (dependent only on $\gamma$), this is sufficient to prove the result. Moving on, consider the strategy in which the adversary chooses the labeling described above, and chooses each marginal ($D_1$ or $D_2$) with probability $1/2$. We’ll break our analysis into two cases dependent on the sample complexity of the learner. If the learner uses $O(1/\gamma')$ examples, then there is a constant probability of drawing a sample only consisting of

\footnote{For simplicity, we’ll assume the adversary is restricted to picking a marginal distribution and labeling rather than any joint distribution over $X \times Y$.}
the point \( x \). Let \( f' \) be the hypothesis returned by the deterministic learner on this sample. By construction, \( f' \) must disagree with either \( h_1 \) or \( h_2 \) on \( x_1 \) or \( x_2 \). Assume \( f' \) differs on \( h_1(x_1) \) (the other cases will follow similarly). When the distribution is \( D_2 \), \( f' \) has error at least \((1 - c_1)\gamma'\). Since this occurs with constant probability independent of the choice of \( c_1 \), choosing \( c_1 \) sufficiently small leads to an expected error of at least \( 3c_1\gamma' \) as desired.

On the other hand, when there are \( n = \Omega(1/\gamma') \) samples, we claim that the adversary can force the following sample to occur with constant probability: \( 2\gamma'n \) instances of \( x_1 \) and \( x_2 \), and \((1 + \gamma)n - 4\gamma'n \) instances of \( x \). This follows from the fact that for the appropriate choice of constant for \( n \), a Chernoff bound gives that both \( x_1 \) and \( x_2 \) occur at most \( 2\gamma'n \) times with constant probability. Since the adversary is allowed to add \( \gamma n \geq 4\gamma'n \) arbitrary examples, they can add instances of \( x_1 \), \( x_2 \), and \( x \) until the above sample is achieved. The remainder of the argument is then the same as the previous case, as any learner response on this sample will incur similarly high expected error.

It is also reasonable to consider distributional corruption in the semi-supervised setting, where the unlabeled and labeled data-sets might have different underlying distributions. We discuss this model in Section 8.4.

8.3 Replacing ERM: Semi-Private Learning

So far we have focused on property generalization for two forms of noise-tolerance—agnostic learning and learning with malicious noise. In this section, we’ll show how to use Algorithm 1 to generalize a broader spectrum of finitely-satisfiable properties through replacing the ERM process with a generic finite learner with the desired property. Our prototypical example will be privacy, which is well known to be finitely-satisfiable via McSherry and Talwar’s [20] exponential mechanism. To start, we’ll cover a few basic privacy definitions.

**Definition 8.11 (Differential Privacy).** A learning algorithm \( A \) is said to be \( \alpha \)-differentially private if for all neighboring inputs \( S, S' \) which differ on a single example:

\[
\Pr[A(S) \in T] \leq e^\alpha \Pr[A(S') \in T],
\]

for all measurable events \( T \) in the range of \( A \).

The exponential mechanism is one of the most widely used techniques in privacy. Informally, the algorithm allows for differentially private selection of a “good” choice from a finite set of objects (potential hypotheses in our case). More formally, let \( s : (X \times Y)^* \times H \rightarrow \mathbb{R} \) be a “score” function, and define “sensitivity” \( \Delta_s \) to be

\[
\Delta_s = \max_{h \in H} \max_{S, S'} |s(S, h) - s(S', h)|
\]

where \( S, S' \) are two neighbouring datasets. The exponential mechanism selects an item with a good score with high probability, while maintaining privacy.

**Definition 8.12 (Exponential Mechanism [20]).** The exponential mechanism \( M_E \) on inputs \( S, H, s \) with privacy parameter \( \alpha \) selects and outputs \( h \in H \) with probability

\[
\frac{\exp\left(\frac{\alpha s(S, h)}{2\Delta_s}\right)}{\sum_{h' \in H} \exp\left(\frac{\alpha s(S, h')}{2\Delta_s}\right)}.
\]

It is well known that the exponential mechanism leads to a private learner for finite hypothesis classes under bounded loss.
Theorem 8.13 (Theorem 3.4 [44]). Let \((\mathcal{D}, X, H, \ell)\) be a finite class with a bounded loss function. Then the sample complexity of \(\alpha\)-differentially private learning \((\mathcal{D}, X, H, \ell)\) is at most:

\[
n_{\text{pri}}(\alpha, \varepsilon, \delta) \leq O\left(\log\left(\frac{|H|}{\delta}\right) \max\{\varepsilon^{-2}, \varepsilon^{-1} \alpha^{-1}\}\right).
\]

We note that [44, Theorem 3.4] only considers classification loss, but the extension to bounded loss is immediate. Unfortunately, even with the power of the exponential mechanism, privacy is a very restrictive condition in the general PAC framework, since we're most often interested in infinite hypothesis sets. Indeed even improper private learning requires finiteness of a highly restrictive measure known as representation dimension [45], which can be infinite for classes of VC dimension 1. As a result, the past decade has seen the introduction of a number of weaker, more practical definitions of privacy. In this section we'll focus on a model introduced in 2013 by Beimel, Nissim, and Stemmer [16] called semi-private learning.

Definition 8.14 (Semi-Private Learning). We call a class \((\mathcal{D}, X, H, \ell)\) semi-private PAC-Learnable if there exists an algorithm \(A\) and two functions \(n_{\text{pub}} = n_{\text{pub}}(\varepsilon, \delta, \alpha)\) and \(n_{\text{pri}} = n_{\text{pri}}(\varepsilon, \delta, \alpha)\) such that for all \(\varepsilon, \delta > 0\) and distributions \(D\) over \(X \times Y\) whose marginal \(D_X\) is in \(\mathcal{D}\), \(A\) satisfies the following:

1. \(A\) outputs a good hypothesis with high probability:

\[
\Pr_{S_U \sim D_X^{\text{pub}}, S_L \sim D_X^{\text{pri}}} [\text{err}_{D, \ell}(A(S_U, S_L)) > \text{OPT} + \varepsilon] \leq \delta.
\]

2. \(A\) is semi-private. That is for all \(S_U \in X^{\text{pub}}:\)

\[
A(S_U, \cdot) \text{ is } \alpha\text{-differentially private}.
\]

In other words, semi-private learning offers a model for applications where labeled data is sensitive, but some (perhaps opt-in) users might not care about their participation itself being released. Unlike standard private learning, distribution-free semi-private classification is known to be characterized by VC dimension, just like realizable PAC-learning [16]. The best sample complexity bounds are due to Alon, Bassily, and Moran (ABM) [17], who use uniform convergence to build a uniform cover for \(H\) from unlabeled data, and then apply the exponential mechanism to the resulting cover.

Due to their reliance on uniform convergence, ABM’s techniques fail in the more general settings we consider. Further, their use of uniform covers results in sub-optimal public sample complexity even for distribution-free classification. We prove in Appendix C that these objects require asymptotically more samples than non-uniform covers (at least in the distribution-family model), and therefore cannot be used to achieve optimal semi-private learning. We circumvent both of these issues by appealing directly to a realizable learner to build a weaker non-uniform cover. For readability, we first restate the algorithm here.

**Algorithm 4: Semi-Private to Realizable Reduction**

**Input:** Realizable PAC-Learner \(A\), Unlabeled Sample Oracle \(O_U\), Labeled Sample Oracle \(O_L\)

**Algorithm:**

1. Draw an unlabeled sample \(S_U \sim O_U\), and labeled sample \(S_L \sim O_L\).

2. Run \textsc{LearningToCover} over \(S_U\) to get \(C(S_U)\).

3. \textbf{Return} the hypothesis in \(C(S_U)\) given by applying the exponential mechanism with respect to \(S_L\).

We prove that Algorithm 4 gives a semi-private agnostic learner in the distribution-family setting.
**Theorem 8.15** (PAC-learning implies Semi-Private Learning). Let $\ell : Y \times Y \to \mathbb{R}_{\geq 0}$ be a bounded $c$-approximate pseudometric. Then the following are equivalent for all triples $(\mathcal{D}, X, H, \ell)$:

1. $(\mathcal{D}, X, H, \ell)$ is discretely-learnable
2. $(\mathcal{D}, X, H, \ell)$ is $c$-agnostically, semi-private learnable.

**Proof.** The proof is essentially the same as Theorem 8.5. The only difference in the argument is to replace the generic ERM learner over the output of LEARNINGTOCOVER with the exponential mechanism [44].

Let’s now take a look at what Theorem 8.15 implies about the special case of distribution-free classification.

**Corollary 8.16.** Let $(X, H)$ be a class of VC-dimension $d$ with sample complexity $n(\varepsilon, \delta)$. The sample complexity of (agnostic) semi-private learning $(X, H)$ is at most:

$$n_{pub}(\varepsilon, \delta, \alpha) \leq n(\varepsilon/2, \delta/2)$$

and

$$n_{pri}(\varepsilon, \delta, \alpha) \leq O \left( \left( d + \log \left( \frac{n(\varepsilon/2, \delta/2)}{\varepsilon} \right) \right) \max\{\varepsilon^{-2}, \varepsilon^{-1}\alpha^{-1}\} \right).$$

Further, the sample complexity of improperly (agnostic) semi-private learning $(X, H)$ is

$$n_{pub}(\varepsilon, \delta, \alpha) \leq O \left( \frac{d + \log(1/\delta)}{\varepsilon} \right)$$

and

$$n_{pri}(\varepsilon, \delta, \alpha) \leq O \left( \left( d \log \left( \frac{1}{\varepsilon} \right) + \log \left( \frac{1}{\delta} \right) \right) \max\{\varepsilon^{-2}, \varepsilon^{-1}\alpha^{-1}\} \right).$$

**Proof.** LEARNINGTOCOVER uses $n(\varepsilon/2, \delta/2)$ samples by definition. In the improper case, Hanneke showed that $n(\varepsilon/2, \delta/2) \leq O \left( \frac{d + \log(1/\delta)}{\varepsilon} \right)$. Since the class has VC dimension $d$, the size of the resulting cover is at most $(e \cdot n(\varepsilon/2, \delta/2))/d^d$, and the private sample complexity bound then follows from Theorem 8.13.

