Quality Improves with More Rays

<table>
<thead>
<tr>
<th>pixelsamples = 1</th>
<th>Area</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 shadow ray</td>
<td>16 shadow rays</td>
</tr>
</tbody>
</table>

Sampling and Reconstruction

- An image is a 2D array of samples
- Discrete samples from real-world continuous signal
(Spatial) Aliasing

- Jaggies probably biggest aliasing problem

Sampling and Aliasing

- Artifacts due to undersampling or poor reconstruction
- Formally, high frequencies masquerading as low
- E.g. high frequency line as low freq jaggies

Image Processing pipeline
Motivation

- Formal analysis of sampling and reconstruction
- Important theory (signal-processing) for graphics
- Also relevant in rendering, modeling, animation
- Note: Fourier Analysis useful for understanding, but image processing often done in spatial domain

Ideas

- Signal (function of time generally, here of space)
- Continuous: defined at all points; discrete: on a grid
- High frequency: rapid variation; Low Freq: slow variation
- Images are converting continuous to discrete. Do this sampling as best as possible.
- Signal processing theory tells us how best to do this
- Based on concept of frequency domain Fourier analysis

Sampling Theory

- Analysis in the frequency (not spatial) domain
- Sum of sine waves, with possibly different offsets (phase)
- Each wave different frequency, amplitude

Fourier Transform

- Tool for converting from spatial to frequency domain
  
  \[ f(x) = \sum_{u=-\infty}^{\infty} F(u)e^{2\pi iux} \]
  
  \[ e^{2\pi iux} = \cos(2\pi ux) + i \sin(2\pi ux) \]
  
- Or vice versa
  
  \[ i = \sqrt{-1} \]
  
- One of most important mathematical ideas
  
- Computational algorithm: Fast Fourier Transform
  
- One of 10 great algorithms scientific computing
  
- Makes Fourier processing possible (images etc.)
  
- Not discussed here, but look up if interested

\[
 f(x) = \sum_{u=-\infty}^{\infty} F(u)e^{2\pi iux} \\
 F(u) = \int_{-\infty}^{\infty} f(x)e^{-2\pi iux}dx
\]

Simple case, function sum of sines, cosines

\[
 f(x) = \sum_{u=-\infty}^{\infty} F(u)e^{2\pi iux} \\
 F(u) = \int_{-\infty}^{\infty} f(x)e^{-2\pi iux}dx
\]

Continuous infinite case

- Forward Transform: \( F(u) = \int_{-\infty}^{\infty} f(x)e^{-2\pi iux}dx \)
- Inverse Transform: \( f(x) = \int_{-\infty}^{\infty} F(u)e^{2\pi iux}du \)

Discrete case

- \( F(u) = \sum_{x=-\frac{N}{2}}^{\frac{N}{2}-1} f(x)\left[ \cos\left(\frac{2\pi ux}{N}\right) - i \sin\left(\frac{2\pi ux}{N}\right) \right] \)
  \( 0 \leq u \leq N - 1 \)
- \( f(x) = \frac{1}{N} \sum_{u=-\frac{N}{2}}^{\frac{N}{2}-1} F(u)\left[ \cos\left(\frac{2\pi ux}{N}\right) + i \sin\left(\frac{2\pi ux}{N}\right) \right] \)
  \( 0 \leq x \leq N - 1 \)
Fourier Transform: Examples 1

Single sine curve
(+constant DC term)

\[ f(x) = \sum_{n=-\infty}^{\infty} F(u) e^{2\pi i u x} \]

\[ F(u) = \int f(x) e^{-2\pi i u x} \, dx \]

Fourier Transform Examples 2

Forward Transform:

\[ F(u) = \int f(x) e^{-2\pi i u x} \, dx \]

Inverse Transform:

\[ f(x) = \int F(u) e^{2\pi i u x} \, du \]

Common examples

- \( f(x) \)
- \( \delta(x-x_0) e^{-2\pi i u x} \)
- \( 1 \)
- \( \delta(u) \)
- \( e^{-\frac{\pi}{a} u^2} \)

Sampling Theorem, Bandlimiting

- A signal can be reconstructed from its samples, if the original signal has no frequencies above half the sampling frequency – Shannon
- The minimum sampling rate for a bandlimited function is called the Nyquist rate

Antialiasing

- Sample at higher rate
  - Not always possible
  - Real world: lines have infinitely high frequencies, can’t sample at high enough resolution
- Prefilter to bandlimit signal
  - Low-pass filtering (blurring)
  - Trade bluriness for aliasing

Fourier Transform Properties

- Forward Transform:
  \[ F(u) = \int f(x) e^{-2\pi i u x} \, dx \]
- Inverse Transform:
  \[ f(x) = \int F(u) e^{2\pi i u x} \, du \]
- Common properties
  - Linearity: \( F(a f(x) + b g(x)) = a F(f(x)) + b F(g(x)) \)
  - Derivatives: \( F(f'(x)) = \int f'(x) e^{-2\pi i u x} \, dx = 2\pi i u F(f(x)) \)
  - 2D Fourier Transform
    \[ F(u,v) = \int f(x,y) e^{-2\pi i (u x + v y)} \, dx \, dy \]
  - Convolution (next)

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**Ideal bandlimiting filter**

- Formal derivation is homework exercise
  - Frequency domain
  - Spatial domain

![Graph of ideal bandlimiting filter](image)

**Convolution 1**

- Spatial domain: output pixel is weighted sum of pixels in neighborhood of input image
  - Pattern of weights is the “filter”

![Convolution 1 example](image)

**Convolution 2**

- Example 1:

![Convolution 2 example](image)

**Convolution 3**

- Example 1:

![Convolution 3 example](image)

**Convolution 4**

- Example 1:

![Convolution 4 example](image)

**Convolution 5**

- Example 1:

![Convolution 5 example](image)
**Convolution in Frequency Domain**

Forward Transform: 
\[ F(u) = \int_{-\infty}^{\infty} f(x)e^{-2\pi iux}dx \]

Inverse Transform: 
\[ f(x) = \int_{-\infty}^{\infty} F(u)e^{2\pi iux}du \]

- Convolution (f is signal; g is filter [or vice versa])
  \[ h(y) = \int f(x)g(y-x)dx = \int g(x)f(y-x)dx \]
  \( h = f \ast g \) or \( f \otimes g \)
- Fourier analysis (frequency domain multiplication) 
  \( H(u) = F(u)G(u) \)

**Practical Image Processing**

- Discrete convolution (in spatial domain) with filters for various digital signal processing operations
- Easy to analyze, understand effects in frequency domain
  - E.g. blurring or bandlimiting by convolving with low pass filter

**Point vs Area Sampling**

- Uniform Supersampling
  - Increasing the number of samples moves each copy of the spectra further apart, thus there is less overlap
  - This reduces, but does not eliminate, aliasing
  \[ \text{Pixel} = \sum \text{Sample} \]

**Non-uniform Sampling**

- Uniform sampling
  - The spectrum of uniformly spaced samples is also a set of uniformly spaced spikes
  - Multiplying the signal by the sampling pattern corresponds to placing a copy of the spectrum at each spike (in freq. space)
  - Aliases are coherent, and very noticeable

- Non-uniform sampling
  - Samples at non-uniform locations have a different spectrum; a single spike plus noise
  - Sampling a signal in this way converts aliases into broadband noise
  - Noise is incoherent, and much less objectionable
  - May cause error in the integral

**Jittered Sampling**

- Add uniform random jitter to each sample
Jittered vs Uniform Supersampling

Distribution of Extrafoveal Cones

Poisson Disk Sampling