Online Learning and Online Investing

Jia Mao

February 20, 2006
Outline

1 Online Investing

2 Constant Rebalanced Portfolios

3 Algorithms competing against best CRP
   • Algorithms competing against best CRP
   • “Universal” algorithm

4 Implementation

5 Semi-Constant-Rebalanced Portfolios (SCRP)
Portfolio Selection

- Consider $n$ stocks
Portfolio Selection

- Consider \( n \) stocks
- Our distribution of wealth is some vector \( \mathbf{b} \)
e.g. \((1/3, 1/3, 1/3)\)
Consider $n$ stocks

Our distribution of wealth is some vector $b$
e.g. $(1/3, 1/3, 1/3)$

At end of one period, we get a vector of “price relatives” $x$
e.g. $(0.98, 1.02, 1.00)$
Portfolio Selection

- Consider \( n \) stocks
- Our distribution of wealth is some vector \( \mathbf{b} \)
  - e.g. (1/3, 1/3, 1/3)
- At end of one period, we get a vector of “price relatives” \( \mathbf{x} \)
  - e.g. (0.98, 1.02, 1.00)
- Our wealth becomes \( \mathbf{b} \cdot \mathbf{x} \)
Log Wealth

- In each period, algorithm A performs 85% as well as algorithm B.
Log Wealth

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- After $t$ steps, we have
  
  $A$’s wealth = $(0.85)^t \ (B$’s wealth)
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  \[ \text{A’s wealth} = (0.85)^t \text{ (B’s wealth)} \]
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  \[ \ln(A) = \ln(B) - 0.16t \]
Log Wealth

- In each period, algorithm A performs 85% as well as algorithm B.
- After $t$ steps, we have
  
  \[
  \text{A's wealth} = (0.85)^t \times \text{(B's wealth)}
  \]
- Nicer if we take logs
  
  \[
  \ln(A) = \ln(B) - 0.16t
  \]
- Problematic when stock price goes to zero.
References

- Tom M. Cover, Erik Ordentlich, *Universal Portfolios with Side Information*, 1996
- Adam Kalai, *slides*, 1997
- Avrim Blum, *slides*, 2000
Online Learning vs. Online Investing

Similarities

- Stocks ↔ Experts
- Wealth allocation ↔ probability distribution (i.e. weights)
- Stock $i$ drops by $l_i$% ↔ Expert $i$ has loss $l_i$

Differences

- Initial wealth allocation (dot) vs. Price relatives vector vs. Sum of losses
- Stock price change automatically changes fraction of wealth vs. Explicit update of weights
Online Learning vs. Online Investing

- Similarities

- Differences
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  - Initial wealth allocation (dot) Price relatives vector vs. Sum of losses
  - Stock price change *automatically* changes fraction of wealth vs. Explicit update of weights
Outline

1. Online Investing

2. Constant Rebalanced Portfolios

3. Algorithms competing against best CRP
   - Algorithms competing against best CRP
   - “Universal” algorithm

4. Implementation

5. Semi-Constant-Rebalanced Portfolios (SCRP)
Compete against the best stock

It's hard, in a sense, because in worst case, we can't hope to do better than \( \frac{1}{n} \times \text{(performance of best stock in hindsight)} \).

However, there is a simple strategy:

1. Initially invest an equal amount in each stock.
2. Let it sit. (no trades)
3. \( \text{wealth of Split} = \text{avg. of stocks} \)
   \[ \geq \frac{1}{n} \times \text{wealth of best stock} \]
4. \( \text{i.e. } \ln(\text{Split}) \geq \ln(\text{best stock}) - \ln(n) \)
5. \( \text{avg. per-day ratio} \geq \left(\frac{1}{n}\right)^{1/t} \to 1 \text{ as } t \to \infty \)
Compete against the best stock

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- It’s hard, in a sense, because in worst case, we can’t hope to do better than \( \frac{1}{n} \times \text{(performance of best stock in hindsight)} \)
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- However, there is a simple strategy *Split* to perform at least this well
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- However, there is a simple strategy **Split** to perform at least this well
  - Initially invest an equal amount in each stock
  - Let it sit. (no trades)
  - wealth of Split = avg. of stocks

\[ \frac{\ln(Split)}{\ln(\text{best stock}) - \ln(n)} \geq \left( \frac{1}{n} \right)^{1/t} \rightarrow 1 \text{ as } t \rightarrow \infty \]
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\[
\frac{\text{wealth of Split}}{\text{wealth of best stock}} \geq \frac{1}{n}
\]
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  - avg. per-day ratio \( \geq (\frac{1}{n})^{1/t} \rightarrow 1 \) as \( t \rightarrow \infty \)
Constant Rebalanced Portfolios (CRPs)

**Definition**

CRP(\(b\)): at end of each period, rebalance back to same distribution of wealth \(b\).
Constant Rebalanced Portfolios (CRPs)

Definition

CRP(b): at end of each period, rebalance back to same distribution of wealth b.