This improves over the recent upper bound of ABM who showed that

$$n_{pub}(\varepsilon, \delta, \alpha) \leq O \left( \frac{d \log(1/\varepsilon) + \log(1/\delta)}{\varepsilon} \right).$$

In fact, for constant $d$ and $\delta$, Corollary 8.16 completely resolves the unlabeled sample complexity of semi-private learning, as ABM [17] prove a matching lower bound.

**Theorem 8.17** (Public Lower Bound [17]). Every class that is not privately learnable requires at least $\Omega(1/\varepsilon)$ unlabeled samples to learn in the semi-private model under classification error.

On the other hand, we note that the private sample complexity remains off by a log factor from the best known lower bounds of Chaudhari and Hsu [21].

**Theorem 8.18** (Private Lower Bound [21]). There exist classes of VC dimension $O(1)$ which require at least:

$$n_{pri}(\varepsilon, \delta, \alpha) \geq \Omega \left( \max\{\varepsilon^{-2}, \varepsilon^{-1}\alpha^{-1}\} \right)$$

private samples to learn.
While we have now resolved the public sample complexity of improper learning, it remains an interesting open problem for the proper regime where certain adversarial examples are known to require an extra \( \log(1/\varepsilon) \) factor in the standard PAC sample complexity [46, 47]. We conjecture that Theorem 8.15 should still be tight in this setting: namely that the unlabeled semi-private sample complexity should always be at least the realizable PAC sample complexity.

**Conjecture 8.19.** Let \((X, H)\) be a hypothesis class which is not privately learnable. Then the realizable sample complexity of \((X, H)\) lower bounds the unlabeled sample complexity of semi-private learning:

\[
\overline{n}_{\text{pub}}(\varepsilon, 1/2) \geq \Omega(n(\varepsilon, 1/2)).
\]

### 8.4 Changing the Base Model: Covariate Shift

One issue with semi-supervised models like semi-private learning is that, in practice, the distribution over unlabeled data probably won’t match the labeled data exactly. In this section, we’ll talk about a final modification to our reduction to tackle such scenarios and more generally to extend property generalization beyond the realizable PAC setting: replacing the base learner. In fact, we already saw this strategy used to a lesser extent in Section 8.1, where we replaced our standard realizable base learner with a discrete learner. Here we’ll look at an application in which we assume our initial learner is robust to covariate shift [48], meaning that even if the distribution underlying the data shifts between train and test time, the algorithm will continue to perform well. This stronger assumption will allow us to build semi-private learners that can handle corruption between the public and private databases. To start, let’s formalize covariate shift in the distribution-family model.

**Definition 8.20 (Covariate Shift).** Let \((\mathcal{D}, X, H, \ell)\) be any class, and for every \(\varepsilon > 0\) let \(C_\varepsilon\) be a “covariate-shift” function that maps every \(D \in \mathcal{D}\) to some family of distributions over \(X\). Given any distribution \(D \in \mathcal{D}\) and any \(h \in H\), let the error of a potential labeling be given by its worst-case error over \(C_\varepsilon(D)\), that is:

\[
\text{CS-err}_{D \times h, \ell, \varepsilon}(c) = \max_{D' \in C_\varepsilon(D)} \left\{ \mathbb{E}_{x \sim D'}[\ell(c(x), h(x))] \right\}.
\]

We say that \((\mathcal{D}, X, H, \ell)\) is realizable learnable under covariate shift \(C = \{C_\varepsilon\}\) if there exists an algorithm \(A\) and function \(n = n(\varepsilon, \delta)\) such that for all \(\varepsilon, \delta > 0\), \(D \in \mathcal{D}\), and \(h \in H\):

\[
\Pr_{S \sim \mathcal{D}^n}[\text{CS-err}_{D \times h, \ell, \varepsilon}(A(S, h(S))) > \varepsilon] \leq \delta.
\]

We call such a learner robust to covariate shift.

We emphasize that in the above definition, the covariate shift family scales with the error parameter \(\varepsilon\). This is a bit different than Shimodaira’s original definition [48], but is a natural choice in our context since we consider algorithms which only use access to the original source distribution (sometimes called “conservative domain adaptation” [49]). In this setting, we’d expect that as we demand higher accuracy, the amount of covariate shift we can tolerate will decrease. Indeed in the agnostic model, it’s clear this scaling is necessary by a similar argument to Proposition 8.10.

The key observation to apply learning under covariate shift in our reduction is simply to notice that the non-uniform cover output by **LearningToCover** not only contains a hypothesis close to \(h_{\overline{\alpha}}\) under the unlabeled distribution \(D\), but also close under any shifted distribution in \(C_\varepsilon(D)\). This can then be used to analyze any semi-supervised model where the marginal of the labeled distribution may be corrupted from \(D\) to any distribution in \(C_\varepsilon(D)\). In this section, we’ll again focus on the setting of semi-private learning. First, let’s formalize what it means to be semi-private learnable under covariate shift.
**Definition 8.21** (Semi-Private Learning under Covariate Shift). We call a class $(\mathcal{D}, X, H, \ell)$ semi-private PAC-learnable under covariate shift $C = \{C_\varepsilon\}$ if there exists an algorithm $A$ and two functions $n_{\text{pub}} = n_{\text{pub}}(\varepsilon, \delta, \alpha)$ and $n_{\text{pri}} = n_{\text{pri}}(\varepsilon, \delta, \alpha)$ such that for all $\varepsilon, \delta > 0$ and distributions $D_X \in \mathcal{D}$ and $D$ over $X \times Y$ whose marginal $D_X' \in C_\varepsilon(D_X)$, $A$ satisfies the following:

1. $A$ outputs a good hypothesis with high probability:

\[
\Pr_{S_U \sim D_X^{n_{\text{pub}}}, S_p \sim D^{n_{\text{pri}}}}[\text{err}_{D,\ell}(A(S_U, S_p)) > \text{OPT} + \varepsilon] \leq \delta.
\]

2. $A$ is semi-private. That is for all $S_U \in X^{n_{\text{pub}}}$:

\[
A(S_U, \cdot) \text{ is } \alpha\text{-differentially private}
\]

In other words, the marginal distribution over private data may be shifted from the public data. This is a realistic scenario in practice, since the distribution of “opt-in” users is likely different from the marginal over the total population. We’ll show that this issue is solvable in the semi-private setting as long as the analogous issue in the non-private setting (distribution shift between train and test time) can be resolved.

**Theorem 8.22.** Let $\ell$ be a $c$-pseudometric and $(\mathcal{D}, X, H, \ell)$ be realizably learnable under covariate shift $C = \{C_\varepsilon\}$, then $(\mathcal{D}, X, H, \ell)$ is semi-private ($c$-agnostically) learnable under covariate shift $C$.

**Proof.** The proof is again almost exactly the same as Theorem 8.5. The only difference is to note that, with high probability, there exists $h'$ in the probabilistic cover output by $\text{LEARNINGTOCOVER}$ on distribution $D$ satisfying:

\[
\forall D_X' \in C_\varepsilon(D_X) : \mathbb{E}_{x \sim D_X'}[\ell(h'(x), h_{\text{OPT}}'(x))] \leq \varepsilon/4c^2.
\]

In the standard argument, this was only true for $D_X$ itself. The rest of the argument then follows exactly as in Theorem 8.5 since the only difference is that the marginal of the adversary’s distribution over private data may be any $D_X' \in C(D_X)$.\qed

Since Theorem 8.22 is a bit abstract, let’s take a look at one concrete application. Given a class $(X, H)$, the class-dependent total variation distance is a metric on distributions measuring the worst case distance across elements of $H \Delta H := \{h \Delta h' : h, h' \in H\}$:

\[
\text{TV}_{H \Delta H}(D, D') := \max_{h \in H \Delta H} \{|D(h) - D'(h)|\}.
\]

It is not hard to see that any realizable learner is robust to $O(\varepsilon)$ covariate shift in $\text{TV}_{H \Delta H}$ distance. We can then apply Theorem 8.22 to build a robust semi-private learner.

**Corollary 8.23.** Let $(X, H)$ be a class of VC-dimension $d$, and for every $\varepsilon > 0$ and distribution $D$ over $X$, define a covariate shift function:

\[
C_\varepsilon(D) := \{D' : \text{TV}_{H \Delta H}(D, D') \leq \varepsilon/2\}.
\]

Then $(X, H)$ is semi-private learnable under covariate shift $C = \{C_\varepsilon\}$ in only $n_{\text{pub}}(\varepsilon, \delta, \alpha) \leq O\left(\frac{d + \log(1/\delta)}{\varepsilon}\right)$ unlabeled samples and $n_{\text{pri}}(\varepsilon, \delta, \alpha) \leq O\left(\left(d \log \left(\frac{1}{\varepsilon}\right) + \log \left(\frac{1}{\delta}\right)\right) \max\{\varepsilon^{-2}, \varepsilon^{-1} \alpha^{-1}\}\right)$ labeled samples.
Finally, we note again that our original learner in these results is robust to covariate shift despite having no access to samples from the new distribution. Unfortunately, this model does come with fairly strong lower bounds regarding the type of covariate shifts to which it is possible to be robust [49]. One solution to this problem is to consider a relaxed variant called (non-conservative) domain adaptation, where the learner additionally has access to a small number of unlabeled samples from the test-time distribution. It is certainly possible to define an analog in the semi-private setting, but naively the use of unlabeled data from the private distribution breaks our reduction since privacy won’t be preserved. We leave as an open question whether any sort of PAC-learner in the non-conservative model could imply semi-private learners with stronger robustness to covariate shift. Some progress has been made in this direction recently by Bassily, Moran, and Nandi [50] for distribution-free classification of halfspaces.

