Sampling and Reconstruction of Visual Appearance: From Denoising to View Synthesis

CSE 274 [Fall 2021], Lecture 3
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Motivation: Monte Carlo Rendering

- Key application area for sampling/reconstruction
- Modern methods for denoising now popular
- 1-3 order of magnitude speedups in mature area
- Denoising now standard in production rendering
  - And in real-time, going down to 1spp
- This, next week: Basic background in rendering
  - Reflection and Rendering Equations
  - Monte Carlo Integration
  - Path Tracing (Basic Monte Carlo rendering method)
  - Also the basics of CSE 168 (163)
- Sign up (email me) re paper presentations

Illumination Models

- Local Illumination
  - Light directly from light sources to surface
  - No shadows (cast shadows are a global effect)

- Global Illumination: multiple bounces (indirect light)
  - Hard and soft shadows
  - Reflections/refractions (already seen in ray tracing)
  - Diffuse and glossy interreflections (radiosity, caustics)

Caustics

- Caustics: Focusing through specular surface
- Major research effort in 80s, 90s till today

Overview of lecture

- **Theory** for all global illumination methods (ray tracing, path tracing, radiosity)
- We derive **Rendering Equation** [Kajiya 86]
  - Major theoretical development in field
  - Unifying framework for all global illumination
  - Introduced Path Tracing: core rendering method
- Discuss existing approaches as special cases

Fairly theoretical lecture (but important). Not well covered in textbooks (though see Eric Veach’s thesis). See reading if you are interested.

Outline

- **Reflectance Equation**
- **Global Illumination**
- **Rendering Equation**
  - As a general Integral Equation and Operator
  - Approximations (Ray Tracing, Radiosity)
  - Surface Parameterization (Standard Form)
Reflection Equation

\[ L_r(x, \omega_r) = L_e(x, \omega_r) + \sum L_i(x, \omega_i) f(x, \omega_i, \omega_r)(\omega_i \cdot n) \]

Reflected Light (Output Image)  Emission  Incident Light (from light source)  BRDF  Cosine of Incident angle

Replace sum with integral

\[ L_r(x, \omega_r) = L_e(x, \omega_r) + \int \int L_i(x, \omega_i) f(x, \omega_i, \omega_r) \cos \theta d\omega \]

Reflection Equation

Environment Maps

- Light as a function of direction, from entire environment
- Captured by photographing a chrome steel or mirror sphere
- Accurate only for one point, but distant lighting same at other scene locations (typically use only one env. map)

Blinn and Newell 1976, Miller and Hoffman, 1984
Later, Greene 86, Cabral et al. 87

The Challenge

- Computing reflectance equation requires knowing the incoming radiance from surfaces
- But determining incoming radiance requires knowing the reflected radiance from surfaces

Rendering Equation

\[ L_r(x, \omega_r) = L_e(x, \omega_r) + \int \int L_i(x, \omega_i) f(x, \omega_i, \omega_r) \cos \theta d\omega \]

Reflected Light (Output Image)  Emission  Reflected Light  BRDF  Cosine of Incident angle
Outline

- Reflectance Equation (review)
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Rendering Equation (Kajiya 86)

![Figure 5. A sample image. All objects are neutral grey. Color on the Stimuli Plate is due to scattering from the green glass walls and color bleeding from the loose pigments.](image)

Rendering Equation as Integral Equation

\[ L(x, \omega_r) = L(x, \omega_r) + \int_{\Omega} K(x', \omega_r, \omega_i) \cos \theta_i d\omega_i \]

Is a Fredholm Integral Equation of second kind [extensively studied numerically] with canonical form

\[ I(u) = e(u) + \int I(v) K(u, v) dv \]

