2.1 Tensor Products of Transformations

Let $T_i \in L(V_i, U_i)$, $i = 1, \ldots, m$. The question arises: Is there a naturally induced linear transformation (l.t.) in $L(\otimes_{i=1}^m V_i, \otimes_{i=1}^m U_i)$ that depends on $T_1, \ldots, T_m$?

**Definition 2.1** (Tensor product of linear transformations)

Let $T_i \in L(V_i, U_i)$, $i = 1, \ldots, m$, and define $\phi \in M(V_1, \ldots, V_m; \otimes_{i=1}^m U_i)$:

$$\phi(v_1, \ldots, v_m) = T_1 v_1 \otimes T_2 v_2 \otimes \cdots \otimes T_m v_m.$$  \hspace{1cm} (1)

The l.t. $h \in L(\otimes_{i=1}^m V_i, \otimes_{i=1}^m U_i)$ that satisfies

$$h(v_1 \otimes \cdots \otimes v_m) = \phi(v_1, \ldots, v_m)$$  \hspace{1cm} (2)

is called the tensor product of the l.t.'s $T_1, \ldots, T_m$ and is denoted by

$$h = \otimes_{i=1}^m T_i$$  \hspace{1cm} (3)

or sometimes by

$$h = \prod_{i=1}^m T_i.$$  \hspace{1cm} (4)

Recall from Exercise 10, Section 1.2 that $h$ is uniquely determined by $\phi$ and hence $\otimes_{i=1}^m T_i$ is unambiguously defined.
Example 1.1  Recall that if \( S: W \rightarrow U \) is a linear mapping then \( S \) defines a unique linear mapping \( S': U^* \rightarrow W^* \) by the formula \((S'f)(w) = f(Sw), \ w \in W, \ f \in U^*\). The l.t. \( S' \) is of course called the dual of \( S \). Now suppose \( S: W \rightarrow U \) and \( T: V \rightarrow Z \) are fixed l.t.'s. Then we can define \( \mathcal{L}: L(U,V) \rightarrow L(W,Z) \) by the formula

\[
\mathcal{L}(A) = T(SA), \quad A \in L(U,V).
\]

As we know (Chapter 1, Theorem 4.1), \( L(U,V) = V \otimes U^* \) in which \((v \otimes f)(u) = f(u)v\), for any \( v \in V, \ f \in U^*, \ u \in U\). Thus for any \( w \in W \)

\[
\mathcal{L}(v \otimes f)(w) = (2(v \otimes f)S)(w)
\]

\[= T(v \otimes f)(Sw)\]

\[= Tf(Sw)v\]

\[= f(Sw)Tv\]

\[= S'(f)(w)Tv\]

\[= (Tv \otimes S'f)w\]

\[= (T \otimes S')(v \otimes f)(w)\].

In other words

\[
\mathcal{L}(v \otimes f) = (T \otimes S')v \otimes f
\]

and hence

\[
\mathcal{L} = T \otimes S',
\]

a linear mapping from \( V \otimes U^* = L(U,V) \) to \( Z \otimes W^* = L(W,Z) \). The example shows that the composition of linear transformations can be regarded as a tensor product of linear transformations. This idea was known to Sylvester but only recently has it been fully exploited.
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Theorem 1.1. If $T_i \in L(V_i, U_i)$ and $S_i \in L(W_i, V_i)$, $i = 1, \ldots, m$, then
\[
\left( \bigotimes_{i=1}^{m} T_i \right) \left( \bigotimes_{i=1}^{m} S_i \right) = \bigotimes_{i=1}^{m} T_i S_i.
\] (6)

Moreover, $\bigotimes_{i=1}^{m} T_i$ has a basis of decomposable tensors and
\[
\rho\left( \bigotimes_{i=1}^{m} T_i \right) = \prod_{i=1}^{m} \rho(T_i).
\] (7)

Proof: Let $w_i \in W_i$, $i = 1, \ldots, m$, and compute
\[
\left( \bigotimes_{i=1}^{m} T_i \right) \left( \bigotimes_{i=1}^{m} S_i \right) w_1 \otimes \cdots \otimes w_m = \bigotimes_{i=1}^{m} T_i S_i w_1 \otimes \cdots \otimes S w_m
\]
\[= T_1 S_1 w_1 \otimes \cdots \otimes T_m S_m w_m
\]
\[= \left( \bigotimes_{i=1}^{m} T_i S_i \right) w_1 \otimes \cdots \otimes w_m.
\]
Since the decomposable tensors span $\bigotimes_{i=1}^{m} W_i$, (6) follows.

To establish (7) let $\{e_{i_1}^1, \ldots, e_{i_1}^{r_1}, e_{i_1}^{r_1+1}, \ldots, e_{i_1}^{s_1} \}$ be a basis of $V_i$ so chosen that $r_1 = \rho(T_i)$, $T_i e_{i_1}^1, \ldots, T_i e_{i_1}^{s_1}$ are l.i. and $T_i e_{i_1}^j = 0$, $j = r_1 + 1, \ldots, s_1$. Then, obviously, $\bigotimes_{i=1}^{m} T_i e_{i_1}^\alpha = 0$ if $\alpha(i) > r_1$ for any $i = 1, \ldots, m$. On the other hand, if $\alpha \in \Gamma(r_1, \ldots, r_m)$ then
\[
\bigotimes_{i=1}^{m} T_i e_{i_1}^\alpha = T_1 e_{1}^\alpha(1) \otimes \cdots \otimes T_m e_{m}^\alpha(m)
\] (8)
and the tensors on the right-hand side in (8) are part of a basis of $\bigotimes_{i=1}^{m} U_i$. Since $|\Gamma(r_1, \ldots, r_m)| = \prod_{i=1}^{m} r_i$, the proof is complete.

The next result deals with a matrix representation of a tensor product. We recall some elementary facts concerning matrix representations: If $T: U \rightarrow V$ and $E = \{e_1, \ldots, e_n\}$ and $F = \{f_1, \ldots, f_m\}$
are ordered bases of $U$ and $V$, respectively, then for each $j = 1, \ldots, n$,

$$T e_j = \sum_{i=1}^{m} a_{ij} f_i.$$ 

The $m \times n$ matrix $A = [a_{ij}]$ is the matrix representation of $T$ and is denoted by

$$A = [T]^F_E.$$ 

If $m = n$ and $S: V \rightarrow U$, $ST = I_U$, $TS = I_V$, and $B = [S]^E_F$ then $AB = BA = I_n$, the $n$-square identity matrix. Also, if $P: V \rightarrow W$ then


where $G$ is an ordered basis of $W$.

**Theorem 1.2** Let $T_i \in L(V_i, U_i)$ and let $E_i = \{e_{i1}, \ldots, e_{iq_i}\}$ and $F_i = \{f_{i1}, \ldots, f_{ip_i}\}$ be ordered bases of $V_i$ and $U_i$, respectively, $i = 1, \ldots, m$. Let $A_i = [T_i]^F_{E_i}$, $i = 1, \ldots, m$. If $E = \{e^\otimes, \alpha \in \Gamma(q_1, \ldots, q_m)\}$ and $F = \{f^\otimes, \alpha \in \Gamma(p_1, \ldots, p_m)\}$, each ordered lexicographically, then for $\alpha \in \Gamma(p_1, \ldots, p_m)$ and $\beta \in \Gamma(q_1, \ldots, q_m)$ the $(\alpha, \beta)$ entry in the matrix representation

$$\begin{bmatrix} T_i \otimes I \end{bmatrix}^F_E$$

is

$$\sum_{i=1}^{m} a_i(\alpha(1), \beta(1))$$

(9)

where $a_i(s,t)$ is the $(s,t)$ entry of $A_i$. 

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Proof: Let $\beta \in \Gamma(q_1, \ldots, q_m)$ and compute that

$$\otimes_{i=1}^{m} T_i^\beta = T_1^{e_1} \otimes \cdots \otimes T_m^{e_m} \beta(m)$$

$$= \sum_{i=1}^{p_1} a_1(i, \beta(1)) f_{1i} \otimes \cdots \otimes \sum_{i=1}^{p_m} a_m(i, \beta(m)) f_{mi}$$

$$= \sum_{\alpha \in \Gamma(p_1, \ldots, p_m)} a_1(\alpha(1), \beta(1)) \cdots a_m(\alpha(m), \beta(m)) f_{\alpha}^\otimes.$$  \hfill 1

Definition 1.2 (Kronecker product) Let $A_i = [a_i(s, t)]$ be a $p_i \times q_i$ matrix, $i = 1, \ldots, m$. Set $p = \prod_{i=1}^{m} p_i$ and $q = \prod_{i=1}^{m} q_i$. Let $A$ be the $p \times q$ matrix whose entries are indexed lexicographically with the elements $\alpha \in \Gamma(p_1, \ldots, p_m)$ and $\beta \in \Gamma(q_1, \ldots, q_m)$ as follows:

$$A_{\alpha, \beta} = \prod_{i=1}^{m} a_i(\alpha(i), \beta(i)). \quad (10)$$

Then $A$ is called the Kronecker or direct product of the $A_i$ and is denoted by

$$A = \bigotimes_{i=1}^{m} A_i$$

or

$$A = \prod_{i=1}^{m} A_i.$$  \hfill 2

Examples 1.2 (a) Let

$$I: \otimes V_i \rightarrow \bigotimes_{i=1}^{p} V_i \otimes \bigotimes_{i=p+1}^{p+q} V_i$$

be the unique bijection (described in Exercise 4, Section 1.3) that satisfies
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\[ i(v_1 \otimes \cdots \otimes v_{p+q}) = (v_1 \otimes \cdots \otimes v_p) \otimes (v_{p+1} \otimes \cdots \otimes v_{p+q}) \]

