The Expressive Power of Planar Flows
Zhifeng Kong, Kamalika Chaudhuri
University of California San Diego
{zkong, kamalika}@eng.ucsd.edu

Abstract
Normalizing flows have received a great deal of recent attention as they allow flexible generative modeling as well as easy likelihood computation. While a wide variety of flow models have been proposed, there is little formal understanding of the representation power of these models. In this work, we study a class of simple normalizing flows called planar flows, and rigorously establish bounds on their expressive power. Our results indicate that while these flows are highly expressive in one dimension, in higher dimensions their representation power may be limited, especially for planar flows of moderate depth.

Background: Normalizing Flows (NF)

Definition (planar flow [Rezende and Mohamed, 2015])
A planar flow \( f \) is defined by an invertible function \( f = \{ f_1, \cdots, f_n \} \), where \( w, z \in \mathbb{R}^d, b \in \mathbb{R} \) with non-linearity \( h : \mathbb{R} \rightarrow \mathbb{R} \) such that \( \|f(z) - p\| \leq \varepsilon \).

Figure: The output distributions transformed from different source distributions with different layers of planar flows (\( h = \tanh \)). Figure in [Rezende and Mohamed, 2015].

Problem Statement
Suppose \( f \) is composed of \( T \) planar flows: \( f = f_T \circ \cdots \circ f_1 \). Let \( q \) be the source (input) distribution and \( p \) the target distribution.

\( f \)−Exact transformation: when does it satisfy
\[ p = f_q(\cdot; a, c) \]

\( f \)−Approximation: given \( \varepsilon > 0 \), is there a bound on \( T \) s.t.
\[ \|f^T_q - p\| \leq \varepsilon \]

Results for high−\( \varepsilon \) approximation: topology matching
Suppose distribution \( q \) is defined on \( \mathbb{R}^d \).

Theorem (planar flows with \( h = \text{ReLU} \))
Suppose \( f \) is composed of finitely many ReLU planar flows. Let \( p = f_q(\cdot; a, c) \). Then, for any \( \varepsilon > 0 \), there exists a zero-measure closed set \( \Omega \subseteq \mathbb{R}^d \) such that for all \( z \in \mathbb{R}^d \setminus \Omega \), we have \( J_f(z)\nabla \log p_q(f(z)) = \nabla \log q(z) \).

Figure: The surface plot of \( q \) (left), a mixture of Gaussian distribution with 4 peaks located at \( \{±1, ±3\} \), and \( p = f_q(\cdot; a, c) \) (right), the transformed distribution of \( q \). The red points correspond to peaks of \( q \) and are mapped to peaks of \( p \).

Reference

Corollary (MoG−MoG, Prod−Prod)
Suppose \( p, q \) are (i) mixture of Gaussian distributions:
\[ p(z) = \sum_{i=1}^{k} w_i \mathcal{N}(z; \mu_i, \Sigma_i), q(z) = \sum_{i=1}^{k} w_i \mathcal{N}(z; \mu_i, \Sigma_i) \]

or (ii) product distributions:
\[ p(z) = \prod_{i=1}^{k} \mathcal{N}(z; \mu_i, \Sigma_i), q(z) = \prod_{i=1}^{k} \mathcal{N}(z; \mu_i, \Sigma_i) \]
where \( \mu \) is a smooth function. Then, there generally does not exist flow \( f \) composed of finitely many ReLU planar flows such that \( p = f_q(\cdot; a, c) \).

Theorem (planar flows with general smooth \( h \))
Suppose \( f = \prod_{i=1}^{k} f_i(\cdot; a_i, c_i) \) where \( f_i(\cdot; a_i, c_i) \) is a zero-measure closed set

\( f_i(\cdot; a_i, c_i) \)−Approximation: given \( \varepsilon > 0 \), is there a bound on \( T \) s.t.
\[ \|f^T_q - p\| \leq \varepsilon \]

Sketch of proof
Let \( \mathcal{L}(p, f) = \sup_{q} \mathcal{L}(p, f) = \sup_{q} \left( \log f_q - \mathbb{E}_q[f(z)] - p(z) \right) \). If we can bound \( \mathcal{L}(p, f) \), then
\[ T(p, f, q \cdot \mathcal{L}(p, f)) = \min_{f} \mathcal{L}(p, f) \]

Examples of \( c_i \)−local non-linearities: tanh \((c_i = -2)\), sigmoid \((c_i = 1)\), and arctan \((c_i = \frac{1}{2})\).