Kernel of equation

Linear Operator Theory

- Linear operators act on functions like matrices act on vectors or discrete representations
  \[ h(u) = (M \circ f)(u) \quad M \text{ is a linear operator} \]
  \[ f \text{ and } h \text{ are functions of } u \]
- Basic linearity relations hold
  \[ a \circ (f + g) = (a \circ f) + (a \circ g) \]
- Examples include integration and differentiation
  \[ (K \circ f)(u) = \int K(u, v) f(v) dv \]
  \[ (D \circ f)(u) = \frac{df}{du} (u) \]

Linear Operator Equation

\[ I(u) = e(u) + \int I(v) K(u, v) dv \]

Kernel of equation

Light Transport Operator

\[ L = E + KL \]

Can be discretized to a simple matrix equation

[L, E are vectors, K is the light transport matrix]

Solving the Rendering Equation

- Too hard for analytic solution, numerical methods
- Approximations, that compute different terms, accuracies of the rendering equation
- Two basic approaches are ray tracing, radiosity. More formally, Monte Carlo and Finite Element. Today Monte Carlo path tracing is core rendering method
- Monte Carlo techniques sample light paths, form statistical estimate (example, path tracing)
- Finite Element methods discretize to matrix equation
Solving the Rendering Equation

- General linear operator solution. Within raytracing:
  - General class numerical Monte Carlo methods
  - Approximate set of all paths of light in scene

\[ L = E + KL \]
\[ (I - K)L = E \]

Binomial Theorem
\[ L = (I + K + K^2 + K^3 + ... )E \]
\[ L = E + KE + K^2E + K^3E + ... \]

Term \( n \) corresponds to \( n \) bounces of light

Ray Tracing

\[ L = E + KE + K^2E + K^3E + ... \]

Emission directly From light sources
Direct Illumination on surfaces
Global Illumination
One bounce indirect
Mirs, Refraction
Two bounce indirect
Caustics etc

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Change of Variables

Integral over angles sometimes insufficient. Write integral in terms of surface radiance only (change of variables)

\[ L_i(x, \omega_i) = L_i(x, \omega_i) + \int L_i(x', \omega_i') \phi(x, \omega_i, \omega_i') \cos \theta d\omega \]

\[ d\omega = \frac{dA' \cos \theta}{|x - x'|^2} \]

Change of Variables

Integral over angles sometimes insufficient. Write integral in terms of surface radiance only (change of variables)

\[ L_i(x, \omega_i) = L_i(x, \omega_i) + \int L_i(x', \omega_i') \phi(x, \omega_i, \omega_i) \cos \theta d\omega \]

\[ d\omega = \frac{dA' \cos \theta}{|x - x'|^2} \]

\[ G(x, x') = G(x', x) = \frac{\cos \theta \cos \theta}{|x - x'|^2} \]

Rendering Equation: Standard Form

Integral over angles sometimes insufficient. Write integral in terms of surface radiance only (change of variables)

\[ L_i(x, \omega_i) = L_i(x, \omega_i) + \int L_i(x', \omega_i') \phi(x, \omega_i, \omega_i) \cos \theta d\omega \]

\[ d\omega = \frac{dA' \cos \theta}{|x - x'|^2} \]

\[ \int_{V(x, x')} dA = G(x, x') = G(x', x) = \frac{\cos \theta \cos \theta}{|x - x'|^2} \]

Summary

- **Theory** for all global illumination methods (ray tracing, path tracing, radiosity)
- We derive **Rendering Equation** [Kajiya 86]
  - Major theoretical development in field
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Motivation: Monte Carlo Integration

Rendering = integration
  - Reflectance equation: Integrate over incident illumination
  - Rendering equation: Integral equation

Many sophisticated shading effects involve integrals
  - Antialiasing
  - Soft shadows
  - Indirect illumination
  - Caustics

Most Sampling/Reconstruction treats actual rendering as a black box. But still helpful to know some basics

Example: Soft Shadows

\[ E(x) = \int L(x, \omega) \cos \theta d\omega \]