If \( E_i = \{ e_{i_1} \otimes \cdots \otimes e_{i_n} \} \) is a basis of \( V_i \), \( i = 1, \ldots, p+q \), then

\[ E = \{ e_\alpha \otimes \gamma \in \Gamma(n_1, \ldots, n_{p+q}) \} \]

is a basis of \( \bigotimes_{i=1}^{p+q} V_i \) and

\[ E' = \{ e_\gamma \otimes e_\tau \gamma \in \Gamma(n_1, \ldots, n_p), \ \tau \in \Gamma(n_{p+1}, \ldots, n_{p+q}) \} \]

is a basis of \( \bigotimes_{i=1}^p V_i \otimes \bigotimes_{i=p+1}^{p+q} V_i \). The ordering in \( E \) is taken to be lexicographic and the ordering in \( E' \) is described as follows:

\( e_\gamma \otimes e_\tau \) comes before \( e_{\gamma'} \otimes e_{\tau'} \), iff \( \gamma \) precedes \( \gamma' \) lexicographically or \( \gamma = \gamma' \) and \( \tau \) precedes \( \tau' \) lexicographically.

What we are after here is the matrix \( [i]_{E}^{E'} \). We have

\[ i(e_\alpha) = e_\gamma \otimes e_\tau \]

where \( \gamma = (\alpha(1), \ldots, \alpha(p)) \) and \( \tau = (\alpha(p+1), \ldots, \alpha(p+q)) \). Now suppose \( i(e_\alpha) \) is the \( k^{th} \) basis element in the ordering in \( E' \).

Let the corresponding \( \gamma \) and \( \tau \) be the \( r^{th} \) and \( s^{th} \) elements, respectively, in the lexicographic orderings in \( \Gamma(n_1, \ldots, n_p) \) and \( \Gamma(n_{p+1}, \ldots, n_{p+q}) \), respectively. A moment's reflection will show that

\[ k = (r-1)n_{p+1} \cdots n_{p+q} + s. \]  \hspace{1cm} (11)

However, the lexicographic ordering in \( \Gamma(n_1, \ldots, n_{p+q}) \) can be achieved by first ordering the \( \alpha \)'s by their initial segments

\( (\alpha(1), \ldots, \alpha(p)) = \gamma \) lexicographically and then for each fixed \( \gamma \) ordering the set of \( \alpha \) of the form
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\[ \sigma = (\gamma(1), \ldots, \gamma(p), \tau(1), \ldots, \tau(q)) \]

lexicographically in \( \tau \). It follows that \( \epsilon^k_\sigma \) is also \( k^{th} \) in the ordering in \( \Gamma(n_1, \ldots, n_{p+q}) \). What we have said can be simply thought of as follows: In order to alphabetize a list of words each involving \( p + q \) letters, first alphabetize according to the first \( p \) letters and then alphabetize each group of words beginning with a fixed \( p \) letter word alphabetically by the last \( q \) letters. We can conclude that the \( k^{th} \) basis vector in \( E \) is mapped into the \( k^{th} \) basis vector in \( F \) by \( \iota \) and hence we have

\[ (\iota)_E^F = I_N \]  

(12)

where \( N = n_1 \cdots n_{p+q} \).

(b) We show that Kronecker product multiplication is associative. In fact, the parentheses in any grouping of factors can be removed:

\[ (A_1 \otimes \cdots \otimes A_p) \otimes (A_{p+1} \otimes \cdots \otimes A_{p+q}) = A_1 \otimes \cdots \otimes A_{p+q}. \]  

(13)

This is easily seen by using the result of the preceding example. First obtain spaces \( V_i \) and \( U_i \), \( T_i \in L(V_i, U_i) \), and bases \( E_i = \{ e_{i1}, \ldots, e_{iq_i} \} \) and \( F_i = \{ f_{i1}, \ldots, f_{ip_i} \} \) of \( V_i \) and \( U_i \), respectively, for which \( [T_i]_{E_i}^{F_i} = A_i \), \( i = 1, \ldots, p+q \). If \( E \) and \( F \) are the bases of \( \otimes \sum \limits_{i=1}^{p+q} V_i \) and \( \otimes \sum \limits_{i=1}^{p+q} U_i \), respectively, described in Theorem 1.2 then

\[ \left[ \begin{array}{c} E \\ \otimes T_i \\ F \\ \sum \limits_{i=1}^{p+q} V_i \end{array} \right] = A_1 \otimes \cdots \otimes A_{p+q}. \]  

(14)

Now let

\[ \delta_{p+q}^{p} V_i \rightarrow \otimes V_i \otimes \otimes V_i \]

\[ \ell_i: \delta_{p+q}^{p} V_i \rightarrow \otimes V_i \otimes \otimes V_i \]
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and

\[ i_2: \bigotimes_{i=1}^{p+q} U_i \rightarrow \bigotimes_{i=1}^{p} U_i \otimes \bigotimes_{i=p+1}^{p+q} U_i \]

be the mappings described in part (a). Then a simple evaluation of both sides on any decomposable tensor \( v_1 \otimes \cdots \otimes v_{p+q} \) shows that

\[ i_2^{-1} \left( \bigotimes_{1}^{p} T_i \otimes \bigotimes_{p+1}^{p+q} T_i \right)_{E} \bigotimes_{1}^{p+q} T_i = \bigotimes_{1}^{p+q} T_i. \]  \( \text{(15)} \)

Now let \( E' \) and \( F' \) be the corresponding bases of \( \bigotimes_{1}^{p} V_i \otimes \bigotimes_{1}^{p+q} V_i \) and \( \bigotimes_{1}^{p} U_i \otimes \bigotimes_{1}^{p+q} U_i \), respectively, ordered as in the preceding example. Then from (12) and (15) and the properties of matrix representations we have

\[ \left[ \bigotimes_{1}^{p+q} T_i \right]_{E} = [i_2^{-1}]_{E'}^{F'} \left[ \bigotimes_{1}^{p} T_i \otimes \bigotimes_{p+1}^{p+q} T_i \right]_{E'} \left[ i_1 \right]_{E}^{F'} \]

\[ = \left[ \bigotimes_{1}^{p} T_i \otimes \bigotimes_{p+1}^{p+q} T_i \right]_{E}. \] \( \text{(16)} \)

According to part (a) the basis \( E' \) is constructed by forming the tensors \( e_{\gamma} \otimes e_{\tau}, \gamma \in \Gamma(q_1, \ldots, q_p) \) and \( \tau \in \Gamma(q_{p+1}, \ldots, q_{p+q}) \) first ordered lexicographically in \( \gamma \), then in \( \tau \), and similar remarks pertain to \( F' \). Thus from Theorem 1.2, in the case of the tensor product of just two transformations, we have

\[ \left[ \bigotimes_{1}^{p} T_i \otimes \bigotimes_{p+1}^{p+q} T_i \right]_{E}' = A \otimes B. \]
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\[(e_\tau^\otimes, \tau \in \Gamma(p_{p+1}, \ldots, q_{p+q}) \text{ and } f_\tau^\otimes, \tau \in \Gamma(p_{p+1}, \ldots, p_{p+q})].\]

But then \(A = A_1 \otimes \cdots \otimes A_p\) and \(B = A_{p+1} \otimes \cdots \otimes A_{p+q}\) and we have the formula (13),

(c) By taking matrix representations in (6) and (7) we have

\[
\left( \bigotimes_{i=1}^{m} A_i \right) \left( \bigotimes_{i=1}^{m} B_i \right) = \bigotimes_{i=1}^{m} A_i B_i, \tag{17}
\]

where each matrix product \(A_i B_i\) is defined, and also

\[
\rho \left( \bigotimes_{i=1}^{m} A_i \right) = \prod_{i=1}^{m} \rho(A_i). \tag{18}
\]

(d) We show that if \(A_1\) is \(p_1 \times q_1\) and \(A_2\) is \(p_2 \times q_2\), then the \(p_1 p_2 \times q_1 q_2\) matrix \(A = A_1 \otimes A_2\) can be partitioned into \(p_2 \times q_2\) blocks so that the \((s, t)\) block is \(a_1(s, t)A_2, s = 1, \ldots, p_1,\)
\(t = 1, \ldots, q_1\). For, the sequences in \(\Gamma(p_1, p_2)\) arranged lexicographically can be subdivided into subsets of \(p_2\) subsequences:

\[
(1, j), j = 1, \ldots, p_2; \ (2, j), j = 1, \ldots, p_2; \quad \ldots; \ (p_1, j), j = 1, \ldots, p_2. \tag{19}
\]

