CSE 105
Theory of Computation

• Professor Jeanne Ferrante

Mary gave a cat to John
Today’s Agenda

• More on Non-Regular Languages
  – The Pumping Lemma Proof
  – More Practice!

• Exam Review Questions
Quote of the Week

• Lazlo Bock, in charge of hiring at Google:
  “I’m not saying you have to be some terrific coder, but to just understand how [these] things work you have to be able to think in a formal and logical and structured way.”

   New York Times, April 19, 2014
Reminders & Announcements

• Office Hours: Some changes this week
  – Mine: Wed Apr 20, 10 am - noon
  – Lots of hours up till 5 pm on Wed

• Exam 1: Wed April 20, 8 pm - 9:50 pm
  – Bring
    • your ID for checking
    • your seat assignment from TritonEd
    • 3 in by 5 in handwritten note card
  – Location: WLH 2001

• Reading Quiz 4: Due Mon Apr 25 by 11:59 pm

• HW 4: Due Wednesday, Apr 27, 11:59 pm
Keeping on beyond Regular!

NONREGULAR

LANGUAGES
We showed in class that there are Non-Regular Languages

- We now have a boundary between these language classes
Regular? Or Non-Regular?

Which of the following languages are regular?

A. The set \( \{ a^n b^n \mid n \geq 0 \} \)
B. The set \( \{ a^n b^{2n} \mid n \geq 0 \} \)
C. The set \( \{ a^n b^m \mid n, m \geq 0 \} \)
D. None or more than one of the above
Pumping Lemma Practice

Show that

$L = \{w \mid w \in \{0,1\}^* \text{ and } w \text{ has an equal number of } 0\text{'s and } 1\text{'s}\}$ is not regular.

Proof (by contradiction):
On beyond Regular!

THE PUMPING LEMMA

PROVING A LANGUAGE IS NOT REGULAR
Pumping Lemma for Regular Languages

For each regular language L, there is a pumping length p for L

1. For every string s in L of length $\geq p$
2. There are strings x, y, z with $s = xyz$, $|y| > 0$, $|xy| \leq p$
3. For each $i \geq 0$, $xy^i z$ is in L.
Pumping Lemma: If $L$ is a regular language, then there is a number $p$, the pumping length, for $L$ such that

1. for every string $s$ in $L$ of length $\geq p$,
2. there are strings $x, y, z$ with $s = xyz$, $|y| > 0$, $|xy| \leq p$ and
3. for each $i \geq 0$, $xy^iz$ is in $L$.

Proof:

Let $M$ be a DFA accepting $L$, and let $p$ be the number of states of $M$. Let $s = s_1 s_2 \ldots s_n$ be any string in $L$ of length $n \geq p$. Then the sequence of states $M$ goes through on processing $s$ is of length $n+1 \geq p+1$, and the last state in the sequence is a final state.

\[ r_0 r_1 r_2 \ldots r_i r_{i+1} \ldots r_j \ldots r_{p+1} \ldots r_n \]

**start** \[ \text{final} \]
• In the first \( p+1 \) states of this sequence, at least 2, \( r_i \) and \( r_j \), must be the same state, by the pigeonhole principle.

• Let \( y \) be the nonempty substring of \( s \) that is read between \( r_i \) and \( r_j \). Then \( s = xyz \), \( |y| > 0 \), and \( |xy| \leq p \). Consider any \( xy^iz \) for \( i \geq 0 \). \( M \) will end in the same final state \( r_n \) for this string, and so \( M \) will accept \( xy^iz \) for \( i \geq 0 \). QED.
Using the Pumping Lemma to show a language L is not Regular

- For each regular language L
  - [We assume L regular and hope to get a contradiction]
- There is a pumping length p for L
  - [Pumping lemma gives you a number p]
- For every string s in L of length \( \geq p \)
  - [You wisely choose a string s in L longer than p]
- There are strings x, y, z with \( s = xyz, \ |y| > 0, |xy| \leq p \)
  - [Given by the pumping lemma]
- For each \( i \geq 0 \), \( xy^i z \) is in L.
  - [You choose an i that leads to contradiction!]
The Pumping Lemma: A One-Act Play

Your Script

• “I’m giving you a language L.”

• “Uh...let’s just say it’s Regular.”

• “Excellent. I’m giving you this string s that I constructed using pumping length p. It is in L and $|s| \geq p$. I think you’ll really like it.”

• “Hm. I followed your directions for xyz, but when I [copy y $N$ times or delete y], the new string is NOT in L! What happened to your 100% Warranty??!”

Pumping Lemma’s Script

• “Is L Regular? In this shop I only work on Regular Languages.”

• “Good. For the Regular language L that you’ve given me, I pick this nice pumping length I call p.”

• “Great string, thanks. I’ve cut s up into parts xyz for you. I won’t tell you what they are exactly, but I will say this: $|y| > 0$ and $|xy| \leq p$. Also, I make you this 100% Lifetime Warranty: you can remove y, or copy it as many times as you like, and the new string will still be in L, I promise!”

