Gaussian Mean Testing Made Simple

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October 26, 2022

Abstract

We study the following fundamental hypothesis testing problem, which we term *Gaussian mean testing*. Given i.i.d. samples from a distribution \( p \) on \( \mathbb{R}^d \), the task is to distinguish, with high probability, between the following cases: (i) \( p \) is the standard Gaussian distribution, \( \mathcal{N}(0, I_d) \), and (ii) \( p \) is a Gaussian \( \mathcal{N}(\mu, \Sigma) \) for some unknown covariance \( \Sigma \) and mean \( \mu \in \mathbb{R}^d \) satisfying \( \|\mu\|_2 \geq \epsilon \). Recent work gave an algorithm for this testing problem with the optimal sample complexity of \( \Theta(\sqrt{d}/\epsilon^2) \). Both the previous algorithm and its analysis are quite complicated. Here we give an extremely simple algorithm for Gaussian mean testing with a one-page analysis. Our algorithm is sample optimal and runs in sample linear time.

*Authors are in alphabetical order.
†Supported by NSF Medium Award CCF-2107079, NSF Award CCF-1652862 (CAREER), a Sloan Research Fellowship, and a DARPA Learning with Less Labels (LwLL) grant.
‡Supported by NSF Medium Award CCF-2107547, NSF Award CCF-1553288 (CAREER), and a Sloan Research Fellowship.
§Supported by NSF Award CCF-1652862 (CAREER), and NSF grants CCF-1841190 and CCF-2011255.
1 Introduction

The paradigmatic problem in distribution testing [GR00, BFR+00] is the following: given sample access to an unknown distribution \( p \), determine whether \( p \) has some global property or is “far” from any distribution having the property. During the past two decades, a wide range of properties have been studied, and we now have sample-optimal testers for many of them [Pan08, CDVV14, VV14, ADK15, DK16, CDGR18, DGPP18, CDKS18, DGK+21].

Without a priori assumptions on the underlying distribution \( p \), at least \( \Omega(\sqrt{N}) \) many samples are required for testing even the simplest properties, where \( N \) is the domain size of \( p \). If \( p \) is either high-dimensional (supported on an exponentially large domain, e.g., \( \{0, 1\}^d \)) or continuous, such a sample bound is prohibitive. This observation has motivated a line of work studying distribution testing of structured distribution families. This includes both nonparametric families in low-dimensions [DKN15b, DKN15a, DKN17, DKP19] and parametric families in high dimensions [CDKS17, DDK19, ABDK18, CCK+21].

This work focuses on the high-dimensional setting. Arguably the most basic high-dimensional testing problem is the following: We assume that \( p \) is an identity covariance Gaussian distribution on \( \mathbb{R}^d \) and the goal is to distinguish between the cases that its mean is zero or at least \( \epsilon \) in \( \ell_2 \)-norm. This is known as the Gaussian sequence model and has a rich history in statistics [Ern91, Bar02, IS03] (see also [DKS17] for a simple algorithm and matching lower bound). In particular, the aforementioned works have established that the sample complexity of this basic problem is \( \Theta(\sqrt{d}/\epsilon^2) \).

Here we consider a generalization of the Gaussian sequence model, recently studied in the TCS literature [CCK+21]. We will call this problem Gaussian-Mean-Testing:

<table>
<thead>
<tr>
<th>Problem: Gaussian-Mean-Testing</th>
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<tr>
<td><strong>Input:</strong> Sample access to a distribution ( p ) supported on ( \mathbb{R}^d ) and ( \epsilon &gt; 0 ).</td>
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<tr>
<td><strong>Output:</strong></td>
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<td>- (Completeness) “ACCEPT” with probability at least ( 2/3 ) if ( p = \mathcal{N}(0, I_d) ),</td>
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<tr>
<td>- (Soundness) “REJECT” with probability at least ( 2/3 ) if ( p = \mathcal{N}(\mu, \Sigma) ) for some ( \mu ) with ( |\mu|_2 \geq \epsilon ).</td>
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The work of [CCK+21] proposed a testing algorithm for this problem with the optimal sample complexity of \( n = \Theta(\sqrt{d}/\epsilon^2) \) for the (restricted) parameter regime of \( \epsilon \in (0, 1] \). The initial algorithm of [CCK+21] had quasi-polynomial time complexity; this was improved to polynomial using an observation from a subsequent work [CJLW21].