Similarly, the sequences in \(\Gamma(q_1, q_2)\) can be subdivided into subsets of \(q_2\) subsequences:

\[
(1, j), j = 1, \ldots, q_2; \ (2, j), j = 1, \ldots, q_2; \quad \ldots; \ (q_1, j), j = 1, \ldots, q_2. \tag{20}
\]

The sequences in the \(s^{th}\) subset in (19) and the \(t^{th}\) subset in (20) define a \(p_2 \times q_2\) submatrix of \(A\), whose \((i, j)\) entry is
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\[ A(s,i),(t,j), \quad i = 1, \ldots, p_2, \quad j = 1, \ldots, q_2: \]

\[ A(s,i),(t,j) = a_1(s,t)a_2(i,j) \]

\[ = (a_1(s,t)A_2)_{i,j}. \]

(e) It follows from (d) that if \( A_1 \) and \( A_2 \) are square matrices then

\[ \text{tr}(A_1 \otimes A_2) = \sum_{s=1}^{p_1} a_1(s,s) \sum_{j=1}^{p_2} a_2(j,j) \]

\[ = \text{tr}(A_1)\text{tr}(A_2). \]

A trivial induction using (b) above shows that if \( A_1, \ldots, A_m \) are square matrices then

\[ \text{tr}\left( \bigotimes_{i=1}^{m} A_i \right) = \prod_{i=1}^{m} \text{tr}(A_i), \quad (21) \]

and from Theorem 1.2, if \( T_i \in L(V_1, V_1), \quad i = 1, \ldots, m \), then

\[ \text{tr}\left( \bigotimes_{i=1}^{m} T_i \right) = \prod_{i=1}^{m} \text{tr}(T_i). \quad (22) \]

(f) As our final example of the use of Theorem 1.2, we compute the value of \( \det( \bigotimes_{i=1}^{m} T_i ) \) if \( T_i \in L(V_1, V_1) \), where \( \dim V_1 = n_1, \quad i = 1, \ldots, m. \)

First let \( A_1 \) be \( n_1 \times n_1 \) and \( A_2 \) be \( n_2 \times n_2 \). Let \( S_1 \) be a nonsingular matrix that brings \( A_1 \) to triangular form with its characteristic roots \( \alpha_1, \ldots, \alpha_{n_1} \) on the main diagonal. Then (see Exercise 2)
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\[(s_1 \otimes I_{n_2})^{-1}(A_1 \otimes A_2)(s_1 \otimes I_{n_2}) = (s_1^{-1} \otimes I_{n_2})(A_1 \otimes A_2)(s_1 \otimes I_{n_2})\]

\[= s_1^{-1} A_1 s_1 \otimes A_2\]

\[
\begin{bmatrix}
\alpha_{1}^{A_2} \\
\alpha_{2}^{A_2} \\
\ddots \\
\alpha_{n_1}^{A_2}
\end{bmatrix}
\]

Thus

\[
\det(A_1 \otimes A_2) = \prod_{i=1}^{n_1} \det(\alpha_i A_2) = \prod_{i=1}^{n_1} \alpha_i \det(A_2) = \det(A_1)^{n_2} \det(A_2)^{n_1}.
\]

If we use (13),

\[
\det(A_1 \otimes A_2 \otimes \cdots \otimes A_m) = \det(A_1 \otimes (A_2 \otimes \cdots \otimes A_m))
\]

\[= \det(A_1)^{n_2 \cdots n_m} \det(A_2 \otimes \cdots A_m)^{n_1}
\]

\[= \cdots
\]

\[= \det(A_1)^{n_2 \cdots n_m} \det(A_2)^{n_1 n_3 \cdots n_m} \det(A_m)^{n_1 \cdots n_{m-1}},
\]

where \(A_i\) is \(n_i\)-square, \(i = 1, \ldots, m\). If we use Theorem 1.2, we then have (setting \(n = n_1 \cdots n_m\))
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\[
\det_T = \prod_{i=1}^{m} \det(T_{i}^{n/n_i}).
\]  

(23)

We recall the notion of an invariant subspace. If \( T: V \rightarrow V \) is linear and \( W \) is a subspace of \( V \) such that \( T(W) \subseteq W \), then \( W \) is an invariant subspace of \( T \). If we set \( T_1 = T|_W \), the restriction of \( T \) to \( W \), then \( T_1 \in L(W,W) \). If \( V = W_1 \oplus W_2 \) in which both \( W_1 \) and \( W_2 \) are invariant subspaces of \( T \), then \( T_1 = T|_{W_1} \) and \( T_2 = T|_{W_2} \) are linear and we say that \( T \) is the direct sum of \( T_1 \) and \( T_2 \) written

\[
T = T_1 \oplus T_2.
\]

In general, if \( V = \bigoplus_{i=1}^{p} W_i \) and \( W_i \) is an invariant subspace of \( T \), we write

\[
T = \bigoplus_{i=1}^{p} T_i,
\]

where \( T_i = T|_{W_i}, \ i = 1, \ldots, m. \)

We have

**Theorem 1.3** Let \( V_i = \bigoplus_{j=1}^{m_i} W_{ij}, \ i = 1, \ldots, p, \) and suppose

\[
T_i \in L(V_i,V_i), \ T_i = \sum_{j=1}^{m_i} T_{ij}, \ T_{ij} = T_i|_{W_{ij}},
\]

i.e., \( W_{ij} \) is an invariant subspace of \( T_i; \ j = 1, \ldots, m; \ i = 1, \ldots, p, \) then

\[
W_\alpha = \bigotimes_{i=1}^{p} W_{i\alpha(i)}, \ \alpha \in \Gamma(m_1, \ldots, m_p)
\]
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is an invariant subspace of

\[ \bigotimes_{l=1}^{p} T_i |_{W_\alpha} = \bigotimes_{i=1}^{p} T_{i\alpha(i)} \]

and

\[ \bigotimes_{l=1}^{p} T_i = \sum_{\alpha \in \Gamma} \bigotimes_{i=1}^{p} T_{i\alpha(i)} \]  \hspace{1cm} (24)

Proof: We know from Theorem 3.3, Chapter I that

\[ \bigotimes_{l=1}^{p} V_i = \sum_{\alpha \in \Gamma} W_\alpha \]  \hspace{1cm} (25)

Let \( W_i \in W_{i\alpha(i)}, \ i = 1, \ldots, p \). Then since \( T_i |_{W_{i\alpha(i)}} = T_{i\alpha(i)} \),

\[ \bigotimes_{l=1}^{p} T_i W = T_{i1} W_1 \otimes \cdots \otimes T_{ip} W_p \]

\[ = T_{i\alpha(1)} W_1 \otimes \cdots \otimes T_{ip\alpha(p)} W_p \]

\[ = \bigotimes_{l=1}^{p} T_{i\alpha(i)} W_l \]  \hspace{1cm} (26)

Thus

\[ \bigotimes_{l=1}^{p} T_i |_{W_\alpha} = \bigotimes_{l=1}^{p} T_{i\alpha(i)} \]  \hspace{1cm} (26)

However, (25) and (26) imply (24).  \[ \square \]

We shall return in a later section to the analysis of the classic invariants of a tensor product of l.t.'a. The result of Theorem 1.3 will be a valuable tool for doing this.

Exercises

1. Show that if \( T_i \in \mathcal{L}(V_1, V_1), \ i = 1, \ldots, m, \) and \( \bigotimes_{l=1}^{m} T_i = 0 \) then

some \( T_i = 0. \)
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Hint: Otherwise there exists \( v_i \neq 0 \) such that \( T_i v_i \neq 0 \). But then \( \otimes_1^{m} T_i v_i \otimes v_m = T_i v_i \otimes \cdots \otimes v_m \neq 0 \) (Why?).

2. Show that if \( T_i \in L(V_i, U_i) \) has an inverse, \( i = 1, \ldots, m \), then so does \( \otimes_1^{m} T_i \) and \( \left( \otimes_1^{m} T_i \right)^{-1} = \otimes_1^{m} T_i^{-1} \).

3. Show that if \( A_1, \ldots, A_m \) are upper (lower) triangular square matrices then so is \( A_1 \otimes \cdots \otimes A_m \).

4. Show that if \( A_1 \otimes \cdots \otimes A_m \) is the 0 matrix then some matrix \( A_i \) must be 0.

5. Show that if any of the square matrices \( A_1, \ldots, A_m \) is nilpotent (i.e., some positive power is the 0 matrix) then \( A_1 \otimes \cdots \otimes A_m \) is nilpotent.

6. Let \( \sigma \in S_m \). Show that there exists a nonsingular linear transformation \( \tau_{\sigma} : \otimes_1^{m} V_i \rightarrow \otimes_1^{m} V_{\sigma^{-1}(i)} \) so that for arbitrary \( T_i \in L(V_i, V_i), \ i = 1, \ldots, m, \)

\[
\tau_{\sigma}^{-1} \otimes_1^{m} T_i \tau_{\sigma} = \otimes_1^{m} T_{\sigma(i)}.
\]

Hint: Define \( \tau_{\sigma} \) by

\[
\tau_{\sigma}(v_1 \otimes \cdots \otimes v_m) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(m)}.
\]

Show that \( \tau_{\sigma^{-1}} = \tau_{\sigma}^{-1} \), i.e., \( \tau_{\sigma^{-1}} \tau_{\sigma} = \text{identity} \).