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References


A Doubly Bounded Loss

In this section we discuss a natural generalization of loss functions over finite label classes we call doubly bounded loss: for all distinct $y, y' \in Y$, we require $\ell(y, y') \in [a, b]$ for some $b \geq a > 0$. This is trivially satisfied by any loss function on a finite label class that satisfies the identity of indiscernibles.

Definition A.1 (Doubly Bounded Loss). We say $\ell : Y \rightarrow Y$ is $(a, b)$-bounded if there exist $b \geq a > 0$ for any distinct $y, y' \in Y$:

$$\ell(y, y') \in [a, b].$$

As discussed in Section 8.1, since we now allow $Y$ to be infinite, we need to work with discrete learnability instead of realizable learnability. We can use a slight modification to the discretization technique in Theorem 8.5 to prove the equivalence of discrete and agnostic learnability for doubly-bounded loss functions. Note that this is stronger than our guarantee for $c$-approximate pseudometrics, which only gives $c$-agnostic learnability.

Theorem A.2. Let $\ell : Y \times Y \rightarrow \mathbb{R}_{\geq 0}$ be an $(a, b)$-bounded loss function. Then for any class $(\mathcal{D}, X, H, \ell)$ the following are equivalent:

1. $(X, H, \mathcal{D}, \ell)$ is (properly) discretely-learnable
2. $(X, H, \mathcal{D}, \ell)$ is (properly) agnostically learnable.

Proof. The proof that agnostic learnability implies discrete learnability is the same as in Theorem 8.5, so we focus only the forward direction. Assume $(\mathcal{D}, X, H, \ell)$ is discretely-learnable. Fix $\varepsilon' = \frac{a \varepsilon}{4b}$, and let $H_{\varepsilon'}$ be a learnable $\varepsilon'$-discretization of $H$. We argue that running LEARNINGTOCOVER on $H_{\varepsilon'}$ (using the promised discrete learner) gives the desired agnostic learner. As before, it is sufficient to prove that $C(S_U)$ contains a hypothesis $h'$ such that $\text{err}_{D, \ell}(h') \leq OPT + \varepsilon/2$. Since $\ell$ is upper bounded, standard empirical risk minimization arguments then give the desired result.

To see why $C(S_U)$ has this property, recall from Lemma 7.3 that for any $h \in H_{\varepsilon'}$, with probability at least $1 - \delta/2$ there exists $h' \in C(S_U)$ that is $\varepsilon'$-close to $h$ in the following sense:

$$\mathbb{E}_{x \sim D_X}[\ell(h'(x), h(x))] \leq \frac{a \varepsilon}{4b}.$$  

Because $\ell$ is $a$-lower bounded, this implies that $h'$ must be close to $h$ in classification error:

$$\Pr_{x \sim D_X}[h(x) \neq h'(x)] \leq \frac{\varepsilon}{4b},$$

and since the loss is bounded by $b$, the risk of $h'$ cannot be much more than of $h$:

$$\text{err}_{D, \ell}(h') \leq \text{err}_{D, \ell}(h) + \varepsilon/4.$$  

Let $h_{OPT} \in H$ be an optimal hypothesis. Since $H_{\varepsilon'}$ $\varepsilon'$-covers $H$, there exists $h_{OPT}' \in H_{\varepsilon'}$ such that:

$$\text{err}_{D, \ell}(h_{OPT}') - \text{err}_{D, \ell}(h_{OPT}) \leq \varepsilon'.$$  

Let $h_{OPT}' \in H_{\varepsilon'}$ denote the output of the base learner $A$ on the labeling given by $h_{OPT}'$. Then by the above, we have that:

\[
\text{err}_{D,\ell}(h_{OPT}') = \mathbb{E}_{(x,y) \sim D} [\ell(h_{OPT}'(x), y)] \\
\leq \mathbb{E}_{(x,y) \sim D} [\ell(h_{OPT}'(x), y)] + \varepsilon/4 \\
\leq \mathbb{E}_{(x,y) \sim D} [\ell(h_{OPT}'(x), y) - \ell(h_{OPT}(x), y) + \ell(h_{OPT}(x), y)] + \varepsilon/4 \\
= \text{err}_{D,\ell}(h_{OPT}') - \text{err}_{D,\ell}(h_{OPT}) + OPT + \varepsilon/4 \\
\leq OPT + \varepsilon/2
\]

as desired. \qed

It is worth noting that the upper bound on the loss can be removed if the adversary is restricted to choosing a marginal over $Y$ which is weakly concentrated.

## B Robust Learning

Robust learning is an extension to the PAC setting that models an adversary with the power to perturb examples at test time. In practice, this corresponds to the fact that we’d like our predictors to be stable to small amounts of adversarial noise—this could range anywhere from a sticker on a stop-sign tricking a self-driving car, to completely imperceptible perturbations that totally fool standard classifiers. The latter was famously demonstrated by Athalye, Engstrom, Ilyas, and Kwok [51], who showed how to generate such perturbations and provided the classic example of tricking a standard ImageNet classifier into thinking a turtle was a gun. Their seminal work caused an explosion of both practical and theoretical research in the area.\footnote{Their work has over 900 citations despite being only four years old.}

Formally, adversarial robustness is modeled simply by changing the error function to be the maximum error over some pre-defined set of neighboring perturbations.

**Definition B.1 (Robust Loss).** Let $X$ be an instance space and $\mathcal{U} : X \rightarrow P(X)$ a “perturbation function” mapping elements to a set of possible perturbations. Given a loss function $\ell : Y \times Y \rightarrow \mathbb{R}_{\geq 0}$, the robust loss of a concept $h : X \rightarrow Y$ with respect to a distribution $D$ over $X \times Y$ is:

\[
R\text{-err}_{\mathcal{U},D}(h) = \mathbb{E}_{(x,y) \sim D} \left[ \max_{x' \in \mathcal{U}(x)} \ell(h(x'), y) \right].
\]

In other words, a hypothesis with low robust loss performs well even against an adversary who can perturb $x$ to any “nearby point” (i.e. any $x' \in \mathcal{U}(x)$). Standard realizable and agnostic Robust PAC-learning are then simply defined by replacing the standard error function with the robust error function. Robust learning in the distribution-family model does require one extra twist: we need to make sure that each hypothesis in the class actually has a corresponding distribution over which it is realizable. To this end, we introduce a basic notion of closure for distribution families.

**Definition B.2 (Robust Closure).** Let $\mathcal{D}$ be a set of distributions over an instance space $X$ and $H$ a concept class. Given any concept $h$, let $X_h$ denote the set of points in $X$ on which $h$ has 0 robust loss with respect to itself, that is:

\[
X_h := \{ x \in X : \forall x' \in \mathcal{U}(x), \ell(h(x'), h(x)) = 0 \}.
\]
For notational simplicity, let $D|_h$ denote the restriction $D|_{X|h}$. The robust closure of $\mathcal{D}$ under $H$ is:

$$\mathcal{D}_H := \mathcal{D} \cup \bigcup_{D \in \mathcal{D}, h \in H} D|_h.$$ 

In the robust distribution-family model, it only really makes sense to define realizable learnability over the robust closure of $\mathcal{D}$, since otherwise there may be hypotheses in the class that are not realizable with respect to any distribution in $\mathcal{D}$ and cannot be chosen by the adversary at all. With this in mind, let’s formalize this model.

**Definition B.3** ((Realizable) Distribution-Family Robust PAC Learning). A class $(\mathcal{D}, X, H, \ell)$ is Robustly PAC-learnable in the realizable setting with respect to perturbation function $U$ if there exists an algorithm $A$ and function $n(\varepsilon, \delta)$ such that for all $\varepsilon, \delta > 0$ and distributions $D$ over $X \times Y$ satisfying:

1. The marginal $D_X \in \mathcal{D}_H$,
2. $\min_{h \in H} R\text{-err}_{U,D}(h) = 0$,

then:

$$\Pr_{S \sim D^{n(\varepsilon, \delta)}}[R\text{-err}_{U,D}(A(S)) > \varepsilon] \leq \delta.$$ 

Agnostic learnability is defined similarly, but since the adversary is unrestricted, there is no longer any need to take the robust closure.

**Definition B.4** ((Agnostic) Distribution-Family Robust PAC Learning). A class $(\mathcal{D}, X, H, \ell)$ is Robustly PAC-learnable in the agnostic setting with respect to perturbation function $U$ if there exists an algorithm $A$ and function $n(\varepsilon, \delta)$ such that for all $\varepsilon, \delta > 0$ and distributions $D$ over $X \times Y$ satisfying $D_X \in \mathcal{D}$, then:

$$\Pr_{S \sim D^{n(\varepsilon, \delta)}}[R\text{-err}_{U,D}(A(S)) > OPT + \varepsilon] \leq \delta,$$

where $OPT = \min_{h \in H} \{R\text{-err}_{U,D}(h)\}$.

We note that different works consider different models of access to the perturbation set $U$ as well (e.g. assuming $U$ is known to the learner [7], or has some type of oracle access [52, 53]). Our reduction requires fairly weak access to $U$—it is enough to be able to estimate the empirical robust loss of a hypothesis $h$ over any finite sample $S \subset X$. With this in mind, let’s now prove realizable and agnostic robust learning are equivalent in the distribution-family model. We’ll focus on the special case of (multi-class) classification, and start by re-stating our modified algorithm for simplicity of presentation.