Before we describe our results, we summarize the approach of [CCK+21].

**Approach of [CCK+21]** The algorithm of [CCK+21] proceeds by reducing Gaussian-Mean-Testing to another high-dimensional testing problem, which we call Hypercube-Mean-Testing, described next. Given sample access to a distribution \( p \) on the hypercube \( \{\pm 1\}^d \) and \( \epsilon \in (0, 1) \), Hypercube-Mean-Testing asks us to distinguish between the cases when \( p \) is uniform and when the mean of the distribution \( p \) has \( \ell_2 \)-norm at least \( \epsilon \).

The relation between Hypercube-Mean-Testing and Gaussian-Mean-Testing is apparent when we look at the function \( F : \mathbb{R}^d \to \{\pm 1\}^d \) that maps \( x \in \mathbb{R}^d \) to \( y \in \{\pm 1\}^d \) coordinatewise by \( y_i = \text{sgn}(x_i) \). For a distribution \( p \), we use \( F(p) \) to denote the distribution of \( F(X) \) when \( X \sim p \). It is not too hard to see that when \( p = \mathcal{N}(0, I_d) \), then \( F(p) \) is the uniform distribution on the hypercube. Similarly, it can be shown that when \( p \sim \mathcal{N}(\mu, \Sigma) \) with \( \|\mu\|_2 \geq \epsilon \) for \( \epsilon \in (0, 1] \) and
max_{i \in [d]} \Sigma_{i,i} \leq 2$, then $F(p)$ is a distribution on hypercube whose mean has euclidean norm larger than $\Omega(\epsilon)$. Thus the algorithm of [CCK+21] works as follows:

1. Check if there is a coordinate $i \in [d]$ such that $\Sigma_{i,i} \geq 2$, which can be tested reliably and efficiently with $O(\log d)$ samples, and

2. Run a tester for Hypercube-Mean-Testing with input distribution $F(p)$ and $\epsilon' = \Theta(\epsilon)$.

One of the main contributions of [CCK+21] is an algorithm for Hypercube-Mean-Testing with sample complexity $n = O(\sqrt{d}/\epsilon^2)$ and runtime\(^1\) $O(n^2d)$. The proposed algorithm and its analysis are somewhat involved. Moreover, the reduction of Gaussian-Mean-Testing to Hypercube-Mean-Testing crucially uses that the parameter $\epsilon$ in Gaussian-Mean-Testing is less than a small enough constant. Our main contribution is a very simple direct algorithm with a compact analysis (by avoiding this reduction, our algorithm works for all values of $\epsilon$).

**Our Result** In this paper, we give a very simple sample-optimal algorithm (Algorithm 1) for Gaussian-Mean-Testing whose full analysis fits in one page. In addition to its simplicity, our algorithm runs in linear time (requires a single pass over the data) and works for all values of $\epsilon$.

In particular, we establish the following result:

**Theorem 1.1.** Algorithm 1 solves Gaussian-Mean-Testing with $n = \Theta\left(\max(1, \sqrt{d}/\epsilon^2)\right)$ samples and can be implemented in $O(n d)$ time.

Our algorithm is extremely easy to describe: we sample two sets of $\Theta\left(\max(1, \sqrt{d}/\epsilon^2)\right)$ samples, compute the sample mean of each of them, and calculate the inner product of the two obtained vectors. The algorithm outputs “ACCEPT” if the inner product has small absolute value, and “REJECT” otherwise. A detailed pseudocode follows.

**Algorithm 1 GaussianMeanTester**

**Input:** Sample access to distribution $p$ on $\mathbb{R}^d$ and $\epsilon > 0$.

**Output:** “ACCEPT” if $p = \mathcal{N}(0, I_d)$, “REJECT” if $p = \mathcal{N}(\mu, \Sigma)$ and $\|\mu\|_2 \geq \epsilon$; both with probability at least $2/3$.