Why CRP?
Constant Rebalanced Portfolios (CRPs)

Definition

CRP\((b)\): at end of each period, rebalance back to same distribution of wealth \(b\).

Why CRP?

Intuition:

Take advantage of market volatility – “Buy low, Sell high”
CRP Example

- Two stocks: first one stays constant (cash), second one alternately doubles and halves

Investing in a single stock will not increase the wealth by more than a factor of $2$

Consider CRP($1/2, 1/2$):

<table>
<thead>
<tr>
<th>Stock #1</th>
<th>Stock #2</th>
<th>Our wealth</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>$1/2 + 1/2$</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>$3/4 + 3/4$</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>$3/4 + 3/8$</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>$(9/8)(3/2)$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

Remark

In hindsight, we see that a ($1/2, 1/2$) CRP is optimal among all CRPs.
CRP Example

- Two stocks: first one stays constant (cash), second one alternately doubles and halves
- Investing in a single stock will not increase the wealth by more than a factor of 2

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<tr>
<td></td>
<td>→</td>
<td>3/4 + 3/4</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>9/8</td>
</tr>
<tr>
<td></td>
<td>→</td>
<td>9/16 + 9/16</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>(3/2)(9/8)</td>
</tr>
<tr>
<td></td>
<td>→</td>
<td>27/32 + 27/32</td>
</tr>
<tr>
<td>1</td>
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<td>(9/8)^2</td>
</tr>
<tr>
<td></td>
<td></td>
<td>27/32 + 27/64</td>
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</tr>
<tr>
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<tr>
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<td></td>
<td>$\vdots$</td>
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Compete against the best CRP
Compete against the best CRP

What if we split up our wealth evenly among all CRPs and let it sit?
Compete against the best CRP

What if we split up our wealth evenly among all CRPs and let it sit?

- [Cover’91], [Cover & Ordentlich’96]
  - algorithm **Universal**
  - wealth \( \geq (\text{best CRP})/(t + 1)^{n-1} \)
  - better split \( \rightarrow \) wealth \( \geq (\text{best CRP})/\sqrt{(t + 1)^{n-1}} \)
  - per-day ratio \( \rightarrow 1 \)
Compete against the best CRP

What if we split up our wealth evenly among all CRPs and let it sit?

- [Cover’91], [Cover & Ordentlich’96]
  - algorithm **Universal**
  - wealth $\geq (\text{best CRP})/(t + 1)^{n-1}$
  - better split → wealth $\geq (\text{best CRP})/\sqrt{(t + 1)^{n-1}}$
  - per-day ratio → 1
- [Blum & Kalai’97]
  - simpler proof of previous result
  - extension to include transaction costs
Compete against the best CRP

What if we split up our wealth evenly among all CRPs and let it sit?

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  - wealth $\geq (\text{best CRP})/(t + 1)^{n-1}$
  - better split $\rightarrow$ wealth $\geq (\text{best CRP})/\sqrt{(t + 1)^{n-1}}$
  - per-day ratio $\rightarrow 1$

- [Blum & Kalai’97]
  - simpler proof of previous result
  - extension to include transaction costs

- [Helmbold, Schapire, Singer, Warmuth’98]
  - “experts”-based algorithm $EG(\eta)$
  - multiplicative update rule, as used in online regression [Kivinen & Warmuth]:
    find new wealth distribution vector $b^{t+1}$ that maximizes
    $\eta \log(w^{t+1} \cdot x^t) - D(w^{t+1}, w^t)$
  - worse guarantees, but better performance on historical data
Universal algorithm

- Split money evenly among all CRPs
- Let it sit (i.e. Do not transfer between CRPs)
Universal algorithm

- Split money evenly among all CRPs
- Let it sit (i.e. Do not transfer between CRPs)
- 4 CRPs

| CRP(1/3, 1/3, 1/3) | CRP(0,0,1) | CRP(0,1,0) | CRP(1,0,0) |

Limit is Universal algorithm. Guarantees: wealth of Universal \( \geq \frac{\text{best CRP}}{t+1} \)
Universal algorithm

- Split money evenly among all CRPs
- Let it sit (i.e. Do not transfer between CRPs)
- 4 CRPs
  - CRP($\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$)
  - CRP(0,0,1)
  - CRP(0,1,0)
  - CRP(1,0,0)
- 100 CRPs
  - CRP($\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$)
  - CRP(0,0,1)

Limit is Universal algorithm.

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More applications: data compression, language modelling, etc.
Universal algorithm

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<table>
<thead>
<tr>
<th>CRP($\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$)</th>
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</tr>
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100 CRPs

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<tr>
<th>CRP($\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$)</th>
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<th>CRP(0,0,1)</th>
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. . . . . . . .
Universal algorithm

- Split money evenly among all CRPs
- Let it sit (i.e. Do not transfer between CRPs)
- 4 CRPs
  \[
  \begin{array}{c|c|c|c}
  \text{CRP}(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) & \text{CRP}(0,0,1) & \text{CRP}(0,1,0) & \text{CRP}(1,0,0) \\
  \end{array}
  \]
- 100 CRPs
  \[
  \begin{array}{c|c|c|c}
  \text{CRP}(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) & \ldots & \text{CRP}(\frac{1}{7}, \frac{2}{7}, \frac{4}{7}) & \ldots & \text{CRP}(0,0,1) \\
  \end{array}
  \]

- Limit is \textbf{Universal} algorithm.