7. Generalize the preceding exercise as follows: Let \( \sigma \in S_m \).

Show that there exist nonsingular linear transformations \( \tau_{\sigma} : \otimes_1^{m} V_i \rightarrow \otimes_1^{m} V_{\sigma^{-1}(i)} \) and \( \tau_{\sigma'} : \otimes_1^{m} U_i \rightarrow \otimes_1^{m} U_{\sigma^{-1}(i)} \) such that
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\[ \tau_{\sigma}^{-1} \otimes_{\lambda} T_{\sigma} = \otimes_{\lambda} T(\sigma). \]

where \( T_\lambda \in L(V_\lambda, U_\lambda), \ i = 1, \ldots, m. \)

8. Using Theorem 1.2 and Exercise 6, show that if \( \sigma \in S_m \) then there exists a permutation matrix \( Q \) depending only on \( \sigma \) such that

\[ Q^t(A_1 \otimes \cdots \otimes A_m)Q = A_{\sigma(1)} \otimes \cdots \otimes A_{\sigma(m)}, \]

where the matrices \( A_1, \ldots, A_m \) are square (\( Q^t \) is the transpose of the matrix \( Q \)).

Hint: the matrix representation of \( \tau_\sigma \) is a permutation matrix.

9. Referring to Example 1.1, prove the following facts about the dual of a linear transformation:

(i) \( (aS_1 + bS_2)^t = aS_1^t + bS_2^t, \ a, b \in \mathbb{R}. \)

(ii) If \( \mathcal{E} = \{ e_1, \ldots, e_n \} \) and \( \mathcal{G} = \{ g_1, \ldots, g_m \} \) are bases of \( \mathcal{W} \) and \( \mathcal{U} \), respectively, and \( \mathcal{F} = \{ f_1, \ldots, f_n \} \) and \( \mathcal{K} = \{ k_1, \ldots, k_m \} \) are dual bases of \( \mathcal{W}^* \) and \( \mathcal{U}^* \), then \( [S]_E^F \) is the transpose of the matrix \( [S]_E^G \).

(iii) Show that if \( A \) is a matrix representation of \( T \) and \( B \) is a matrix representation of \( S \), then bases of \( V \otimes U^* = L(U, V) \) and \( Z \otimes \mathcal{W}^* = L(N, Z) \) may be chosen so that \( Z \) has the matrix representation \( A \otimes B^t \).

10. Show that if \( A = A_1 \otimes \cdots \otimes A_m \neq 0 \) is a diagonal matrix (i.e., no nonzero off-diagonal entries), then each \( A_\lambda \) is a diagonal matrix.
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Hint: Suppose \( a_h(i,j) \neq 0, \ i \neq j \). Choose \( \alpha, \beta \) such that 
\[ a_h(\alpha(t), \beta(t)) \neq 0, \ t \neq k, \ \text{and} \ \alpha(k) = i \ \text{and} \ \beta(k) = j. \] Then 
\[ A_{\alpha, \beta} \neq 0 \ \text{and} \ \alpha \neq \beta. \]

11. Show that if \( T \in \mathcal{L}(V, V) \), \( \dim V = n \), then
\[
\det \bigotimes_{l=1}^{m} T = \left( \det T \right)^{m-1}.
\]

Hint: This follows from formula (23).

2.2 Properties of Mappings on Tensor Spaces

Recall that if \( T \in \mathcal{L}(V, V) \) then a triangular basis for \( T \) is 
an ordered basis \( E = \{ e_1, \ldots, e_n \} \) of \( V \) for which \( T e_i = \lambda_i e_i + v_i \), 
\( v_i \in \langle e_1, \ldots, e_{i-1} \rangle, \ i = 1, \ldots, n \). It is an easy argument to see that 
if the underlying field \( \mathbb{R} \) contains every eigenvalue (e.v.) \( \lambda_i \) of 
\( T \) then a triangular basis always exists (see Exercise 1). Moreover, 
if \( V \) is a unitary space then the triangular basis can be chosen to 
be o.n. (This is the Schur triangularization theorem, see Exercise 2.) 
These properties are inherited by the tensor product of l.t.'s.

Theorem 2.1 Let \( T_i \in \mathcal{L}(V_i, V_i) \), and let \( E_i = \{ e_{i1}, \ldots, e_{in_i} \} \) 
be a basis of \( V_i, \ i = 1, \ldots, m \). Let \( E = \{ e_{\alpha} \otimes \ \alpha \in \Gamma(n_1, \ldots, n_m) \} \)
be the corresponding basis of \( \bigotimes_{l=1}^{m} V_i \). If \( E_i \) is a triangular basis 
for \( T_i \), then \( E \) (with the usual lexicographic ordering) is a 
triangular basis for \( T = \bigotimes_{l=1}^{m} T_i \). Moreover, if each \( E_i \) is an o.n. 
triangular basis for \( T_i \), then \( E \) is an o.n. triangular basis for \( T \).

Proof. Let \( \lambda_{ij}, \ j = 1, \ldots, n_i \), be the e.v.'s of \( T_i \), 
\( i = 1, \ldots, m \). If \( E_i \) is triangular for \( T_i \), then for \( \alpha \in \Gamma \) we
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have

$$T_\alpha \otimes = T_1 e_\alpha(1) \otimes \cdots \otimes T_m e_\alpha(m)$$

$$= (\lambda_1 e_\alpha(1) + v_1) \otimes \cdots \otimes (\lambda_m e_\alpha(m) + v_m)$$

in which $$v_t \in \langle e_{t,j} ; j < \alpha(t) \rangle$$, $$t = 1, \ldots, m$$ (i.e., $$v_1 = 0$$). Continuing,

$$T_\alpha \otimes = \left( \prod_{t=1}^m \lambda_t e_\alpha(t) \right) e_\alpha + w, \quad (1)$$

where $$w$$ is a linear combination of tensor products of vectors

$$x_\otimes = x_1 \otimes \cdots \otimes x_m$$

in which $$x_t$$ is either $$e_\alpha(t)$$ or

$$x_t = v_t \in \langle e_{t,j} ; j < \alpha(t) \rangle$$

and the latter alternative must hold for at least one $$t$$. It is clear then that $$x_\otimes$$ is a linear combination of basis elements $$e_\beta$$ in which $$\beta$$ precedes $$\alpha$$ in lexicographic order and hence the same statement holds for $$w$$. Thus from (1) we see that $$E$$ is a triangular basis for $$T$$. We also observe the important fact that the e.v.'s of $$T$$ are the $$n_1 \cdots n_m$$ numbers

$$\prod_{t=1}^m \lambda_t e_\alpha(t), \quad \alpha \in \Gamma. \quad (2)$$

(We also remark that the ordering of each set $$\{\lambda_{t_1}, \ldots, \lambda_{t_n} \}$$ may be prescribed in advance.) The second assertion of the theorem follows immediately from Theorem 4.5 in Chapter 1.

Example 2.1(a) The converse of Theorem 2.1 is not true, i.e., the fact that $$E$$ is a triangular basis for $$T$$ does not imply that $$E_i$$ is a triangular basis for $$T_i$$, $$i = 1, \ldots, m$$. It clearly suffices to exhibit two matrices whose Kronecker product is upper triangular, one of which is not upper triangular. Thus take
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\[ A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}. \]

Then

\[ A \otimes B = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \]

However a partial converse of Theorem 2.1 is contained in Exercise 3.

(b) Observe that if \( T_{i}^{u_{ij}} = \lambda_{i,j}^{u_{ij}} \), then

\[ \bigotimes_{1}^{m} T_{i}^{u_{\alpha}} = \prod_{i=1}^{m} \lambda_{i\alpha(i)}^{u_{\alpha}} \]

and hence \( u_{\alpha}^{\otimes} \) is an eigenvector of \( \bigotimes_{1}^{m} T_{i} \) corresponding to

\[ m \prod_{i=1}^{m} \lambda_{i\alpha(i)}. \]

We are now in a position to prove a number of interesting results about the tensor product.

**Theorem 2.2** Let \( T_{i} \in \mathbb{L}(V_{i}, V_{i}) \) and assume that \( V_{i} \) is a unitary space, \( i = 1, \ldots, m \). If each \( T_{i} \) is (a) normal; (b) hermitian; (c) positive-definite (p.d.); (d) positive semi-definite (p.s.d.); (e) unitary; (f) skew hermitian (and \( m \) is odd), then \( \bigotimes_{1}^{m} T_{i} \) has the corresponding property. Conversely, if \( 0 \neq \bigotimes_{1}^{m} T_{i} \) is normal, then so is every \( T_{i}, \ i = 1, \ldots, m \).