1: Set $n = 25C^2\sqrt{d}/\epsilon^2$, where $C_*$ is the absolute constant from Fact 2.1.
2: Sample $2n$ i.i.d. points from $p$ and denote them by $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$.
3: Define $Z = (1/n^2)(\sum_{i=1}^n X_i)^\top(\sum_{i=1}^n Y_i)$.
4: if $|Z| \leq \sqrt{3d}/n$ then
5: return “ACCEPT”
6: else
7: return “REJECT”
8: end if

We now describe the high-level idea of our proof.

**Our Technique** A natural first attempt at a tester would be to attempt to approximate $\|\mu\|_2^2$ and to reject if the answer is too large. A reasonable way to do this is to compute two independent estimates of $\mu$ and take their inner product. For example, as $Z = (\sum_{i=1}^n X_i/n)^\top(\sum_{i=1}^n Y_i/n)$, where $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ are all independent samples. (Note that the two estimates above should

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\(^1\)This follows by the reduction procedure of [CCK+21] with a more careful runtime analysis by [CJLW21].
be independent: one cannot take \((\sum_{i=1}^{n} X_i/n)^\top (\sum_{i=1}^{n} X_i/n)\) as a reliable estimate of \(\|\mu\|_2^2\) because the correlation between the two halves will bias the final estimate.) Computing the variance of \(Z\), \(\text{Var}[Z]\), it is not hard to see that in the completeness case, \(|Z| = O(\sqrt{d}/n)\) with high probability. We would like to be able to claim that in the soundness case the quantity \(|Z - \|\mu\|_2^2|\) is similarly small; this would give us an easy separation, so long as \(\|\mu\|_2^2\) is large, it means that the values of \(p\) will be larger than required. Fortunately, in this case we are rescued by another argument. If \(\text{Var}[Z]\) is large, it means that the values of \(Z\) will be spread over a very large range. This in turn will make it unlikely that \(Z\) will be in the narrow range of values where \(|Z| = O(\sqrt{d}/n)\). In particular, we can formalize this by noting that \(Z\) is a degree-2 polynomial in Gaussian inputs, and applying the Carbery-Wright anti-concentration inequality (see Fact 2.1). Thus, we end up with the simple tester of compute \(Z\), and then check whether or not \(|Z| < C\sqrt{d}/n\) for an appropriate constant \(C\).

Remark 1.2. Our testing algorithm applies for the following more general testing problem:

1. (Completeness) \(p\) is a distribution with mean \(\mu\) and covariance \(\Sigma\) satisfying \(\|\mu\|_2 \leq c \cdot \epsilon\) for a small enough positive constant \(c < 1\) and \(\|\Sigma\|_F \leq \sqrt{d}\), where \(\|\cdot\|_F\) is the Frobenius norm.
2. (Soundness) \(p\) is a log-concave distribution with mean \(\mu\) satisfying \(\|\mu\|_2 \geq \epsilon\).

2 Proof of Theorem 1.1

Notation and Background We use \(I_d\) to denote the \(d \times d\) identity matrix. For a vector \(x \in \mathbb{R}^d\), we use \(\|x\|_2\) to denote its Euclidean norm. For a univariate random variable \(X\), we use \(\mathbb{E}[X]\) and \(\text{Var}[X]\) to denote its mean and variance, respectively. The multivariate Gaussian distribution with mean \(\mu\) and covariance \(\Sigma\) is denoted by \(\mathcal{N}(\mu, \Sigma)\). For two matrices \(A\) and \(B\), we use \(\langle A, B \rangle\) to denote the trace inner product, i.e., \(\langle A, B \rangle = \text{tr}(A^\top B)\).

We will require the following well-known fact.

Fact 2.1 (Carbery-Wright inequality for quadratics [CW01]). There exists a \(C_* > 0\) such that the following holds: Let \(G \sim \mathcal{N}(0, I_d)\) in \(\mathbb{R}^d\), \(p : \mathbb{R}^d \to \mathbb{R}\) be a degree-2 polynomial, and \(\alpha \in (0, \infty)\). Then we have that \(\mathbb{P}(|p(G)| \leq \alpha \sqrt{\mathbb{E}[p^2(G)]}) \leq C_* \sqrt{\alpha}\).

We are now ready to prove our main result.