• “Well, then L wasn’t a Regular Language. Since you lied, the Warranty was void. Thanks for playing.”
Pumping Lemma Practice

Thm. $L = \{0^i 1^j \mid i, j \geq 0 \text{ and } i \geq j\}$ is not regular.

Proof (by contradiction):
Assume (towards contradiction) that $L$ is regular. Then the pumping lemma applies to $L$. Let $p$ be the pumping length for $L$. Choose $s$ to be the string $\underline{\text{_________}}$, $|s| \geq p$. The pumping lemma guarantees $s$ can be divided into parts $xyz$ s.t. for any $i \geq 0$, $xy^iz$ is in $L$, and that $|y| > 0$ and $|xy| \leq p$. But if we let $i = \underline{\text{_____}}$, we get the string XXXX, which is not in $L^*$, a contradiction. Therefore the assumption is false, and $L$ is not regular. Q.E.D.

* Note: more proof required!

A. $s = 0^p1^p$, $i = 2$
B. $s = 0^p1^p$, $i = 0$
C. $s = 0^p1^p$, $i = p$
D. None or more than one of the above
Pumping Lemma Practice

• Thm. \( L = \{ a^n b^m \mid n,m \geq 1 \text{ and } m > n \} \) is not regular.
• Proof (by contradiction):
• Assume (towards contradiction) that \( L \) is regular. Then the pumping lemma applies to \( L \). Let \( p \) be the pumping length. Choose \( s \) to be the string \__________. The pumping lemma guarantees \( s \) can be divided into parts \( xyz \) s.t. for any \( i \geq 0 \), \( xy^i z \) is in \( L \), and that \( |y| > 0 \) and \( |xy| \leq p \). But if we let \( i = \_____ \), we get the string XXXX, which is not in \( L^* \), a contradiction. Therefore the assumption is false, and \( L \) is not regular. Q.E.D.

\* Note: more proof required!

A. \( s = a^p b^{p+1} \), \( i = 2 \)
B. \( s = a^p b^{p+1} \), \( i = 0 \)
C. \( s = a^p b^p \), \( i = 2 \)
D. None or more than one of the above
Closure Proof Review

Show that the set of regular languages is closed under reversal: if $L$ is regular, then $L^r = \{ w \mid w^r \in L \}$ is also regular.

Given: Assume $L$ is regular. Then there is a DFA $M = (Q, \Sigma, \delta, q_0, F)$ that recognizes $L$.

Want to Show: $L^r$ is regular.

Construction: We define an NFA $N$ that recognizes $L^r$ by adding a new start state $q_1$ to $Q$, with an epsilon edge to each state in $F$, and making $\{q_0\}$ the single final state of $N$. In $N$, we also reverse the direction of the original edges in $M$ as follows. $N = (Q \cup \{q_1\}, \Sigma, \delta', q_1, \{q_0\})$ where $\delta'$ is ...
Proof, continued

\[ \delta' (q_1, \varepsilon) = F \]
\[ \delta' (q_1, a) = \emptyset \quad \text{for } a \in \Sigma \]
\[ \delta' (q, a) = \{ r \mid \delta (r, a) = q \} \quad \text{for } q \in Q, a \in \Sigma \]
\[ \delta' (q, \varepsilon) = \emptyset \quad \text{for } q \in Q \]

**Correctness:** \( w = w_1 \ldots w_n \in L(N) \) \IFF 

N accepts input \( w \), starting from \( q_1 \) in N and ending in the single final state \( q_0 \) in N \IFF 

In N on input \( w \), there is a sequence of states \( q_1, f = r_n, r_{(n-1)}, \ldots, r_0 = q_0 \), with \( r_i \) in \( Q \) for \( 0 \leq i \leq n \) such that there is an epsilon edge from \( q_1 \) to \( f \in F \), and 

\[ \delta (r_i, w_{i+1}) = r_{(i+1)} \quad \text{for } 0 \leq i \leq n - 1 \] \IFF 

M accepts input \( w^r = w_n \ldots w_1 \) with the sequence of states \( q_0, r_1, \ldots r_n = f \) in \( Q \), for some \( f \) in \( F \) \IFF 

\( w \in L^r \)
Test Tips

- Get a good night’s sleep night before
- Practice writing out solutions beforehand

At the Test

- Remember to breathe!
- Read through all the problems before starting, decide which are harder
- Recommended strategy:
  - Start harder problem but don’t get stuck there!
    Only spend a few minutes
    - E.g. write down what is known, what want to show
  - Jump to easier
  - Repeat
You can do this hard thing!

See Amy Cuddy’s Ted Talk, 2012
Exam 1  Proof Practice

• Write your proofs individually.
• When done, you’ll exchange papers and give feedback:

Imagine you are a TA reading the proof. Think about the following questions:
1. Can you read the proof easily? Is the notation clear?
2. Are the assumptions of the proof made clear?
3. Does each step logically follow, or are there theorems cited so that the steps logically follow?
4. Is the overall proof correct, that is, does it prove what was asked?
5. Are there any suggestions you can make for improving the proof?