**Proof:** (a) Choose an o.n. basis \( E_{i} = \{ e_{1i}, \ldots, e_{n_{i}} \} \) of eigenvectors of \( T_{i} \):

\[ T_{i} e_{ij} = \lambda_{i,j}^{e_{ij}}, \quad j = 1, \ldots, n_{i}, \quad i = 1, \ldots, m. \]

Let \( E = \{ e_{\alpha}^{\otimes}, \ \alpha \in \Gamma(n_{1}, \ldots, n_{m}) \} \) be the corresponding o.n. basis of
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\[ m \otimes V_i \]. Then obviously

\[ \otimes_{\lambda_i \alpha(i)} = \prod_{i=1}^{m} \lambda_i \alpha(i) \alpha_i \]  \hspace{1cm} (3)  

and thus \( \otimes_{\lambda_i} \) possesses an o.n. basis of eigenvectors and thereby is normal. Conversely, suppose \( \otimes_{\lambda_i} \) is normal and let the basis \( \E_i \) of \( V_i \) be an o.n. triangular basis for \( T_i \), \( i = 1, \ldots, m \). Then, by Theorem 2.1, \( E \) is an o.n. triangular basis of \( \otimes_{\lambda_i} \).

However, a triangular o.n. basis for a normal transformation must in fact be an o.n. basis of eigenvectors. This means that if \( A_i = [T_i]_{E_i} \), then

\[ A_1 \otimes \cdots \otimes A_m = [\otimes_{\lambda_i}]_E \]

is a diagonal matrix and hence that \( E_i \) is an o.n. basis of eigenvectors of \( T_i \), \( i = 1, \ldots, m \); but then each \( T_i \) is normal.

(b) This follows from (a) and the fact that all the \( \lambda_{ij} \) are real. For, from (3), \( \otimes_{\lambda_i} \) is normal and has real eigenvalues.

Parts (c), (d), (e), and (f) all follow in a similar way. \( \Box \)

The extent to which properties (b) - (f) of Theorem 2.2 can be inferred for each \( T_i \) from the corresponding properties of \( \otimes_{\lambda_i} \) is dealt with in Exercises 4 - 8.

Parts of Theorem 2.2 can be proved quite easily by using the following elementary formula.

\[ (\otimes_{\lambda_i})^* = \otimes_{\lambda_i}^* \]  \hspace{1cm} (4)  

In (4) we are assuming that \( T_i \in L(U_i, V_i) \), \( U_i \) and \( V_i \) are unitary spaces, and \( T_i^* \) is the conjugate dual of \( T_i \) satisfying
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\[ (T_i u_i, v_i) = (u_i, T_i^* v_i), \quad u_i \in U_i, \quad v_i \in V_i \]

and finally that the conjugate dual on the left in (4) is computed with respect to the induced inner products in \( \otimes_{i=1}^m U_i \) and \( \otimes_{i=1}^m V_i \) given by Theorem 4.5, Chapter 1. The formula (4) is trivial to establish:

\[
\left( \left( \otimes_{i=1}^m T_i \right) u_1 \otimes \cdots \otimes u_m, v_1 \otimes \cdots \otimes v_m \right) = \prod_{i=1}^m (T_i u_i, v_i) \\
= \prod_{i=1}^m (u_i, T_i^* v_i) \\
= (u, \otimes_{i=1}^m T_i^* v). \quad (5)
\]

Since decomposable elements span, the computation (5) establishes (4).

**Example 2.2** We use formula (4) to prove Theorem 2.2(a). This is easy using Theorem 1.1:

\[
\left( \otimes_{i=1}^m T_i \right)^* \left( \otimes_{i=1}^m T_i \right) = \otimes_{i=1}^m T_i^* \otimes_{i=1}^m T_i \\
= \otimes_{i=1}^m T_i^* T_i \\
= \otimes_{i=1}^m T_i^* T_i \\
= \otimes_{i=1}^m T_i \otimes_{i=1}^m T_i^* \\
= \left( \otimes_{i=1}^m T_i \right) \left( \otimes_{i=1}^m T_i \right)^*.
\]
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At this point we define a linear operator on $\otimes_{l}^{m} V$ associated with a permutation $\sigma$ in $S_{m}$, the symmetric group of degree $m$.

**Definition 2.1 (Permutation operator)** Let $\sigma \in S_{m}$ and define

$$\varphi \in \mathcal{M}(V, \ldots, V; \otimes_{l}^{m} V)$$

by

$$\varphi(v_{1}, \ldots, v_{m}) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(m)}.$$ 

Let $P(\sigma)$ denote the unique linear map in $\mathcal{L}(\otimes_{l}^{m} V, \otimes_{l}^{m} V)$ satisfying

$$P(\sigma) \otimes = \varphi.$$ 

Then $P(\sigma)$ is called the **permutation operator** associated with $\sigma$.

**Example 2.3 (a)** The space of $n$-square matrices $M_{n,n}(\mathbb{R})$, is a tensor product of $V_{n}(\mathbb{R})$ with itself (see Exercise 2, Section 1.2). Then if $\sigma = (12) \in S_{2}$,

$$P(\sigma) x \otimes y = y \otimes x = (x \otimes y)^{T}.$$ 

Hence $P(\sigma) A = A^{T}$ for any $n$-square matrix $A$. Thus a permutation operator can be thought of as a generalization of the notion of the transpose of a matrix.

(b) The correspondence $\sigma \mapsto P(\sigma)$ forms a **faithful representation** of $S_{m}$ in terms of linear operators in $\mathcal{L}(\otimes_{l}^{m} V, \otimes_{l}^{m} V)$. This means simply that:

(i) $P(\sigma \theta) = P(\sigma) P(\theta)$;  

(ii) $P(e) = I$, the identity mapping on $\otimes_{l}^{m} V$;  

(iii) $P(\sigma)$ is nonsingular and $P(\sigma)^{-1} = P(\sigma^{-1})$.
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(iii) $F(\sigma) = F(\theta)$ iff $\sigma = \theta$ (this requires that $\dim V > 1$).

For, from (6),

$$F(\sigma \theta) v^{\otimes} = v^{(\sigma \theta^{-1}(1) \otimes \cdots \otimes v^{(\sigma \theta^{-1}(m)$$

$$= v^{\theta^{-1}(\sigma^{-1}(1)) \otimes \cdots \otimes v^{\theta^{-1}(\sigma^{-1}(m)$$

$$= F(\sigma) v^{\theta^{-1}(1) \otimes \cdots \otimes v^{\theta^{-1}(m)$$

$$= F(\sigma) F(\theta) v^{\otimes}.$$  

The decomposable elements span so (7) follows. The proof of (ii) is trivial and (iii) follows from (i) and (ii): $F(\sigma) F(\sigma^{-1}) = F(\sigma \sigma^{-1}) = F(\theta) F(\theta) = I$. The proof of (iv) uses the following fact, which is of some interest in itself.

Theorem 2.3 Let $v_i$ and $u_i$ be vectors in $V_i$, $i = 1, \ldots , m$.

Then

(a) $v^{\otimes} = 0$, iff some $v_i = 0$.

(b) If $v^{\otimes} \neq 0$, then $v^{\otimes} = u^{\otimes}$, iff $v_i = c_i u_i$, $i = 1, \ldots , m$, and $\prod_{i=1}^{m} c_i = 1$.

Proof: (a) Clearly, if some $v_i = 0$ then $v^{\otimes} = 0$. Conversely, if $v^{\otimes} = 0$ and every $v_i \neq 0$, choose $f_i \in V_i^*$ such that $f_i(v_i) = 1$, $i = 1, \ldots , m$. Then there exists a linear $h \in \left( \bigotimes_{i=1}^{m} V_i \right)^*$ such that $h \otimes = f_1 \cdots f_m$. We conclude that $0 = h(v^{\otimes}) = \prod_{i=1}^{m} f_i(v_i) = 1$.

(b) Suppose $0 \neq v^{\otimes} = u^{\otimes}$, then by (a) every $v_i$ and $u_i$ is different from 0. For a fixed $k$ let $f_k \in V_k^*$ be arbitrary and choose $f_i \in V_i^*$, $i \neq k$, such that $f_i(v_i) = 1$. Then there exists a linear $h$ such that
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\[ h(v^\otimes) = \prod_{i=1}^{m} f_i(v_i) = f_k(v_k). \]

Also, \( h(u^\otimes) = f_k(u_k) \prod_{i \neq k} f_i(u_i). \) Letting \( c_k = \prod_{i \neq k} f_i(u_i) \) we have

\[ f_k(v_k) = h(v^\otimes) = h(u^\otimes) = c_k f_k(u_k) = f_k(c_k u_k). \]

Since \( f_k \) is arbitrary, \( v_k = c_k u_k \) and \( 0 \neq v^\otimes = \prod_{i=1}^{m} c_i u_i^\otimes \) implies \( \prod_{i=1}^{m} c_i = 1. \) The converse statement in (b) is obvious. \[ \square \]

We can now complete Example 2.3. Clearly, we need only show that \( P(\sigma) = I \) implies that \( \sigma \) is the identity. But

\[ v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(m)} = v_1 \otimes \cdots \otimes v_m \]

implies that for nonzero \( v_i, v_{\sigma^{-1}(i)} = c_i v_i, i = 1, \ldots, m. \) If we select \( m \) nonzero vectors in \( V, \) no one of which is a multiple of any other, then we conclude that \( \sigma \) must be the identity. The fact that such a selection can always be made is elementary and is contained in Exercise 10.