Proof. (of Theorem 1.1) Algorithm 1 samples \(2n\) points, where \(n \geq 25C_*^2 \sqrt{d}/\epsilon^2\) and \(C_*\) is the constant in the Carbery-Wright Theorem (cf. Fact 2.1); without loss of generality, we will assume that \(n \geq 1\). Let the \(2n\) samples be \(X_1, \ldots, X_n\) and \(Y_1, \ldots, Y_n\), where each \(X_i\) and \(Y_i\) is distributed as \(\mathcal{N}(\mu, \Sigma)\). We define our test statistic to be \(Z := (1/n^2)(\sum_{i=1}^{n} X_i)^\top (\sum_{i=1}^{n} Y_i)\). Algorithm 1 outputs “ACCEPT” if \(|Z| \leq \sqrt{3d}/n\) and outputs “REJECT” otherwise. The claim on the running time is immediate. We now analyze the completeness (the algorithm outputs “ACCEPT” with probability \(2/3\) when \(p = \mathcal{N}(0, I_d)\)) and soundness (the algorithm outputs “REJECT” with probability \(2/3\) when \(p = \mathcal{N}(\mu, \Sigma)\) and \(\|\mu\|_2 > \epsilon\)) of the proposed test.

Completeness We begin by calculating the mean and variance of \(Z\) when \(p = \mathcal{N}(0, I_d)\). Let \(G_1\) and \(G_2\) be two independent \(\mathcal{N}(0, I_d)\) random variables. Since \((\sum_{i} X_i)/\sqrt{n}\) and \((\sum_{i} Y_i)/\sqrt{n}\) are independently distributed as \(\mathcal{N}(0, I_d)\), it follows that \(Z\) has the same distribution as \((1/n)(G_1^\top G_2)\). This directly gives use that \(\mathbb{E}[Z] = 0\). The variance of \(Z\) can be calculated as follows:

\[
\text{Var}[Z] = \frac{1}{n^2} \mathbb{E} \left[ (G_1^\top G_2)^2 \right] = \frac{1}{n^2} \mathbb{E} \left[ (G_1 G_1^\top, G_2 G_2^\top) \right] = \frac{1}{n^2}(I_d, I_d) = \frac{d}{n^2},
\]
where we use that $G_1$ and $G_2$ are independent. By Chebyshev’s inequality, with probability at least $2/3$, $|Z| \leq \sqrt{3d}/n$. Therefore, the probability of acceptance when $p = \mathcal{N}(0, I_d)$ is at least $2/3$.

**Soundness** We now consider the setting when $\|\mu\|_2 \geq \epsilon$ and $\Sigma$ is an arbitrary positive semidefinite matrix. Since $X_i$’s and $Y_i$’s are i.i.d. with mean $\mu$, we get that $E[Z] = \|\mu\|_2^2$, which is larger than $\epsilon^2$. Our goal will be to show that, with probability at least $2/3$, $|Z| > 2\sqrt{d}/n$. We will use the Carbery-Wright inequality (Fact 2.1). Observe that $Z$ is a quadratic polynomial of a Gaussian distribution. Let $\|Z\|_{L^2}$ denote $\sqrt{E[Z^2]}$ and observe that $\|Z\|_{L^2} \geq E[Z] \geq \epsilon^2$. By Fact 2.1, we obtain the following:

$$\mathbb{P} \left( |Z| \leq 2\sqrt{d}/n \right) = \mathbb{P} \left( |Z| \leq \frac{2\sqrt{d}/n}{\|Z\|_{L^2}} \cdot \|Z\|_{L^2} \right) \leq C_s \sqrt{\frac{2\sqrt{d}/n}{\|Z\|_{L^2}}} \leq C_s \sqrt{\frac{2\sqrt{d}/n}{\epsilon^2}} < \frac{1}{3},$$

where the second inequality uses that $\|Z\|_{L^2} \geq \epsilon^2$ and the last inequality uses that $n \geq 25C_s^2\sqrt{d}/\epsilon^2$. Therefore, the probability of rejection, i.e., of the event where $|Z| > \sqrt{3d}/n$, is at least $2/3$. Combining the above, it follows that the algorithm correctly rejects with probability at least $2/3$. □

**References**