**Algorithm 5**: Agnostic to Realizable Reduction Robust Setting

**Input**: Realizable Robust PAC-Learner $A$

**Algorithm**:

1. Draw an unlabeled sample $S_U$, and labeled sample $S_L$.
2. Run $A$ over all possible subsets and labelings of $S_U$ to get:

$$C(S) := \{A(S, h(S)) \mid S \subseteq S_U, h \in H|_S\}.$$ 

3. **Return** the hypothesis in $C(S)$ with lowest empirical robust error over $S_L$. 

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Theorem B.5. If \((\mathcal{D}, X, H)\) is robustly PAC-learnable in the realizable setting with sample complexity \(n(\varepsilon, \delta)\), then Algorithm 5 robustly learns \((\mathcal{D}, X, H)\) in the agnostic setting in:

\[
m_U(\varepsilon, \delta) \leq O \left( \max_{\mu \in [0,1-\varepsilon]} \left\{ \frac{n(\varepsilon/(2(1-\mu)), \delta/3)}{1-\mu} \right\} \right)
\]

unlabeled samples, and

\[
m_L(\varepsilon, \delta) \leq O \left( \frac{m_U(\varepsilon, \delta) + \log(1/\delta)}{\varepsilon^2} \right)
\]

labeled samples.

Proof. The proof is similar to Theorem 7.4 but like malicious noise (Theorem 8.8), requires the use of subsampling. The key issue is that for a distribution \(D\) over \(X \times Y\), the optimal hypothesis \(h_{OPT}\) may not be realizable with respect to \(D_X\), that is we may have:

\[
\text{R-err}_{U,D_X \times h_{OPT}}(h_{OPT}) > 0.
\]

As a result, our realizable learner (and therefore \textsc{LearningToCover}) has no guarantees over this distribution. On the other hand, our realizable learner does have good guarantees over the restricted marginal \(D_X|_{h_{OPT}}\). We can then fix the above issue by running \textsc{LearningToCover} of all subsets of \(S\) including its restriction to \(X_{h_{OPT}}\). We will see that this essentially simulates running the realizable learner on the realizable restriction \(D_X|_{h_{OPT}}\) and recovers the desired guarantees.

Let’s take a look at this more formally. As in our previous arguments it is enough to prove that \(C(S)\) contains a hypothesis \(h'\) with robust error at most \(OPT + \varepsilon/2\), since a standard Chernoff bound tells us that \(O(\log(|C(S)|/\delta)/\varepsilon^2)\) labeled examples are enough to estimate the robust loss of every hypothesis in \(C(S)\) with high probability. We note that this is the only step in our reduction which requires access to the perturbation set \(\mathcal{U}\).

It is left to show that \(C(S)\) satisfies this property. Let \(D|_{h_{OPT}}\) denote the restriction of \(D\) to \(X_{h_{OPT}}\), the points in \(X\) on which \(h_{OPT}\) has 0 robust loss with respect to itself. Let \(\bar{D}|_{h_{OPT}}\) be the restriction to the complement, that is, \(X\setminus X_{h_{OPT}}\). The idea is to decompose our analysis into two separate parts over \(D|_{h_{OPT}}\) and \(\bar{D}|_{h_{OPT}}\). With this in mind, let \(\mu^*\) denote the mass of \(D_X\) on \(X_{h_{OPT}}\), and let \(OPT'\) denote the robust error of \(h_{OPT}\) over \(D|_{h_{OPT}}\). Since we are restricting our attention to classification error, notice that we can decompose \(OPT\) as:

\[
OPT = \text{R-err}_{U,D}(h_{OPT})
= \Pr_{(x,y) \sim D} [\exists x' \in U(x) : h_{OPT}(x') \neq y]
= \mu^* \Pr_{(x,y) \sim D|_{h_{OPT}}} [\exists x' \in U(x) : h_{OPT}(x') \neq y] + (1-\mu^*) \Pr_{(x,y) \sim D|_{h_{OPT}}} [\exists x' \in U(x) : h_{OPT}(x') \neq y]
= \mu^* + (1-\mu^*)OPT',
\]

where the last step follows from noting that by definition for all \(x\) in the support of \(\bar{D}|_{h_{OPT}}\), \(h_{OPT}\) is not constant on \(U(x)\). To get a function within \(\varepsilon/2\) robust loss of \(OPT\), we claim it is sufficient to prove \(C(S)\) contains some \(h\) within robust error \(\varepsilon/(2(1-\mu^*))\) of \(h_{OPT}\) over \(D|_{h_{OPT}}\), that is some \(h\) satisfying:

\[
\Pr_{x \sim D_X|_{h_{OPT}}} [\exists x' \in U(x) : h(x') \neq h_{OPT}(x')] \leq \varepsilon/(2(1-\mu^*)).
\]

This follows from a similar analysis to the above. Namely, letting \(R(h, x, y)\) denote the event

\[
R(h, x, y) := \exists x' \in U(x) : h(x') \neq y
\]

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for notational simplicity, we can break down $R$-err$_{U,D}(h)$ as:

$$R$-err$_{U,D}(h) = \mu^* \Pr_{(x,y) \sim D|h_{OPT}} [R(h,x,y)] + (1 - \mu^*) \Pr_{(x,y) \sim D|h_{OPT}} [R(h,x,y)]$$

$$\leq \mu^* + (1 - \mu^*) \Pr_{(x,y) \sim D|h_{OPT}} [R(h,x,y)]$$

$$\leq \mu^* + (1 - \mu^*) \left( \Pr_{(x,y) \sim D|h_{OPT}} [R(h_{OPT},x,y)] + \Pr_{(x,y) \sim D|h_{OPT}} [\exists x' \in U(x) : h(x') \neq h_{OPT}(x')] \right)$$

$$\leq \mu^* + (1 - \mu^*) \left( \text{OPT}^* + \frac{\varepsilon}{2(1 - \mu^*)} \right)$$

$$= \text{OPT}^* + \varepsilon/2.$$ 

It remains to prove that $C(S)$ contains a hypothesis satisfying Equation (3) with high probability. To see this, note that by definition of realizable robust learning, on a labeled sample $(S,h_{OPT}(S)) \sim D|h_{OPT} \times h_{OPT}$ of size $n(\varepsilon/(2(1 - \mu^*)), \delta/3)$ our learner will output $h$ satisfying:

$$\Pr_{x \sim D_X|h_{OPT}} [\exists x' \in U(x) : h(x') \neq h_{OPT}(x)] \leq \varepsilon/(2(1 - \mu^*))$$

with probability at least $1 - \delta/3$. To get Equation (3), it is then enough to note that $h_{OPT}$ is constant on $U(x)$ for all $x$ in the support of $D_X|h_{OPT}$ by definition.

The idea is now to draw a large enough unlabeled sample such that with probability at least $1 - \delta/3$, the restriction to $X|h_{OPT}$ is at least this size. By a Chernoff bound, it is enough to draw $c_1 \frac{n(\varepsilon/(1 - \mu^*), \delta/3)}{1 - \mu^*}$ points to achieve this for some large enough constant $c_1 > 0$\footnote{We’ve assumed for simplicity that $n(\varepsilon, \delta) \geq \Omega(\log(1/\delta))$. This assumption can be removed by including an extra additive factor of $\log(1/\delta)$.}. Since we do not know $\mu^*$, we’ll need to draw $c_1 \max_{\mu \in [0,1]} \left\{ \frac{n(\varepsilon/(1 - \mu), \delta/3)}{1 - \mu} \right\}$ points to ensure this property holds (if $\mu^* \geq 1 - \varepsilon$, note that any hypothesis gives a valid solution). By a union bound we have that this overall process succeeds with probability at least $1 - 2\delta/3$, and outputting the hypothesis in $C(S)$ with the lowest empirical robust risk then succeeds with probability $1 - \delta$ as desired. $\square$

Theorem [B.5] can be extended to many of the generic property generalization results in the main body, including approximate pseudometric loss, malicious noise, and semi-private learning, though the exact parameters may be somewhat weaker (e.g. learning over non-binary loss may incur additional factors and lead to $c$-agnostic rather than truly agnostic learning).

C Partial PAC-Learning

Partial PAC-learning is an extension of the standard PAC model to functions that are only defined on a certain portion of the input. Originally introduced by Long [10] and recently developed in greater depth by Alon, Hanneke, Holzman, and Moran (AHHM) [11], this model allows for the theoretical formalization of popular data-dependent assumptions such as margin that have no known analog in the PAC model. Combined with the distribution-family framework, this captures a significant portion of learning assumptions studied in both theory and practice (e.g. learning halfspaces with margin and distributional tail bounds). Let’s formalize this model, starting with partial functions.

**Definition C.1** (Partial Function). Let $X$ be an instance space and $Y$ a label space. A partial function is a labeling $f : X \to Y \cup \{\ast\}$, where elements labeled “$\ast$” are thought as of undefined. The support of $f$, denoted $\text{supp}(f)$, is the set of elements $x \in X$ s.t. $f(x) \neq \ast$. 
Standard Partial PAC-learning is defined much like the standard model with the simple modification that “∗” labels are always considered to be incorrect. As a result, in the realizable case, when the adversary selects a particular partial function $f$, their marginal distribution over the instance space $X$ must be restricted to lying on $\text{supp}(f)$. This makes formalizing data-dependent assumptions easy. If one wanted to consider halfspaces with margin $\gamma$ for instance, one simply labels every point within $\gamma$ of the decision boundary as “∗.” Interestingly, much like the distribution-family setting, Partial-PAC learning falls outside both the uniform convergence and the sample compression paradigm [54]. AHHM also show a dramatic failure of empirical risk minimization: not only does naively applying an ERM to the partial class fail, it will also fail on any total extension of the class. Despite the lack of these standard tools, both Long and AHHM were able to show that distribution-free classification of partial classes is still controlled by VC dimension, and as a result that the equivalence of realizable and agnostic learnability extends to this setting. In this section, we’ll discuss how a variant of our reduction shows that this result extends to the distribution-family model, extended loss function, and to properties beyond agnostic learning.