**Theorem 2.4** Let \( V \) be a unitary space. Then the permutation operators \( P(\sigma) \) are unitary transformations with respect to the inner product induced in \( \bigotimes_{1}^{m} V. \) In particular,

\[ P(\sigma)^* = P(\sigma^{-1}). \]  

(8)

**Proof:** For arbitrary \( x_i \) and \( y_i \) in \( V \) we have

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\[(x^\otimes, P(\sigma^{-1})y^\otimes) = (x^\otimes, Y_{\sigma(1)} \otimes \cdots \otimes Y_{\sigma(m)})\]

\[= \prod_{i=1}^{m} (x_1^{i}, y_{\sigma(i)})\]

\[= \prod_{i=1}^{m} (x_{\sigma^{-1}(i)}, y_1^{i})\]

\[= (P(\sigma)x^\otimes, y^\otimes).\]

Permutation operators will be of considerable interest to us in studying symmetry properties of multilinear functions. In our next result we list two of their elementary properties.

Theorem 2.5  If \( T \in L(V, V) \) and \( P(\sigma) \) is a permutation operator on \( \bigotimes_{1}^{m} V \) then

\[P(\sigma) \otimes T = \bigotimes_{1}^{m} TP(\sigma). \quad (9)\]

Also, if \( c(\sigma) \) denotes the number of cycles in the disjoint cycle decomposition of \( \sigma \) (including cycles of length 1) then

\[\text{tr}(P(\sigma)) = n^{c(\sigma)}. \quad (10)\]

Proof: To prove (9) simply evaluate both sides on a decomposable element \( v^\otimes \):

\[P(\sigma) \otimes T v^\otimes = P(\sigma) T v_{1}^\otimes \otimes \cdots \otimes T v_{m}^\otimes = T v_{\sigma^{-1}(1)}^\otimes \otimes \cdots \otimes T v_{\sigma^{-1}(m)}^\otimes = \bigotimes_{1}^{m} TP(\sigma)v^\otimes.\]

To verify (10) let \( \{e_1, \ldots, e_n\} \) be a basis of \( V \) and \( \{f_1, \ldots, f_n\} \) a dual basis of \( V^* \). Let \( E = \{e_{\alpha}^\otimes, \alpha \in \mathbb{F}_n^m\} \) be the corresponding


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basis in $\otimes V$. For each $\beta \in \Gamma_n^m$, let $h_\beta \in (\otimes V)^*_n$ be the linear functional defined by

$$h_\beta \otimes = f_{\beta(1)} \cdots f_{\beta(m)};$$

it is trivial to see that $\{h_\beta; \beta \in \Gamma_n^m\}$ is the dual basis of $E$. Then the main diagonal elements of $[P(\sigma)]^E_E$ are the numbers

$$h_\alpha(P(\sigma)e_\alpha^\otimes) = \prod_{j=1}^m f_{\alpha(j)}(e_{\alpha\sigma^{-1}(j)})$$

$$= \prod_{j=1}^m \delta_{\alpha(j), \alpha\sigma^{-1}(j)}$$

$$= \delta_{\alpha, \alpha\sigma^{-1}}.$$

Thus $tr(P(\sigma))$ is precisely the number of $\alpha$ for which $\alpha\sigma^{-1} = \alpha$, or, equivalently,

$$\alpha = \alpha\sigma. \tag{11}$$

Suppose $(t \sigma(t) \cdots \sigma^{p-1}(t))$ is a typical cycle in $\sigma$; then the condition (11) requires that

$$\alpha(t) = \alpha\sigma(t)$$

$$\alpha(\sigma(t)) = \alpha\sigma^2(t)$$

$$\vdots$$

$$\alpha(\sigma^{p-2}(t)) = \alpha\sigma^{p-1}(t)$$

$$\alpha(\sigma^{p-1}(t)) = \alpha\sigma^p(t)$$

$$= \alpha(t),$$

and hence $\alpha(t) = \cdots = \alpha\sigma^{p-1}(t)$. In other words, the sequence $\sigma$ must have exactly the same integer $k$ appearing in each of the
positions $t, \sigma(t), \ldots, \sigma^{p-1}(t)$. Thus for each cycle in $\sigma$ there are $n$ choices for $k$ and altogether $n^{c(\sigma)}$ choices for $\sigma$. 

In a later section we will show that Theorem 2.5 has a converse: namely, any l.t. on $\bigotimes_{l}^{m} V$ that commutes with every permutation operator must in fact be a linear combination of transformations of the form $\bigotimes_{l}^{m} T, \ T \in L(V, V)$. (This result is important in the analysis of the representations of the full linear group.)

Linear combinations of permutation operators will play an important role in our study of symmetry properties of multilinear functions. In order to accommodate a number of rather widely differing special cases we allow a large class of operators to be included in the following definition.

**Definition 2.2 (Symmetrizer)** Let $\chi: \mathbb{S}_{m} \rightarrow \mathbb{R}$ be an arbitrary scalar valued function defined on the symmetric group of degree $m$. The linear transformation defined on $\bigotimes_{l}^{m} V$ by

$$
\sum_{\sigma \in \mathbb{S}_{m}} \chi(\sigma)P(\sigma)
$$

is called a symmetrizer.

Of considerable interest is a particular instance of (12), which requires the following definition.

**Definition 2.3 (Character of degree 1)** Let $H \subseteq \mathbb{S}_{m}$ be a group of permutations. If $\chi: H \rightarrow \mathbb{R}$ is a function defined on $H$, which satisfies

(i) $\chi(e) = 1$ ($e$ is the identity permutation) and
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(ii) $\chi(\sigma_1 \sigma_2) = \chi(\sigma_1) \chi(\sigma_2)$ for all $\sigma_1$ and $\sigma_2$ in $H$, then $\chi$ is called a character of degree 1 on $H$.

Given a group $H \subset S_m$ and a character $\chi$ of degree 1 on $H$, we shall say that the symmetrizer

$$\frac{1}{|H|} \sum_{\sigma \in H} \chi(\sigma) P(\sigma)$$

is defined by $H$ and $\chi$.

Example 2.4 (a) Let $H = S_m$ and $\chi = \epsilon$, the sign of the permutation $\sigma$.

(b) Let $H = S_m$ and $\chi \equiv 1$.

(c) Let $H$ be the cyclic group generated by powers of the cycle $\sigma = (1 2 \cdots m)$. Define $\chi(\sigma^t) = e^{i(2\pi/m)t}$.

(d) A character on $H$ of degree 1 is never 0. For,

$$1 = \chi(e) = \chi(\sigma \sigma^{-1}) = \chi(\sigma) \chi(\sigma^{-1})$$

for any $\sigma \in H$.

(e) If $R = \mathbb{C}$, the complex numbers, then $\chi(\sigma^{-1}) = \overline{\chi(\sigma)}$.

For, again $\chi(\sigma) \chi(\sigma^{-1}) = 1$, and since $\sigma^p = e$ for some positive integer $p$, $|\chi(\sigma)| = 1$.

(f) $\sum_{\sigma \in H} \chi(\sigma)$ is $|H|$ or 0 according as $\chi \equiv 1$ or not.

For, let $s = \sum_{\sigma \in H} \chi(\sigma)$, multiply $s$ by $\chi(\theta)$, $\theta \in H$, and sum over $H$. Then $s^2 = |H|s$, $s(s - |H|) = 0$, so $s = 0$ or $s = |H|$.

If $\chi \equiv 1$, clearly $s = |H|$. If $s = |H|$, then since $\chi(\theta)s = s$, $\chi(\theta) = 1$.

Theorem 2.6 Let

$$S = \frac{1}{|H|} \sum_{\sigma \in H} \chi(\sigma) P(\sigma)$$

(14)
be the symmetrizer defined by $H$ and $\chi$. Then $S$ is idempotent and

$$\rho(S) = \frac{1}{|H|} \sum_{\sigma \in H} \chi(\sigma)n^{c(\sigma)},$$  \hspace{1cm} (15)$$

where $c(\sigma)$ is defined in Theorem 2.5. Moreover, if $R = \mathbb{C}$ and $V$ is a unitary space, then $S$ is p.s.d. hermitian with respect to the inner product induced in $\otimes^m V$.

Proof: Observe that if $\theta \in H$ then from Example 2.3 (b)

$$P(\theta)S = \frac{1}{|H|} \sum_{\sigma \in H} \chi(\sigma)P(\theta)P(\sigma)$$

$$= \frac{1}{|H|} \sum_{\sigma \in H} \chi(\sigma)P(\theta \sigma)$$

$$= \frac{1}{|H|} \sum_{\varphi \in H} \chi(\theta^{-1} \varphi)P(\varphi)$$

$$= \frac{1}{|H|} \sum_{\varphi \in H} \chi(\theta^{-1})\chi(\varphi)P(\varphi)$$

$$= \chi(\theta^{-1})S.$$  

Hence

$$\chi(\theta)P(\theta)S = S$$

and summing on $\theta \in H$ and dividing by $|H|$ yields $S^2 = S$.

