Problem Set 5 Solutions

In all problems the languages are over the alphabet $\Sigma = \{0, 1\}$. You may use without proof the NP-completeness of the following problems, but no others:

- SAT, 3-SAT, INDEPENDENT-SET, CLIQUE, VERTEX-COVER, 3-COLORING,
- HAMILTONIAN-PATH, HAMILTONIAN-CYCLE, SUBSET-SUM, INTEGER-PROGRAMMING, SET-COVER.

Problem 1. [30 points] Recall that graph $G = (V, E)$ is $k$-colorable if there exists a map $c: V \to \{1, \ldots, k\}$ such that $c(i) \neq c(j)$ for every edge $\{i, j\} \in E$. Prove that the following 4-COLORING problem is NP-complete:

Input: $\langle G \rangle$ where $G$ is a graph
Question: Is $G$ 4-colorable?

The following is a verifier for 4-COLORING, showing the latter is in NP:

Verifier $V((G), (c))$, where $G = (V, E)$
- Accept iff
  - $c$ is a map from $V$ to $\{1, 2, 3, 4\}$
  - $c(i) \neq c(j)$ for all $i, j \in E$

We now show that 3-COLORING $\leq_p$ 4-COLORING. The reduction function $f$ takes input a graph $G = (V, E)$ and produces another graph $G' = (V', E')$ as follows. Introduce a vertex $w$ that is new (meaning not in $V$) and let $V' = V \cup \{w\}$. Let $E' = E \cup \{\{v, w\} : v \in V\}$.

If $c: V \to \{1, 2, 3\}$ is a 3-coloring of $G$ then define $c': V' \to \{1, 2, 3, 4\}$ by $c'(v) = c(v)$ for $v \in V$ and $c'(w) = 4$. This is a 4-coloring of $G'$. This shows that if $G$ is 3-colorable then $G'$ is 4-colorable.

Conversely, suppose $c': V' \to \{1, 2, 3, 4\}$ is a 4-coloring of $G'$. Without loss of generality, the color assigned to $w$ is 4. But since $w$ is connected to all vertices in $V$, no vertex in $v$ has $c'(v) = 4$. So the map $c: V \to \{1, 2, 3\}$ defined by $c(v) = c'(v)$ for all $v \in V$ is a 3-coloring of $G$. This shows that if $G'$ is 4-colorable then $G$ is 3-colorable.

Problem 2. [30 points] Let $G = (V, E)$ be a graph and $I, W \subseteq V$. We say that $I$ covers $W$ if for every $w \in W$ there exists $i \in I$ such that $\{i, w\} \in E$. Prove that the following IS-COVER problem is NP-complete:
**Input:** \( \langle G, U, W \rangle \) where \( G = (V, E) \) is a graph and \( U, W \) are disjoint subsets of \( V \)

**Question:** Does there exist an independent set \( I \) in \( G \) such that \( I \subseteq U \) and \( I \) covers \( W \)?

The following is a verifier for IS-Cover, showing the latter is in \( \text{NP} \):

Verifier \( V((G, U, W), \langle I \rangle) \) where \( G = (V, E) \) and \( U, W \) are disjoint subsets of \( V \)

Accept iff all the following are true:
- \( I \subseteq U \)
- \( i, j \notin E \) for all \( i, j \in I \)
- For all \( w \in W \) there exists \( i \in I \) such that \( \{i, w\} \in E \).

We now show that 3-SAT \( \leq_p \) IS-Cover. The reduction function \( f \) takes input a 3-CNF formula \( \varphi = C_1 \land \cdots \land C_m \) over variables \( x_1, \ldots, x_k \), and outputs \( \langle G, U, W \rangle \) where \( G = (V, E) \), these quantities being defined as follows:

\[
U = \{ x_1, \overline{x}_1, \ldots, x_k, \overline{x}_k \}
\]

\[
W = \{ C_1, \ldots, C_m \}
\]

\[
V = U \cup W
\]

\[
E_1 = \{ \{ x_i, \overline{x}_i \} : 1 \leq i \leq k \}
\]

\[
E_2 = \{ \{ \ell, C_j \} : 1 \leq j \leq m \text{ and literal } \ell \text{ is in clause } C_j \}
\]

\[
E = E_1 \cup E_2
\]

Let \( a = a[1] \ldots a[k] \in \{0, 1\}^k \) be a satisfying assignment to \( \varphi \). Let \( I = \{ x_i : a[i] = 1 \}\} \cup\{ \overline{x}_i : a[i] = 0 \}. \) Then \( I \subseteq U \) is an independent set in \( G \) that covers \( W \).

Conversely let \( I \subseteq U \) be an independent set in \( G \) that covers \( W \). Let \( a[i] = 1 \) if \( x_i \in I \) and 0 otherwise. Then \( a = a[1] \ldots a[k] \in \{0, 1\}^k \) is a satisfying assignment to \( \varphi \).