In the distribution-family model, formalizing realizable learnability requires some slight changes from the standard model, since we need to make sure our hypotheses are actually realizable over some distribution in the family (this is automatic in the distribution-free setting). To this end, we introduce a basic notion of closure for distribution families.

**Definition C.2 (Partial Closure).** Let $\mathcal{D}$ be a set of distributions over an instance space $X$ and $H$ a concept class. Given any concept $h$, and distribution $D$ over $X$, let $D|_h$ denote the restriction $D|_{\text{supp}(f)}$. The partial closure of $\mathcal{D}$ under $H$ is:

$$\mathcal{D}_H := \mathcal{D} \cup \bigcup_{D \in \mathcal{D}, h \in H} D|_h.$$  

In the realizable model it makes more sense to work with the closure of $\mathcal{D}$ than $\mathcal{D}$ itself, since otherwise the class $H$ may contain hypotheses that cannot be realized over any distribution, and therefore cannot be accessed by the adversary at all. For simplicity, we’ll also restrict our attention to (multi-class) classification where the label space $Y = [m]$, and recall that the loss of any undefined point is always 1.

**Definition C.3 ((Realizable) Distribution-Family Partial PAC Learning).** A partial class $(\mathcal{D}, X, H)$ is PAC-learnable in the realizable setting if there exists an algorithm $A$ and function $n(\varepsilon, \delta)$ such that for all $\varepsilon, \delta > 0$ and distributions $D$ over $X \times Y$ satisfying:

1. The marginal $D_X \in \mathcal{D}_H$,
2. $\min_{h \in H} \text{err}_D(h) = 0$,

then:

$$\Pr_{S \sim D^{n(\varepsilon, \delta)}} \left[ \text{err}_D(A(S)) > \varepsilon \right] \leq \delta,$$

where the error $\text{err}_D(h)$ is standard classification error:

$$\text{err}_D(h) = \Pr_{(x, y) \sim D} [h(x) \neq y].$$

Agnostic learnability is defined analogously, but since the adversary is unrestricted, there is no need to move to the closure of $\mathcal{D}$.

**Definition C.4 ((Agnostic) Distribution-Family Partial PAC Learning).** A partial class $(\mathcal{D}, X, H)$ is PAC-learnable in the agnostic setting if there exists an algorithm $A$ and function $n(\varepsilon, \delta)$ such that for all $\varepsilon, \delta > 0$ and distributions $D$ over $X \times Y$ satisfying $D_X \in \mathcal{D}$, then:

$$\Pr_{S \sim D^{n(\varepsilon, \delta)}} \left[ \text{err}_D(A(S)) > \text{OPT} + \varepsilon \right] \leq \delta.$$
The issue with our standard reduction strategy for partial functions is that in the agnostic model, the adversary’s marginal distribution over $X$ might have support outside of $\text{supp}(h_{OPT})$, which causes LEARNINGTOCOVER to lose its guarantee of outputting a non-uniform cover. This can be dealt with by a variant of our subsampling technique. If we run LEARNINGTOCOVER over all subsamples of the unlabeled sample $S_U$, one of these subsamples must match the support of $h_{OPT}$. This is in fact the same strategy used for adversarial robustness in Appendix [B] but we will include the algorithm again to make this section self-contained.

**Algorithm 6: Agnostic to Realizable Reduction Partial PAC Setting**

**Input:** Realizable Robust PAC-Learner $A$

**Algorithm:**

1. Draw an unlabeled sample $S_U$, and labeled sample $S_L$.
2. Run $A$ over all possible subsets and labelings of $S_U$ to get:
   
   $C(S) := \{A(S, h(S)) \mid S \subseteq S_U, h \in H|S\}$.
3. Return the hypothesis in $C(S)$ with lowest empirical error over $S_L$.

**Theorem C.5.** If $(\mathcal{D}, X, H)$ is a realizable PAC-learnable partial class with sample complexity $n(\epsilon, \delta)$, then Algorithm 6 agnostically learns $(\mathcal{D}, X, H)$ in

$$m_U(\epsilon, \delta) \leq O\left(\max_{\mu \in [0, 1-\epsilon]} \left\{\frac{n(\epsilon/(2(1-\mu)), \delta/3)}{1-\mu}\right\}\right)$$

unlabeled samples, and

$$m_L(\epsilon, \delta) \leq O\left(\frac{m_U(\epsilon, \delta) + \log(1/\delta)}{\epsilon^2}\right)$$

labeled samples.

**Proof.** The proof is essentially the same as for Theorem [B.5] but we repeat it here for completeness. As always, it is enough to prove that $C(S)$ (from Algorithm 6) contains a hypothesis $h'$ with error at most $OPT + \epsilon/2$. The key issue with our standard reduction is that the optimal hypothesis $h_{OPT}$ may be undefined on certain examples in the unlabeled sample $S_U$. By running over all subsamples of $S_U$, we in essence simulate pulling samples only from the support of $h_{OPT}$, which is enough to get the desired guarantee.

More formally, let $D|_{h_{OPT}}$ be the restriction of $D$ to $\text{supp}(h_{OPT})$, and $\bar{D}|_{h_{OPT}}$ the restriction to its complement $X \setminus \text{supp}(h_{OPT})$. The idea is to decompose our analysis into two separate parts over $D|_{h_{OPT}}$ and $\bar{D}|_{h_{OPT}}$. With this in mind, let $\mu^*$ denote the mass of $D_X$ on the undefined portion of $h_{OPT}$, and let $OPT'$ denote the error of $h_{OPT}$ over $D|_{h_{OPT}}$. Since we have restricted our attention to classification, notice that we can decompose $OPT$ as:

$$\text{err}_D(h_{OPT}) = \Pr_{(x,y) \sim D}[h_{OPT}(x) \neq y]$$

$$= \mu^* \Pr_{(x,y) \sim \bar{D}|_{h_{OPT}}}[h_{OPT}(x) \neq y] + (1 - \mu^*) \Pr_{(x,y) \sim D|_{h_{OPT}}}[h_{OPT}(x) \neq y]$$

$$= \mu^* + (1 - \mu^*)OPT'.$$
We’d like to prove that $C(S)$ contains a hypothesis $h$ within $\varepsilon/2$ error of optimal. We claim it is sufficient to show that $C(S)$ contains a hypothesis within $\varepsilon/(2(1-\mu^*))$ classification distance of $h_{OPT}$, since:

$$
\text{err}_{D}(h) = \mu^* \left( \mathbb{E}_{(x,y) \sim D|h_{OPT}} [h(x) \neq y] + (1 - \mu^*) \mathbb{E}_{(x,y) \sim D|h_{OPT}} [h(x) \neq y] \right)
\leq \mu^* + (1 - \mu^*) \mathbb{E}_{(x,y) \sim D|h_{OPT}} [h(x) \neq y]
\leq \mu^* + (1 - \mu^*) \left( \mathbb{E}_{(x,y) \sim D|h_{OPT}} [h_{OPT}(x) \neq y] + \frac{\varepsilon}{2(1 - \mu^*)} \right)
= \mu^* + (1 - \mu^*) OPT + \frac{\varepsilon}{2}
= OPT + \varepsilon/2,
$$

where the third line follows from the fact that $h'$ and $h_{OPT}$ only differ on a $\frac{\varepsilon}{2(1-\mu^*)}$ fraction of inputs over $D|h_{OPT}$.

It is left to argue that $C(S)$ contains such a hypothesis $h$. Recall that on a labeled sample $(S, h(S)) \sim D|h_{OPT} \times h_{OPT}$ of size $n\varepsilon/(2(1-\mu^*)), \delta/3$, LEARNINGTOCOVER will contain $h$ that is $\varepsilon/(2(1-\mu^*))$-close to $h_{OPT}$ in classification error over $D|h_{OPT}$ with probability at least $1 - \delta/3$. The idea is then to draw a large enough unlabeled sample such that with probability at least $1 - \delta/3$, the restriction of the sample to $D|h_{OPT}$ is at least this size (since we run over every subsample, we will always hit this restriction). By a Chernoff bound, it is enough to draw $c_1 n\varepsilon/(2(1-\mu^*)), \delta/3)$ points to achieve this for some large enough constant $c_1 > 0$.\footnote{As in Theorem B.5, we’ve assumed for simplicity that $n(\varepsilon, \delta) \geq \Omega(1/\delta)$. This assumption can be removed by including an extra additive factor of $\log(1/\delta)$.}\footnote{As in Theorem B.5, we’ve assumed for simplicity that $n(\varepsilon, \delta) \geq \Omega(1/\delta)$. This assumption can be removed by including an extra additive factor of $\log(1/\delta)$.} Since we do not know $\mu^*$, we’ll need to draw $c_1 \max_{\mu \in [0,1]} \left\{ \frac{n\varepsilon/(1-\mu), \delta/3}{1-\mu} \right\}$ points to ensure this property holds (if $\mu^* \geq 1 - \varepsilon$, note that any hypothesis gives a valid solution). By a union bound we have that this overall process succeeds with probability at least $1 - 2\delta/3$, and outputting the hypothesis in $C(S)$ with the lowest empirical risk then succeeds with probability $1 - \delta$ as desired. \hfill \qed

Like Theorem B.5, Theorem C.5 can be extended to many of the generic property generalization results in the main body, including approximate pseudometric loss, malicious noise, and semi-private learning, though it may experience some degradation of parameters (e.g. $c$-agnostic rather than truly agnostic learning) depending on how the loss of “$*$” values are formalized in these settings.