To prove (15) we use the standard fact that the rank of an idempotent linear transformation is equal to its trace (see Exercise 11). Then formula (10) immediately yields formula (15).

Finally, from (8) we have $P(\sigma)^* = P(\sigma^{-1})$, and since
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\[ \chi(\sigma) = \chi(\sigma^{-1}), \]

\[ S^* = \frac{1}{|H|} \sum_{\sigma \in H} \chi(\sigma^{-1})P(\sigma^{-1}) = S. \]

Since \( S \) is idempotent, its eigenvalues must be 1 with multiplicity \( \rho(S) \) and 0 with multiplicity \( n^m - \rho(S) \).

**Example 2.5 (a)** Let \( S_c \) be the symmetrizer defined by \( H = S_m \) and \( \chi = e \). Then if \( \dim V = n < m \), \( S_c = 0 \), i.e., \( S_c \) is the 0 transformation on \( \otimes V \). To see this let \( e_1, \ldots, e_n \) be a basis of \( V \) and \( f_1, \ldots, f_n \) a dual basis of \( V^* \). If \( E = \{ e_\alpha, \alpha \in \mathcal{G}_n \} \), then a typical main diagonal element of \( [S_c]^E \) is \( h_\alpha (S_c e_\alpha^\otimes) \), where \( h_\alpha \in (\otimes V)^* \) and \( h_\alpha^\otimes = \prod_{j=1}^{m} e_\alpha(j) \). Then

\[ h_\alpha (S_c e_\alpha^\otimes) = h_\alpha \left( \frac{1}{m!} \sum_{\sigma \in S_m} \varepsilon(\sigma) e_\alpha^{\otimes^{-1}} \right) \]

\[ = \frac{1}{m!} \sum_{\sigma \in S_m} \varepsilon(\sigma) \prod_{t=1}^{m} f_\alpha(t)(e_\alpha^{-1}(t)) \]

\[ = \frac{1}{m!} \det A, \]

where \( A \) is the \( m \)-square matrix whose \((i,j)\) entry is \( f_\alpha(i)(e_\alpha(j)) \). Since \( n < m \), the sequence \( \alpha \) must involve some integer at least twice, and thus \( f_\alpha(1), \ldots, f_\alpha(m) \) is a linearly dependent set. It follows that \( \det A = 0 \). Thus each diagonal entry of \( [S_c]^E \) is 0 so that \( \text{tr}(S_c) = 0 \). But since \( S_c \) is idempotent, \( S_c = 0 \).

(b) It follows from the preceding example and formula (15) that if \( m > n \):
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\[ \sum_{\sigma \in S_m} \epsilon(\sigma) n^{c(\sigma)} = 0, \]

e.g., take

\[ m = 3, \quad n = 2, \quad S_3 = \{ (e, (12), (13), (23), (123), (132)) \}, \]

\[ c(e) = 3, \quad c((12)) = c((13)) = c((23)) = 2, \]

\[ c((123)) = c((132)) = 1. \]

Then the sum becomes

\[ 1 \cdot 2^3 - 1(2^2 + 2^2 + 2^2) + 1(2 + 2) = 0. \]

(c) Let \( S_1 \) be the symmetrizer defined by \( H = S_m \) and \( \chi = 1 \). We observe that \( S_1 S_\epsilon = S_\epsilon S_1 = 0 \). For, as we saw in the proof of Theorem 2.6,

\[ P(\theta) S_\epsilon = \epsilon(\theta^{-1}) S_\epsilon = \epsilon(\theta) S_\epsilon. \]

If we sum on \( \theta \in S_m \) and divide by \( \frac{1}{m!} \), we obtain \( S_1 S_\epsilon = 0 \).

Similarly, \( S_\epsilon S_1 = 0 \).

(d) We note that

\[ (S_1 - S_\epsilon) S_\epsilon x^{\otimes} = (S_1 S_\epsilon - S_\epsilon^2) x^{\otimes} = -S_\epsilon x^{\otimes}. \]

Thus if \( S_\epsilon x^{\otimes} \neq 0 \), it is an eigenvector of \( S_1 - S_\epsilon \) corresponding to the e.v. \(-1\). It is easy to see that if \( x_1, \ldots, x_m \) are l.i., then \( S_\epsilon x^{\otimes} \neq 0 \) (see Exercise 12).

As a final topic in this section we establish a connection between tensor products of vector spaces and tensor products of l.t.'s.
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**Theorem 2.7** Let \( V_i \) and \( U_i \) be vector spaces over \( R \) and let \( L_i = L(V_i, U_i), \ i = 1, \ldots, p \). Define

\[
\kappa: L_1 \times \cdots \times L_p \to L(\bigotimes_{l=1}^p V_{i,l}, \bigotimes_{l=1}^p U_{i,l}) = P
\]

by

\[
\kappa(T_1, \ldots, T_p) = T_1 \otimes \cdots \otimes T_p.
\]

Then the pair \((P, \kappa)\) is a tensor product of the spaces \( L_1, \ldots, L_p \).

**Proof:** Clearly \( \kappa \) is multilinear. Let \( n_l = \dim V_i \) and \( m_l = \dim U_i \) so that \( \dim L_i = n_l m_l \), \( i = 1, \ldots, p \). Then \( \dim P = \prod_{l=1}^p n_l m_l = n \), and in view of the result contained in Exercise 12, Section 1.3 it suffices to show that \( \dim(\text{Im} \ \kappa) = n \) to conclude the result. Let \( \{v_{i,1}, \ldots, v_{i,n_i}\} \) and \( \{u_{i,1}, \ldots, u_{i,m_i}\} \) be bases of \( V_i \) and \( U_i \), respectively. For fixed \( \alpha \in \Gamma(n_1, \ldots, n_p) \) and \( \beta \in \Gamma(m_1, \ldots, m_p) \), let \( T_i \in L_i \) be defined by

\[
T_i v_{i,k} = \delta_{\alpha(i)} u_{i,\beta(i)}, \quad k = 1, \ldots, n_i, \quad (16)
\]

and set \( T_{\alpha, \beta} = \kappa(T_1, \ldots, T_p) \). Then for \( \gamma \in \Gamma(n_1, \ldots, n_p) \),

\[
T_{\alpha, \beta} v_{\gamma} = T_1 v_{\gamma(1)} \otimes \cdots \otimes T_p v_{\gamma(p)}
\]

\[
= \delta_{\gamma(1), \alpha(1)} \cdots \delta_{\gamma(p), \alpha(p)} u_{\beta(1)} \otimes \cdots \otimes u_{\beta(p)}
\]

\[
= \delta_{\alpha, \gamma} u_{\beta}.
\]

In other words, \( T_{\alpha, \beta} \) maps \( v_{\alpha}^{\otimes} \) into \( u_{\beta}^{\otimes} \) and all other \( v_{\gamma}^{\otimes} \) into 0. Since \( \{v_{\alpha}^{\otimes}, \alpha \in \Gamma(n_1, \ldots, n_p)\} \) and \( \{u_{\beta}^{\otimes}, \beta \in \Gamma(m_1, \ldots, m_p)\} \) are bases of \( \bigotimes_{l=1}^p V_i \) and \( \bigotimes_{l=1}^p U_i \), respectively,

\[
\{T_{\alpha, \beta}, \alpha \in \Gamma(n_1, \ldots, n_p), \beta \in \Gamma(m_1, \ldots, m_p)\}
\]
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is a basis of $P$ containing $\prod_{i=1}^{p} n_i \prod_{i=1}^{p} m_i = n$ elements.

Example 2.6 (a) We show that if $A_1, \ldots, A_p$ are matrices and $A_1 \otimes \cdots \otimes A_p = 0$, then some $A_i = 0$ (see Exercise 4, Section 2.1). To see this, simply combine Theorem 2.3 (a) and Theorem 2.7.

(b) If $A_i$ and $B_i$ are in $M_{n_i} \otimes \cdots \otimes M_{n_p}$, $i = 1, \ldots, p$, and

$$0 \neq A_1 \otimes \cdots \otimes A_p = B_1 \otimes \cdots \otimes B_p$$

then $A_i = c_i B_i$, $i = 1, \ldots, p$, and $\prod_{i=1}^{p} c_i = 1$. For, this is an application of Theorem 2.3 (b) and Theorem 2.7.