**Challenges**
- Visibility and blockers
- Varying light distribution
- Complex source geometry

Source: Agrawala. Ramamoorthi, Heirich, Mull, 2000
Monte Carlo Algorithms

Advantages
- Robust for complex integrals in computer graphics (irregular domains, shadow discontinuities and so on)
- Efficient for high-dimensional integrals (common in graphics: time, light source directions, and so on)
- Quite simple to implement
- Work for general scenes, surfaces
- Easy to reason about (but care taken re statistical bias)

Disadvantages
- Noisy
- Slow (many samples needed for convergence)
- Not used if alternative analytic approaches exist (but those are rare)

Integration in 1D

$$\int_{x=1}^{x=1} f(x) \, dx = ?$$

We can approximate

$$\int_{x=0}^{x=1} f(x) \, dx = \frac{1}{N} \sum_{i=1}^{N} f(x_i)$$

Monte Carlo methods (random choose samples)

Advantages:
- Robust for discontinuities
- Converges reasonably for large dimensions
- Can handle complex geometry, integrals
- Relatively simple to implement, reason about
Other Domains

\[ \int_a^b f(x) \, dx = \frac{b-a}{N} \sum_{i=1}^N f(x_i) \]

Multidimensional Domains

Same ideas apply for integration over …
- Pixel areas
- Surfaces
- Projected areas
- Directions
- Camera apertures
- Time
- Paths

Random Variables

- Describes possible outcomes of an experiment
- In discrete case, e.g. value of a dice roll \( [x = 1-6] \)
- Probability \( p \) associated with each \( x \) \( (1/6 \text{ for dice}) \)
- Continuous case is obvious extension

Expected Value

- Expectation
  - Discrete: \( E(f) = \sum_{i=1}^n p_i f(x_i) \)
  - Continuous: \( E(f) = \int_0^1 p(x) f(x) \, dx \)

For Dice example:
\[
E(x) = \sum_{i=1}^6 \frac{1}{6} x_i = \frac{1}{6} (1+2+3+4+5+6) = 3.5
\]

Continuous Probability Distributions

<table>
<thead>
<tr>
<th>PDF ( p(x) )</th>
<th>CDF ( P(x) )</th>
</tr>
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<tbody>
<tr>
<td>( p(x) \geq 0 )</td>
<td>( P(x) = \int_0^x p(x) , dx )</td>
</tr>
<tr>
<td>( P(x) = \Pr(X &lt; x) )</td>
<td>( P(1) = 1 )</td>
</tr>
<tr>
<td>( \Pr(\alpha \leq X \leq \beta) = \int_\alpha^\beta p(x) , dx )</td>
<td>( = P(\beta) - P(\alpha) )</td>
</tr>
</tbody>
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Sampling Techniques

Problem: how do we generate random points/directions during path tracing?
- Non-rectilinear domains
- Importance (BRDF)
- Stratified
Generating Random Points

**Uniform distribution:**
- Use random number generator

<table>
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<tr>
<th>Probability</th>
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Generating Random Points

**Specific probability distribution:**
- Function inversion
- Rejection
- Metropolis

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<th>Probability</th>
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Common Operations

Want to **sample** probability distributions
- Draw samples distributed according to probability
- Useful for integration, picking important regions, etc.

**Common distributions**
- Disk or circle
- Uniform
- Upper hemisphere for visibility
- Area luminaire
- Complex lighting like an environment map
- Complex reflectance like a BRDF

Sampling Continuous Distributions

Cumulative probability distribution function

\[
P(x) = Pr(X < x)
\]

Construction of samples
- Solve for \( X = P^{-1}(U) \)

Must know:
1. The integral of \( p(x) \)
2. The inverse function \( P^{-1}(x) \)

Example: Power Function

Assume
\[
p(x) = (n+1)x^n
\]

\[
P(x) = x^{n+1}
\]

\[
X = p(x) \Rightarrow X = P^{-1}(U) = \sqrt[n+1]{U}
\]