### D Uniform Stability

Uniform stability, originally introduced by Bousquet and Elisseeff\footnote{As in Theorem B.5, we’ve assumed for simplicity that $n(\varepsilon, \delta) \geq \Omega(1/\delta)$. This assumption can be removed by including an extra additive factor of $\log(1/\delta)$.}, is a useful algorithmic property that is closely tied to both generalization and privacy. Informally, an algorithm $A$ is said to be uniformly stable if for all elements $x \in X$, the probability that $A$ changes its output on $x$ over neighboring datasets is small.

**Definition D.1** (Uniform Stability). A learning algorithm is said to be $\alpha$-uniformly stable if for all neighboring inputs $S, S'$ which differ on a single example, all $x \in X$, and all $y \in Y$:

$$
\Pr[A(S)(x) = y] \leq \Pr[A(S')(x) = y] + \alpha.
$$

Uniform stability can also be thought of as a form of private prediction\footnote{As in Theorem B.5, we’ve assumed for simplicity that $n(\varepsilon, \delta) \geq \Omega(1/\delta)$. This assumption can be removed by including an extra additive factor of $\log(1/\delta)$.}, which protects against adversaries who have restricted access to a model only through prediction responses on individual points (this is often the case in practice since it is common to release APIs with query access rather than full models). Like semi-privacy, this definition has the benefit of maintaining practicality in a reasonable range...
of circumstances while weakening the stringent requirements of standard private learning. Indeed, it is well known that in the distribution-free classification setting, uniformly stable learning and private prediction are both possible for any class with finite VC dimension \[57, 56, 58\]. Unsurprisingly, these previous works (at least those working in the agnostic model), rely on uniform convergence and uniform covers. We’ll show these can be replaced with a variant of our standard reduction. The argument is otherwise similar to the proof in \[58\].

**Theorem D.2.** Let \((\mathcal{D}, X, H)\) be a realizably learnable class with sample complexity \(n(\varepsilon, \delta)\). Then there exists an \(\alpha\)-uniformly stable, \(\alpha\)-semi private algorithm that agnostically learns \((\mathcal{D}, X, H)\) in only

\[
m_U(\varepsilon, \delta, \alpha) \leq O \left( \alpha \frac{n(\varepsilon/2, \delta/2)}{\varepsilon} \right)
\]

unlabeled samples, and

\[
m_L(\varepsilon, \delta, \alpha) \leq O \left( \frac{\log (\Pi_H(n(\varepsilon, \delta)))}{\min\{\alpha \varepsilon, \varepsilon^2\}} \right)
\]

labeled samples.

**Proof.** The proof boils down to a standard subsampling trick first noted by \[57\]. Instead of drawing our standard unlabeled sample of size \(n(\varepsilon/2, \delta/2)\), we draw a sample of size \(\alpha/2 \cdot n(\varepsilon/2, \delta/2)\) and run **LearningToCover** over a random \(\alpha/2\) fraction of the sample. This ensures that swapping out any individual sample can only effect the result with probability \(\alpha/2\). Since this subsample is of size \(n(\varepsilon/2, \delta/2)\), **LearningToCover** keeps its standard guarantees and the output \(C(S_U)\) has a hypothesis within \(\varepsilon/2\) of optimal with probability \(1 - \delta/2\). We can now apply the exponential mechanism with privacy parameter \(\alpha/4\), which ensures the algorithm is \(\alpha/2\)-uniformly stable with respect to the labeled sample as well. The sample complexity bounds come from standard analysis of the exponential mechanism and the size of \(C(S_U)\). Semi-privacy comes for free due to our use of the exponential mechanism.

As in previous sections, Theorem D.2 can be extended to any of the generic property generalization results in the main body, including for instance \(\varepsilon\)-approximate pseudometric loss, malicious noise, and robustness to covariate shift.

### E Statistical Query Model

Kearns’ \[59\] statistical query model is a popular modification of PAC learning where the sample oracle is replaced with the ability to ask noisy statistical questions about the data.

**Definition E.1** (Realizable SQ-learning). Given a distribution \(D\) over \(X\) and \(h \in H\), let \(STAT(D, h)\) be an oracle which upon input of a function \(\psi : X \times Y \to [-1, 1]\) and tolerance \(\tau \in \mathbb{R}_{\geq 0}\) may output any estimate of the expectation of \(\psi\) up to \(\tau\) error, that is:

\[
STAT(D, h)(\psi, \tau) \in \mathbb{E}_{x\sim D}[\psi(x, h(x))] \pm \tau.
\]

We call a class \((\mathcal{D}, X, H, \ell)\) SQ-learnable if for all \(\varepsilon > 0\), there exists some tolerance \(\tau = \tau(\varepsilon)\), query complexity \(n(\varepsilon, \tau)\), and an algorithm \(A\) such that for all \(D \in \mathcal{D}\) and \(h \in H\), \(A\) achieves \(\varepsilon\) error in at most \(n(\varepsilon, \tau)\) oracle calls to \(STAT(D, h)\) with tolerance at worst \(\tau\).

Agnostic learning is then defined analogously where \(D, h\) is replaced with a generic distribution over \(X \times Y\) whose marginal lies in \(\mathcal{D}\). We can use a basic form of discretization to prove property generalization in the SQ model.

\[18\] While we generally think of \(\tau\) as being at worst polynomial in \(\varepsilon\), this is not strictly necessary for the model.
Theorem E.2. Let $\ell$ be a $c$-approximate pseudometric and $(\mathcal{D}, X, H, \ell)$ a realizable SQ-learnable class with query complexity $n(\varepsilon, \tau)$. Then $(\mathcal{D}, X, H, \ell)$ is $c$-agnostically SQ-learnable up to $\varepsilon + \tau$ error in $(1/\tau)^{n(\varepsilon, \tau)}$ statistical queries of tolerance at worst $\tau$.

Proof. The idea is similar to our discretization in Theorem 8.5. The realizable SQ-learner $A$ makes some finite $n(\varepsilon, \tau)$ queries. Let $C_A$ denote the set of outputs of $A$ when fed every possible combination of responses from the discretized set $\{-1, -1 + 2\tau, \ldots, 1 - 2\tau, 1\}$. For every $D \in \mathcal{D}$ and $h \in H$, one of these combinations must be a valid query response in the realizable model, so $C_A$ covers $(\mathcal{D}, X, H, \ell)$. By the same arguments of Theorem 8.5, $C_A$ must contain a hypothesis with error at most $c \cdot OPT + \varepsilon$. Since we can directly compute the loss of every element in $C_A$ up to $\tau$ error in the SQ model simply by querying the loss function, this gives the desired result in $|C_A| = (1/\tau)^{n(\varepsilon, \tau)}$ queries.

We note that while our reduction in this model experiences exponential blowup in the number of queries, this should really be thought of as corresponding to a blow up in run-time instead of “sample complexity” in the standard sense (which corresponds more closely to $\tau$).

F Fairness

Recent years have seen rising interest in an algorithmic property called fairness. Informally, fairness tries to tackle the issue that “well-performing” classifiers in the standard sense may actually be discriminatory against certain individuals or subgroups. We will consider a form of fair learning introduced by Rothblum and Yona [60] called Probably Approximately Correct and Fair (PACF) learning. Their definition is based off of a notion of fairness that ensures that similar individuals are treated similarly with respect to a fixed metric.

Definition F.1 (Metric Fairness). Let $d : X \times X \to \mathbb{R}_{\geq 0}$ be a similarity measure on $X$ and $D$ a distribution over $X$. A classifier $h : X \to Y_{out}$ is called $(\alpha, \gamma)$-fair with respect to $d$ and $D$ if $h$ acts similarly on most similar individuals:

$$\Pr_{x,x' \sim D}[|h(x) - h(x')| > d(x, x') + \gamma] \leq \alpha.$$  

We note that the output space $Y_{out}$ may differ from the label space $Y$ in general learning problems.

In fact, this definition only really makes sense when the output classifier $h$ is allowed to be real-valued (as this allows for some flexibility in the $|h(x) - h(x')|$ term). As such, when considering settings such as binary classification where $Y = \{0, 1\}$ is discrete, Rothblum and Yona’s initial formalization considers returning probabilistic classifiers with $Y_{out} = [0, 1]$. Here $h(x) = y \in [0, 1]$ is taken to be the probability of the label being 1. The error of a probabilistic classifier $h$ with respect to any distribution $D$ over $X \times \{0, 1\}$ is then given by its expected $\ell_1$ distance:

$$err_D(h) = \mathbb{E}_{(x,y) \sim D} [|h(x) - y|].$$

For simplicity we’ll focus in this section on this same regime extended to the distribution-family model.

In broad strokes, the goal of Fair PAC learning is to output a fair classifier satisfying standard PAC guarantees. Practically this requires a few modifications. First, since there may be no fair classifier satisfying these guarantees, we will only require our output to be as good as the best fair classifier. Second, we will actually allow some slack in the fairness parameters, which Rothblum and Yona [60] show is a practical way to ensure that fair learnability remains possible across a broad range of classes.
Definition F.2 (PACF-learning\[60\]). We say $(\mathcal{D}, X, H)$ is (agnostically) $(\alpha, \gamma)$-PACF-learnable with respect to a similarity metric $d : X \times X \to Y$ if there exists an algorithm $A$ and function $n = n(\varepsilon, \varepsilon_\alpha, \varepsilon_\gamma, \delta)$ such that for all $\varepsilon, \varepsilon_\alpha, \varepsilon_\gamma, \delta > 0$, and distributions $D$ over $X \times Y$ such that $D_X \in \mathcal{D}$, $A(S)$ satisfies the following guarantees with probability $1 - \delta$ over samples $S$ of size $n$:

1. $A(S)$ is accurate:
   $$\text{err}_{D, \ell}(A(S)) \leq \text{OPT}_{\alpha, \gamma} + \varepsilon$$

2. $A(S)$ is $(\alpha + \varepsilon_\alpha, \gamma + \varepsilon_\gamma)$-fair.