(c) If $A_1, \ldots, A_p$ are square complex matrices and $0 \neq A_1 \otimes \cdots \otimes A_p$ is hermitian, then $A_i = c_i H_i$, $H_i$ is hermitian, and $\prod_{i=1}^{p} c_i$ is real. For,

$$(A_1 \otimes \cdots \otimes A_p)^* = A_1^* \otimes \cdots \otimes A_p^* = A_1 \otimes \cdots \otimes A_p.$$

Thus $A_i^* = d_i A_i$ [from (b)]. Now $A_i \neq 0$, say $a_{st} \neq 0$, and thus $d_i a_{st} = \overline{a}_{ts} \neq 0$, $\overline{a}_{st} = d_i a_{ts}$, and $a_{st} = \overline{d_i} a_{ts} = \overline{d_i} d_i a_{st}$,

$|d_i|^2 = 1$. Set $d_i = e^{i\theta}$ and consider

$$(e^{i\theta/2} A_i)^* = e^{-i\theta/2} \overline{d_i} A_i = e^{i\theta/2} A_i.$$

Hence $e^{i\theta/2} A_i$ is hermitian, i.e., $A_i = c_i H_i$, $H_i$ is hermitian. Then

$$A_1 \otimes \cdots \otimes A_p = (c_1 \cdots c_p) H_1 \otimes \cdots \otimes H_p$$

and

$$(A_1 \otimes \cdots \otimes A_p)^* = \overline{c_1} \cdots \overline{c_p} H_1 \otimes \cdots \otimes H_p.$$

Thus $\prod_{i=1}^{p} c_i = \prod_{i=1}^{p} \overline{c_i}$ and $\prod_{i=1}^{p} c_i$ is real (compare this with Exercise 4).
Exercises

1. Show that if \( T \in L(V, V) \) and every e.v. of \( T \) is contained in \( R \), then \( T \) possesses a triangular basis.

Hint: This is the same as showing that if \( A \) is an \( n \)-square matrix over \( R \) and every e.v. of \( A \) is in \( R \), then \( A \) is similar to an upper triangular matrix. To do this let \( \lambda_1 \) be an e.v. of \( A \) with corresponding eigenvector \( u_1 \). Let \( S \) be a nonsingular matrix with \( u_1 \) as its first column and observe that

\[
(S^{-1}A S)_{i,1} = \sum_k S^{-1}_{ik} (A S)_{k1} = \sum_k S^{-1}_{ik} (A S^{(1)})_{k,1} = \sum_k S^{-1}_{ik} \lambda_1 S_{k1} = \lambda (S^{-1}S)_{i1} = \lambda \delta_{i1}.
\]

\((S^{(1)})_k \) is column \( k \) of \( S \) and \((A S^{(1)})_k \) is the \( k^{th} \) entry in the indicated \( n \)-tuple.) Thus \( S^{-1}A S \) has \( \lambda_1 \) in the \((1,1)\) position and is 0 in the rest of the first column. The argument is completed by an induction on the size of the matrix. Observe that the e.v.'s of \( A \) can be arranged in any prescribed order along the main diagonal. In terms of \( T \) this means that the triangular basis \( \{e_1, \ldots, e_n\} \) can be so chosen that \( T e_j = \lambda_j e_j + u_j, \quad u_j \in \langle e_1, \ldots, e_{j-1} \rangle \) in which the order of the \( \lambda_j \) can be arbitrarily prescribed.

2. Prove that if \( V \) is a unitary space, then the triangular basis in Exercise 1 can be chosen to be o.n.

3. Show that if \( \sigma \neq T \in L(V, V) \) and \( \mathbf{E} = \{e_1, \ldots, e_n\} \) is a basis of \( V \) for which \( \{e_\alpha^\otimes, \alpha \in \mathbb{F}_n^m\} \) is a triangular basis for \( T \),
then $E$ is a triangular basis for $T$.

**Hint:** Let $[T]_E^E = A$. Then we can show that if $0 \neq A \otimes \cdots \otimes A$ is upper triangular, then $A$ must be upper triangular. For, if some $a_{ij} \neq 0$, $i > j$, then $\alpha = (i, \ldots, i)$ follows $\beta = (j, \ldots, j)$ in the lexicographic ordering and

$$(A \otimes \cdots \otimes A)_{\alpha, \beta} = a_{ij} \neq 0,$$

a contradiction.

4. Referring to Theorem 2.2, show that if $0 \neq \bigotimes_{i=1}^{m} T_i$ is hermitian, then $T_i = c_i H_i$, where $H_i$ is hermitian and $\prod_{i=1}^{m} c_i$ is a real number.

**Hint:** From Theorem 2.2 each $T_i$ is normal. Now since $T_i \neq 0$, it follows that $T_i$ has a nonzero e.v., say $\lambda_{i\alpha(i)} \neq 0$, $i = 1, \ldots, m$. Let $k$ be fixed and consider $\prod_{i \neq k}^{m} \lambda_{i\alpha(i)} \lambda_{kt}$, $t = 1, \ldots, n_k$. This number is an e.v. of $\bigotimes_{i=1}^{m} T_i$, and since these are all real, it follows that $a_k \lambda_{kt}$ is real, $t = 1, \ldots, n_k$, where $a_k = \prod_{i \neq k}^{m} \lambda_{i\alpha(i)} \neq 0$. Thus $a_k T_k$ is hermitian and the result follows.

5. Show that if $0 \neq \bigotimes_{l=1}^{m} T_l$ is p.d. as in Exercise 4, then each $H_i$ can be chosen p.d. and $\prod_{i=1}^{m} c_i > 0$.

**Hint:** The same argument shows that $a_k \lambda_{kt}$ is a positive number.

6. Show that if $0 \neq \bigotimes_{l=1}^{m} T_l$ is p.s.d. as in Exercise 4, then each $H_i$ can be chosen p.s.d. and $\prod_{i=1}^{m} c_i > 0$. 

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7. Show that if \( \otimes_{l=1}^{m} T_{i} \) is unitary, then each \( T_{i} = c_{i}U_{i}, \quad U_{i} \) unitary and \( \prod_{l=1}^{m} c_{i} = 1 \).

Hint: The same argument shows that \( |a_{k}x_{l}| = 1 \), so that each \( T_{k} \) is a multiple of a unitary transformation.

8. State and prove the appropriate result in case \( 0 \neq \otimes_{l=1}^{m} T_{i} \) is skew-hermitian.

9. Use formula (4) to establish parts (b) - (f) of Theorem 2.2.

10. Show that if \( \dim V > 1 \) and \( m \) is a positive integer, then there always exist \( m \) nonzero vectors in \( V \), no one of which is a multiple of any other.

Hint: Let \( e_{1} \) and \( e_{2} \) be l.i. and define \( v_{k} = ke_{1} + (k+1)e_{2}, \quad k = 1, \ldots, m. \)

11. Prove that if \( T \in L(V,V) \) and \( T^{2} = T \), then \( \rho(T) = \text{tr}(T) \).

Hint: By Sylvester's law of nullity obtain a basis \( \{e_{1}, \ldots, e_{n}\} \) of \( V \) such that \( Te_{1}, \ldots, Te_{r} \) is a basis of \( \text{Im } T \) and \( e_{r+1}, \ldots, e_{n} \) is a basis of \( \ker T \), the kernel of \( T \). Then clearly \( E = \{Te_{1}, \ldots, Te_{r}, e_{r+1}, \ldots, e_{n}\} \) is a basis of \( V \);

moreover, \( T^{2} = T \) implies that the matrix representation of \( T \) has precisely \( r \) 1's down the main diagonal and 0's elsewhere.

12. In the notation of Example 2.5 show that if \( x_{1}, \ldots, x_{m} \) are l.i. in \( V \), then \( S_{\mathcal{E}} x_{\otimes} \neq 0. \)

Hint: Let \( f_{1}, \ldots, f_{m} \) be dual to \( x_{1}, \ldots, x_{m} \) and let \( h \) satisfy \( h \otimes \prod_{i=1}^{m} f_{i}. \) Then if \( S_{\mathcal{E}} x_{\otimes} = 0, \) we compute that
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\[ 0 = h(S \otimes x) = \frac{1}{m!} \sum_{\sigma \in S_m} \varepsilon(\sigma) h(x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(m)}) \]

\[ = \frac{1}{m!} \sum_{\sigma \in S_m} \varepsilon(\sigma) \prod_{t=1}^{m} f_{t}(x_{\sigma^{-1}(t)}) = \frac{1}{m!} \det A, \]

where

\[ A = [f_{i}(x_{j})] = [\delta_{ij}] = I_m. \]

13. Show that \( \chi = \varepsilon \) and \( \chi = 1 \) are the only two characters of degree 1 on \( S_m \).

2.3 Symmetry Classes

In the preceding section we introduced the notion of a symmetrizer defined by a group \( H \) and character \( \chi \) of degree 1.

The range of such a symmetrizer will be crucial in the study of multilinear functions with certain symmetry properties.

Definition 3.1 (Symmetric multilinear function) Let \( U \) and \( V \) be vector spaces over \( \mathbb{R} \) and let

\[ \varphi: \times_{1}^{m} V \rightarrow U, \]

be multilinear. Let \( H \) be a subgroup of \( S_m \) and \( \chi \) a character of degree 1 on \( H \). Then \( \varphi \) is symmetric with respect to \( H \) and \( \chi \) if

\[ \varphi(v_{\sigma(1)}, \ldots, v_{\sigma(m)}) = \chi(\sigma)\varphi(v_1, \ldots, v_m) \quad (1) \]

holds for all \( \sigma \in H \) and all \( v_i \in V \). The totality of such \( \varphi \) is denoted by \( M_m(V, U, H, \chi) \).

Example 3.1 (a) Let \( V = V_n(\mathbb{R}) \), \( m = 2 \), and \( U = M_n(\mathbb{R}) \)