Guarantee: \[\text{wealth of Universal} \geq \frac{(\text{best CRP})}{(t+1)^n-1}\]

More applications: data compression, language modelling, etc.
Universal algorithm

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  - CRP(0,0,1)
  - CRP(0,1,0)
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Simple analysis (without commission)

Universal achieves avg. wealth of all CRP's. "Near" CRP's do nearly as well. Lots of CRP's are "near" the optimal CRP.
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Simple analysis (without commission)

- **Universal** achieves avg. wealth of all CRP’s.
- “Near” CRP’s do nearly as well.
- Lots of CRP’s are “near” the optimal CRP.
Proof.

1. We have $b$ is "near" $b_\ast$ if $b = (1 - \alpha) b_\ast + \alpha z$.

2. We require that $\text{Wealth of CRP}_b \geq (1 - \alpha) t + \alpha z$.

3. The probability of a random $b$ being "near" $b_\ast$ is $\text{Vol}(\alpha z | z \in \beta)$.

4. $\text{Vol}_\beta = \alpha t - 1$ for $\alpha = \frac{1}{t+1}$, we get $\text{Wealth of CRP}_x \geq \frac{1}{e}$.

5. Wealth of Universal $\geq \frac{1}{1+1}$.

A more refined analysis can get rid of the $\frac{1}{e}$.
Proof.

\[ \text{1. } \text{b is "near" } b^* \text{ if } b = (1 - \alpha)b^* + \alpha z \]
Proof.

1. **b** is “near” **b** if \( b = (1 - \alpha)b^* + \alpha z \)

2. \[
\frac{\text{Wealth of CRP}_b}{\text{Wealth of CRP}_{b^*}} \geq (1 - \alpha)^t
\]
Proof.

1. \( b \) is "near" \( b^* \) if \( b = (1 - \alpha)b^* + \alpha z \)

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\]

3. \[
\text{Prob}\{\text{a random } b \text{ is "near" } b^*\} \text{ is}
\]

\[
\frac{\text{Vol}\{(1 - \alpha)b^* + \alpha z | z \in \beta\}}{\text{Vol}\beta} = \frac{\text{Vol}\{\alpha z | z \in \beta\}}{\text{Vol}\beta} = \alpha^{t-1}
\]
Proof.

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4. if we choose \( \alpha = \frac{1}{t+1} \), we will get
\[
\frac{\text{Wealth of CRP}_x}{\text{Wealth of CRP}_y} \geq \left(\frac{t}{t+1}\right)^t \geq \frac{1}{e}
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Transaction Costs

Fixed % commission charged on purchases, paid for by sales CRPs pay commission (c < 1) as well

\[
x = (1 - \alpha) y + \alpha z \geq (1 - \alpha)(1 - \alpha c) \geq (1 - \alpha)^2 (1 + c)\]

Wealth of CRP x Wealth of CRP y \geq (1 - \alpha)(1 + c)^t \geq \text{Wealth of Universal}

So with commission, \[t \to t (1 + c)\]

Wealth of best CRP \[\geq (1 + c^t + 1)^n - 1\]
Transaction Costs

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Outline

1. Online Investing

2. Constant Rebalanced Portfolios

3. Algorithms competing against best CRP
   - Algorithms competing against best CRP
   - “Universal” algorithm

4. Implementation

5. Semi-Constant-Rebalanced Portfolios (SCRP)
Implementing “Universal”

- Uniform randomized approximation [Blum & Kalai]

Chebyshev’s inequality ensures that using $N \geq \left( \frac{R - 1}{\epsilon \delta} \right)$ random CRPs, with probability at least $1 - \delta$, wealth of approximation is at least $(1 - \epsilon) \cdot (\text{wealth of Universal})$.

For a given market, can determine in hindsight the optimal CRP [Helmbold] and estimate $R$ in the worst case, $R$ grows like $t^{n-1}$.

In practical experiments on stock market data, $R < 2$ for various combinations of two stocks.

Non-uniform randomized approximation [Kalai & Vempala]

Same performance guarantees with runtime polynomial in $\log(1/\eta)$, $1/\epsilon$, $t$, and $n$. 
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- Uniform randomized approximation [Blum & Kalai]
  - if the best CRP achieves wealth $R \cdot \text{Universal}$
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Semi-Constant-Rebalanced Portfolios (SCRP)

Definition

SCRP(\(b\)): at end of any subset of the periods, rebalance back to same distribution of wealth \(b\).

- Proposed as a good strategy in the presence of transaction costs [Helmbold]
- Flexible: one may prefer not to rebalance if transaction costs outweigh the benefits of rebalancing
- No strategy can guarantee the exponential growth rate of the best SCRP in hindsight, even without commission [Blum&Kalai]