Here $\text{OPT}_{\alpha, \gamma}$ is the optimal error of any $(\alpha, \gamma)$-fair classifier, that is:

$$\text{OPT}_{a, b} := \min_{h \in H_{D_X, a, b}} \{\text{err}_{D, \ell}(h)\},$$

and

$$H_{D_X, a, b}^d = \{h \in H : h \text{ is (a, b)-fair with respect to } d \text{ and } D_X\}$$

Realizable learnability is defined similarly, where the adversary is constrained to picking distributions which have 0 error with respect to some $(\alpha, \gamma)$-fair classifier in $H$. We show that property generalization holds for the PACF model.

Theorem F.3 (Agnostic $\to$ Realizable (PACF Setting)). Let $(\mathcal{D}, X, H)$ be any class that is realizably $(\alpha, \gamma)$-PACF learnable with sample complexity $n(\varepsilon, \varepsilon_\alpha, \varepsilon_\gamma, \delta)$. Then $(\mathcal{D}, X, H)$ is agnostically $(\alpha, \gamma)$-fair-PACF learnable in only

$$m_U(\varepsilon, \varepsilon_\alpha, \varepsilon_\gamma, \delta) \leq n(\varepsilon/2, \varepsilon_\alpha, \varepsilon_\gamma, \delta/2)$$

unlabeled samples, and

$$m_L(\varepsilon, \varepsilon_\alpha, \varepsilon_\gamma, \delta) \leq O\left(\frac{\log(\Pi_{H}(n(\varepsilon/2, \varepsilon_\alpha, \varepsilon_\gamma, \delta/2))) + \log(1/\delta)}{\varepsilon^2}\right)$$

labeled samples.

Proof. The key observation is that the definition of fairness depends only on the classifier $h$ and the marginal distribution $D_X$. Let $h_{OPT}$ be the hypothesis achieving the minimum error over $H_{D_X, a, \gamma}$. By the above observation, with probability $1 - \delta$ the hypothesis set $C(S_U)$ returned by LEARNINGTOCOVER contains an $(\alpha + \varepsilon_\alpha, \gamma + \varepsilon_\gamma)$-fair $h$ satisfying:

$$\mathbb{E}_{x \sim D_X}[|h(x) - h_{OPT}(x)|] \leq \varepsilon/2.$$ 

Since $\ell_1$ error is a metric (and therefore satisfies the triangle inequality), we can use our argument for $c$-pseudometric loss functions from Theorem [8.5] to argue that choosing the lowest empirical risk $(\alpha + \varepsilon_\alpha, \gamma + \varepsilon_\gamma)$-fair classifier in $C(S_U)$ with respect to a sufficiently large labeled sample $S_L$ gives the desired learner.

With care, this result can be extended to a broader range of loss functions as well as to other finitely-satisfiable properties covered in this work.

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\[19\] We note that our presentation of this definition differs slightly from [60]. Their $(\alpha, \gamma)$-PACF-learnability formally corresponds to $(\alpha - \varepsilon_\alpha, \gamma - \varepsilon_\gamma)$-PACF-learnability in our version.
G    Notions of Coverability

In this section we discuss the connection between non-uniform covers and several previous notions of cov-
erability used in various learning applications. For simplicity, we’ll restrict our attention to covering with
respect to standard classification distance, that is given a distribution \(D\) and hypotheses \(h\) and \(h'\) over some
instance space \(X\):

\[
d_D(h, h') = \Pr_{x \sim D}[h(x) \neq h'(x)].
\]

To start, let’s recall the basic notion of an \(\varepsilon\)-cover specified to this measure for simplicity.

**Definition G.1** (\(\varepsilon\)-cover). Let \(X\) be an instance space, \(Y\) a label space, and let \(L_{X,Y}\) denote the set of all
labelings \(c : X \to Y\). A set \(C \subset L_{X,Y}\) is said to form an \(\varepsilon\)-cover for \((D, X, H)\) if for every hypothesis
\(h \in H\), there exists \(c \in C\) such that

\[
d_D(c, h) \leq \varepsilon.
\]

\(C\) is called proper if \(C \subset H\).

Finite \(\varepsilon\)-covers are exceedingly useful in learning theory. As discussed in Section 5, a common strategy
in the literature is to use unlabeled samples to construct an \(\varepsilon\)-cover with high probability [29, 35, 17, 61]. This results in a distribution over potential covers we call a uniform
\((\varepsilon, \delta)\)-cover.

**Definition G.2** (Uniform \((\varepsilon, \delta)\)-cover). Let \(X\) be an instance space, \(Y\) a label space, and let \(L_{X,Y}\) denote
the set of all labelings \(c : X \to Y\). A distribution \(D_C\) over the power set \(P(L_{X,Y})\) is said to form a uniform
\((\varepsilon, \delta)\)-cover for \((D, X, H)\) if:

\[
\Pr_{C \sim D_C}[C\text{ is an }\varepsilon\text{-cover for } (D, X, H)] \geq 1 - \delta.
\]

\(D_C\) is called proper if its support lies entirely in \(H\).

In this work, we introduce a weaker non-uniform variant of this notion where each \(h\) has an individual
guarantee of being covered by the distribution, but it is not necessarily the case that a sample will cover all
\(h \in H\) simultaneously.

**Definition G.3** (Non-Uniform \((\varepsilon, \delta)\)-cover). Let \(X\) be an instance space, \(Y\) a label space, and let \(L_{X,Y}\) denote
the set of all labelings \(c : X \to Y\). A distribution \(D_C\) over the power set \(P(L_{X,Y})\) is said to form a non-uniform
\((\varepsilon, \delta)\)-cover for \((D, X, H)\) if for every fixed hypothesis \(h \in H\),

\[
\Pr_{C \sim D_C}[C\text{ is an }\varepsilon\text{-cover for } (D, X, \{h\})] \geq 1 - \delta.
\]

\(D_C\) is called proper if its support lies entirely in \(H\).

In the context of learning, we are usually interested not just in the existence of these covers, but in
the more challenging problem of constructing them from a small number of unlabeled samples. In other
words, given a class \((\mathcal{D}, X, H)\), we’d like to know how many unlabeled samples from an adversarially
chosen distribution \(D \in \mathcal{D}\) are necessary to build a uniform (or non-uniform) \((\varepsilon, \delta)\)-cover for \((D, X, H)\).

In Section 8.3, we saw that the ability to construct a non-uniform \((\varepsilon, \delta)\)-cover from \(O\left(\frac{\log(1/\delta)}{\varepsilon}\right)\) samples
was crucial to give a semi-private learner with optimal public sample complexity. This improved over recent
work of Alon, Bassily, and Moran (ABM) [17], who showed that it is possible to build a uniform \((\varepsilon, \delta)\)-cover
in \(O\left(\frac{\log^{3/2}(1/\varepsilon) + \log(1/\delta)}{\varepsilon^{3/2}}\right)\) samples.

It is interesting to ask whether non-uniformity is really necessary here, or whether ABM’s analysis is
simply sub-optimal. We’ll show that the former is true, at least in the proper distribution-family setting:
the \(\log(1/\varepsilon)\) gap between these models is necessary and uniform covers cannot be used to build optimal
semi-private learners.
Theorem G.4 (Separation of Uniform and Non-Uniform Covers). There exists an instance space $X$, hypothesis class $H$, and family of distributions $\mathcal{D}$ such that for any sufficiently small $\varepsilon > 0$, the following statements holds:

1. Any algorithm which returns a finite proper uniform $(\varepsilon, 1/3)$-cover for $(\mathcal{D}, X, H)$ requires at least $\Omega(1/\varepsilon \log(1/\varepsilon))$ samples.

2. There exists an algorithm which returns a finite proper non-uniform $(\varepsilon, \delta)$-cover for $(\mathcal{D}, X, H)$ in $O(\log(1/\delta)/\varepsilon)$ samples.

Proof. Let the instance space $X = \mathbb{N}$ and $H$ be the class of indicators along with the all 0’s function, that is $H = \{h_i : i \in \mathbb{N}\} \cup \{h_0\}$ where $h_i(x) = 1 \{x = i\}$ and $h_0$ is 0 everywhere. We consider the family of distribution $\mathcal{D} = \{\mathcal{D}_{n,k}\}_{n,k>0}$ given by $k$-sets of $[n]$ where

$$\mathcal{D}_{n,k} = \{\text{unif}(T) : T \subset [n] \text{ and } |T| = k\},$$

where unif($T$) is a uniform distribution over $T$.

We start with the first claim, that building a bounded uniform $(\varepsilon, 1/2)$-cover needs at least $\Omega(1/\varepsilon \log(1/\varepsilon))$ samples. More formally, for any error parameter $\varepsilon > 0$ and size bound $m = m(\varepsilon) \in \mathbb{N}$, let $k = \lfloor 1/(2\varepsilon) \rfloor$. We will show that for any algorithm $\mathcal{A}$ on $k \log(k)$ samples that outputs at most $m$ hypotheses, $\mathcal{A}$ must fail to output an $\varepsilon$-cover with probability at least $1/2$.

Let $n \gg m, k$ be some natural number to be fixed later and consider the family of distributions $\mathcal{D}_{n,k}$. By Yao’s minimax principle, it is sufficient to show that there exists a distribution over the elements in $\mathcal{D}_{n,k}$ such that any deterministic algorithm over $k \log(k)$ samples outputting a set of (at most) $m$ hypotheses fails to give a proper $\varepsilon$-cover with probability $1/2$. We claim that taking the uniform distribution over $\mathcal{D}_{n,k}$ suffices. To formalize this, it is useful to observe the following claim.