**Trick**
\[
Y = \text{max}(U_1, U_2, \ldots, U_i, U_{i+1})
\]

\[
Pr(Y < x) = \prod_{i=1}^{n} Pr(U_i < x) = x^n
\]
Sampling a Circle

\[ A = \int_0^\pi r \, d\theta \int_0^r 2r \, dr \theta = \frac{r^2 \theta}{2} \int_0^\pi = \pi \]

\[ p(r, \theta) \, d\theta \, dr = \frac{1}{\pi} r \, dr \, d\theta \Rightarrow p(r, \theta) = \frac{r}{\pi} \]

\[ p(\theta) = \frac{1}{2\pi} \]

\[ p(r) = 2r \]

\[ r = \sqrt{U_1} \]

\[ \theta = 2\pi U_2 \]

Rejection Sampling

\[ I = \int_0^1 f(x) \, dx = \iint_{y \leq f(x)} \, dx \, dy \]

Algorithm

Pick \( U_2 \) and \( U_2 \)

Accept \( U_2 \) if \( U_2 < f(U_2) \)

Wasteful? Efficiency = Area / Area of rectangle

Sampling a Circle: Rejection

do {
\[ X = 1 - 2 \times U_1 \]
\[ Y = 1 - 2 \times U_2 \]
while (\( X^2 + Y^2 > 1 \))

May be used to pick random 2D directions

Circle techniques may also be applied to the sphere

More formally

Definite integral

\[ I(f) = \int_0^1 f(x) \, dx \]

Expectation of \( f \)

\[ E[f] = \int_0^1 f(x) \, p(x) \, dx \]

Random variables

\[ X_i \sim p(x) \]
\[ Y_i = f(X_i) \]

Estimator

\[ F_N = \frac{1}{N} \sum_{i=1}^{N} Y_i \]
Importance Sampling

1. **Unbiased Estimator**

\[
\mathbb{E}[F_N] = I(f)
\]

\[
\mathbb{E}[\sum_i Y_i] = \sum_i \mathbb{E}[Y_i]
\]

\[
\mathbb{E}[a Y] = a \mathbb{E}[Y]
\]

**Properties**

Assume uniform probability distribution for new

\[
\int_\Omega f(x) dx = 1
\]

\[
\int_\Omega \sum_i Y_i = \sum_i \int_\Omega Y_i
\]

\[
\int_\Omega \sum_i f(x) p(x) dx
\]

Direct Lighting – Directional Sampling

\[
E(x) = \int_\Omega L(x, \omega) \cos \theta d\omega
\]

Ray intersection \( x' (x, \omega) \)

Sample \( \omega \) uniformly by \( \Omega \)

\[
Y_i = L(x' (x, \omega), -\omega) \cos \theta 2\pi
\]

Direct Lighting – Area Sampling

\[
E(x) = \int_\Omega L(x, \omega) \cos \theta d\omega = \int_\Omega L' (x', \omega', \omega) V(x, x') \frac{\cos \theta \cos \theta}{|x - x'|^2} dA
\]

Ray direction \( \omega' = x - x' \)

Sample \( x' \) uniformly by \( A \)

Importance Sampling

- This is still unbiased

\[
\mathbb{E}[Y_i] = \int_\Omega Y(x)p(x) dx
\]

\[
\mathbb{E}[f(x) p(x) dx]
\]

\[
\mathbb{E}[f(x) dx]
\]

for all \( N \)

Importance Sampling

- Zero variance if \( p(x) \sim f(x) \)

\[
p(x) = cf(x)
\]

\[
Y_i = \frac{f(x)}{p(x)} \frac{1}{c}
\]

\[
\text{Var}(Y) = 0
\]

Less variance with better importance sampling
Stratified Sampling

- Estimate subdomains separately

\[ E_2(f(x)) \]

Stratified Sampling

- Less overall variance if less variance in subdomains

\[ \text{Var} \left[ F_N \right] = \frac{1}{N} \sum_{i=1}^{\mu} N \text{Var} \left[ F_i \right] \]

More Information

- Veach PhD thesis chapter (linked to from website)
- Course Notes (links from website)