Claim G.5. Any subset of hypotheses $C \subset H$ of size $m$ can be a proper $\varepsilon$-cover for $H$ under at most $\binom{m}{k}$ distributions in $\mathcal{D}_{n,k}$.

Let’s prove the result under this assumption. The key observation is that by standard lower bounds on the coupon collector problem, a sample $S$ of $k \log(k)$ points from any uniform $T \in \mathcal{D}_{n,k}$ will not include unif($T$)’s entire support with probability at least $1/2$. With this in mind, assume that the input sample $S$ contains only $\text{supp}(S) = j < k = \text{supp}(\text{unif}(T))$ elements. As a result, there are $\binom{n-j}{k-j}$ consistent distributions with $S$, and by Claim G.5, $\mathcal{A}(S)$ is a proper $\varepsilon$-cover for at most $\binom{m}{k}$ of them. Since $S$ is equally likely to have been sampled from any of these distributions, the probability that $\mathcal{A}(S)$ is a proper $\varepsilon$-cover is at most:

$$\Pr[\mathcal{A} \text{ fails given } \text{supp}(S) = j < k] \geq \frac{\binom{n-j}{k-j} - \binom{m}{k}}{\binom{n-j}{k-j}}.$$

Taking $n$ sufficiently larger than $m$ and $k$, we can make this probability as close to 1 as desired for any $0 < j < k$. Finally, since samples of this form occur with probability at least $1/2$, the algorithm fails with probability at least $1/3$ as desired. It is left to prove Claim G.5.

Proof of Claim G.5. Notice that for any distribution unif($T$) $\in \mathcal{D}_{n,k}$, any $i \in T$ and any $j \neq i$, $d_{\text{unif}(T)}(h_i, h_j) > 2\varepsilon$. Let $C$ be any proper $\varepsilon$-cover of $H$ under distribution unif($T$). Then, by the above argument, it must contain $\{h_i : i \in T\}$. Since $|T| = k$, $C$ can be a proper $\varepsilon$-cover of $H$ under at most $\binom{m}{k}$ distributions in $\mathcal{D}_{n,k}$. \qed

We now move to proving that a proper non-uniform $(\varepsilon, \delta)$-cover can be built in only $O(\log(1/\delta)/\varepsilon)$ samples. This follows from the fact that for any $n \geq k$ and distribution unif($T$) $\in \mathcal{D}_{n,k}$, each $i \in T$ is
in the random sample $S$ with probability $1 - \delta$. Since each $h_j$ for $j \notin T$ is covered by $h_0$, outputting 
\{h_i : i \in S\} \cup \{h_0\}$ generates a proper non-uniform $(\varepsilon, \delta)$-cover. 

The construction in Theorem \textit{G.4}20 can easily be modified to give a class with the same gap which is not privately learnable (say by embedding a single copy of a threshold over $[0, 1]$). Since any such class requires at least $\Omega(\frac{1}{\varepsilon})$ public samples to semi-privately learn by Theorem \textit{S.17}12 theorem then provides a separation between using uniform and non-uniform covers in semi-private learning: the former provably requires an extra log factor, while the latter matches the lower bound exactly. Unfortunately, our proof of this result only holds in the proper setting, as Claim \textit{G.5} fails when improper hypotheses are allowed. We conjecture that this is not an inherent barrier: the separation should continue to hold in the improper case, albeit with some different analysis.

We have now seen a weak separation between uniform and non-uniform covers, but one might reasonably wonder whether a much stronger separation is possible. In particular, all previous constructions of uniform covers use uniform convergence, but there exist simple examples of learnable classes in the distribution-family model that fail this property: do such classes provide an example of objects which are non-uniformly coverable but not uniformly coverable? Surprisingly, the answer is no! It turns out that an algorithm for non-uniform covering can always be used to construct a uniform covering without too much overhead. Moreover, we’ll see that the $\log(1/\varepsilon)$ gap is tight when $(X, H)$ has finite VC dimension.

To prove this, it will actually be useful to make a brief aside and introduce another closely related notion of covering called fractional covers. These objects are essentially a form of non-uniform covering which output a single hypothesis instead of a set of them.

**Definition G.6** (Fractional cover). Let $X$ be an instance space, $Y$ a label space, and let $L_{X,Y}$ denote the set of all labelings $c : X \to Y$. A distribution $D_C$ over $L_{X,Y}$ is said to form a fractional $(\varepsilon, p)$-cover for a hypothesis class $H$ for $(D, X, H)$ if for any fixed $h \in H$, a sample from $D_C$ covers $h$ with probability $p$:

$$\Pr_{c \sim D_C}[d(c, h) \leq \varepsilon] \geq p.$$ 

Fractional covers are closely connected to non-uniform covers. In fact, one can easily move between the two by sampling or subsampling.

**Proposition G.7** (Non-uniform cover $\iff$ Fractional cover). Let $(D, X, H)$ be any class, $C_{\text{frac}}$ a fractional $(\varepsilon, p)$-cover, and $C_{n-u}$ a non-uniform $(\varepsilon, 1/2)$-cover. Then the following hold:

1. **Drawing** $\log_{1/(1-p)} (1/\delta)$ samples from $C_{\text{frac}}$ gives a non-uniform $(\varepsilon, \delta)$-cover.
2. **Choosing a random hypothesis** from $C_{n-u}$ gives a fractional $(\varepsilon, 1/2|C|)$-cover.

**Proof.** Both statements are essentially immediate from definition. For any fixed $h \in H$, if we draw $M$ samples from $C_{\text{frac}}$, the probability we fail to cover $h$ is $(1 - p)^M$, so setting $M = \log_{1/(1-p)} (1/\delta)$ gives the desired non-uniform cover. On the other hand, for any fixed $h \in H$, a sample from $C \sim C_{n-u}$ contains $c$ $\varepsilon$-close to $h$ with probability $1/2$. Outputting a uniformly random element of $C$ then gives an element within $\varepsilon$ of $h$ with probability $1/2|C|$ as desired.  

It will also be useful to note a classical relation between covers and fractional covers.

**Lemma G.8.** If there exists a fractional $(\varepsilon, p)$-cover for $(D, X, H)$, then there exists a $2\varepsilon$-cover of size $1/p$.

20While Alon, Bassily, and Moran only state this result for the distribution-free setting, it holds in the distribution-family model as well.
Proof. This follows from classical packing-covering duality. The existence of a fractional \((\varepsilon, p)\)-cover implies there cannot exist a \(2\varepsilon\)-packing of size \(1/p\) (that is, a set of \(1/p\) hypotheses in \(H\) that are pairwise \(2\varepsilon\)-separated with respect to \(D\)). By packing-covering duality, this implies the existence of a \(2\varepsilon\)-cover of size \(1/p\).

With this in hand, let’s show that uniform covers can be constructed for any realizably learnable class, regardless of whether we have uniform convergence.

**Theorem G.9** (Realizable learning \(\rightarrow\) Uniform cover). Let \((\mathcal{D}, X, H)\) be realizably PAC-learnable with sample complexity \(n(\varepsilon, \delta)\). Then it is possible to construct a uniform \((\varepsilon, \delta)\)-cover for \((\mathcal{D}, X, H)\) in \(n(\varepsilon/2, \delta')\) samples where \(\delta' = O\left(\frac{\delta}{\Pi_H(n(\varepsilon/2, 1/2))}\right)\).

Proof. We’ll start by proving a slightly more general fact. If for every \(D \in \mathcal{D}\), \((D, X, H)\) has a proper \((\varepsilon/2)\)-cover \(C_D\) of size at most \(C = C(\varepsilon/2)\), then it is possible to construct a uniform \((\varepsilon, \delta)\)-cover in \(n(\varepsilon/2, \delta/C)\) samples. This is essentially immediate from Lemma 7.3 which states that running LEARNINGTOCOVER over a sample of size \(n(\varepsilon, \delta/C)\) gives a non-uniform \((\varepsilon, \delta/C)\)-cover. Union bounding over \(C_D\) then gives that a sample from the non-uniform cover \((\varepsilon/2)\)-covers \(C_D\) with probability at least \(1 - \delta\). Since \(C_D\) is itself an \((\varepsilon/2)\)-cover, this implies that the entire class \(H\) \(\varepsilon\)-covered by the sample with probability at least \(1 - \delta\) as desired.

It remains to show that for every \(D \in \mathcal{D}\), \((D, X, H)\) has a proper \((\varepsilon/2)\)-cover of size \(O(\Pi_H(n(\varepsilon/2, 1/2)))\). This follows from combining Proposition G.7 and Lemma G.8. In particular, Lemma G.7 implies that running LEARNINGTOCOVER over a sample of size \(n(\varepsilon/2, 1/2)\) produces a non-uniform \((\varepsilon/2, 1/2)\)-cover of size at most \(\Pi_H(n(\varepsilon/2, 1/2))\). Proposition G.7 states that subsampling from this cover gives a fractional \((\varepsilon/2, 1/(2\Pi_H(n(\varepsilon/2, 1/2))))\)-cover, which in turn implies the existence of a \((\varepsilon/2)\)-cover of size \(O(\Pi_H(n(\varepsilon/2, 1/2)))\) as desired. We note that this last argument is similar to an observation made in Benedek and Itai’s seminal work on the distribution-dependent model.

When \((X, H)\) has finite VC-dimension \(d\), note that Theorem G.9 exactly matches the lower bound exhibited in Theorem G.4 as the required number of samples for a uniform \((\varepsilon, \delta)\)-cover becomes:

\[
n(\varepsilon/2, \delta') \leq O\left(\frac{d \log(1/\varepsilon) + \log(1/\delta)}{\varepsilon}\right).
\]

This also matches the bound given by ABM [17] using uniform convergence.