Computer Graphics
CSE 167 [Win 17], Lecture 2: Review of Basic Math
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To Do
- Complete Assignment 0 (due Jan 18)
- Get help if issues with compiling, programming
- Any problems with edX edge?
- Any confusion on course requirements?
- Textbooks: access to OpenGL references
- About first few lectures
  - Somewhat technical: core math ideas in graphics
  - HW1 is simple (only few lines of code): Lets you see how
    to use some ideas discussed in lecture, create images

Motivation and Outline
- Many graphics concepts need basic math like linear algebra
  - Vectors (dot products, cross products, …)
  - Matrices (matrix-matrix, matrix-vector mult., …)
  - E.g: a point is a vector, and an operation like translating or
    rotating points on object can be matrix-vector multiply
- Should be refresher on very basic material for most of you
  - Only basic high school math required
  - If you don’t understand, talk to me (review in office hours)

Vectors
- Usually written as \( \vec{a} \) or in bold. Magnitude written as \( \| \vec{a} \| \)
- Length and direction. Absolute position not important
- Use to store offsets, displacements, locations
  - But strictly speaking, positions are not vectors and cannot be added:
    a location implicitly involves an origin, while an offset does not

Vector Addition
- Geometrically: Parallelogram rule
- In cartesian coordinates (next), simply add coords

Cartesian Coordinates
- \( X \) and \( Y \) can be any (usually orthogonal unit) vectors
- \( A = \begin{bmatrix} x \\ y \end{bmatrix} \) \( A^T = \begin{bmatrix} x & y \end{bmatrix} \) \( \| A \| = \sqrt{x^2 + y^2} \)
Vector Multiplication

- Dot product
- Cross product
- Orthonormal bases and coordinate frames

Note: Some books talk about right and left-handed coordinate systems. We always use right-handed

**Dot (scalar) product**

\[ a \cdot b = b \cdot a = ? \]

\[ a \cdot (b + c) = a \cdot b + a \cdot c \]

\[ (ka) \cdot b = a \cdot (kb) = k(a \cdot b) \]

\[ \phi = \cos^{-1} \left( \frac{a \cdot b}{\|a\| \|b\|} \right) \]

**Dot product in Cartesian components**

\[
\begin{pmatrix} x_a \\ y_a \end{pmatrix} \cdot \begin{pmatrix} x_b \\ y_b \end{pmatrix} = ?
\]

\[
a \cdot b = x_a x_b + y_a y_b
\]

**Dot product: some applications in CG**

- Find angle between two vectors (e.g. cosine of angle between light source and surface for shading)
- Finding projection of one vector on another (e.g. coordinates of point in arbitrary coordinate system)
- Advantage: computed easily in cartesian components

**Projections (of b on a)**

\[ \|b \rightarrow a\| = ? \]

\[ b \rightarrow a = \|b\| \cos \phi = \frac{a \cdot b}{\|a\|} \]

**Vector Multiplication**

- Dot product
- Cross product
- Orthonormal bases and coordinate frames

Note: Some books talk about right and left-handed coordinate systems. We always use right-handed
Cross (vector) product

- Cross product orthogonal to two initial vectors
- Direction determined by right-hand rule
- Useful in constructing coordinate systems (later)

\[
a \times b = -b \times a
\]

\[
\|a \times b\| = \|a\| \|b\| \sin \phi
\]

Cross product: Properties

- \(x \times y = +z\)
- \(y \times x = -z\)
- \(y \times z = +x\)
- \(z \times y = -x\)
- \(a \times (b + c) = a \times b + a \times c\)
- \(z \times x = +y\)
- \(x \times z = -y\)

Cross product: Cartesian formula?

\[
a \times b = \begin{vmatrix}
    x & y & z \\
    x_a & y_a & z_a \\
    x_b & y_b & z_b
\end{vmatrix}
\]

\[
a \times b = a' b = \begin{bmatrix}
    0 & -z_a & x_a \\
    z_a & 0 & -x_a \\
    -y_a & x_a & 0
\end{bmatrix}
\]

Dual matrix of vector \(a\)

Vector Multiplication

- Dot product
- Cross product
- Orthonormal bases and coordinate frames

Note: book talks about right and left-handed coordinate systems. We always use right-handed coordinate systems.

Orthonormal bases/coordinate frames

- Important for representing points, positions, locations
- Often, many sets of coordinate systems (not just X, Y, Z)
  - Global, local, world, model, parts of model (head, hands, ...)
- Critical issue is transforming between these systems/bases
  - Topic of next 3 lectures

Coordinate Frames

- Any set of 3 vectors (in 3D) so that

\[
\|u\| = \|v\| = \|w\| = 1
\]

\[
u \cdot v = v \cdot w = u \cdot w = 0
\]

\[
w = u \times v
\]

\[
p = (p \cdot u)u + (p \cdot v)v + (p \cdot w)w
\]
Constructing a coordinate frame

- Often, given a vector \( \mathbf{a} \) (viewing direction in HW1), want to construct an orthonormal basis
- Need a second vector \( \mathbf{b} \) (up direction of camera in HW1)
- Construct an orthonormal basis (for instance, camera coordinate frame to transform world objects into in HW1)

We want to associate \( \mathbf{w} \) with \( \mathbf{a} \), and \( \mathbf{v} \) with \( \mathbf{b} \)
- But \( \mathbf{a} \) and \( \mathbf{b} \) are neither orthogonal nor unit norm
- And we also need to find \( \mathbf{u} \)

\[
\mathbf{w} = \frac{\mathbf{a}}{||\mathbf{a}||}
\]

\[
\mathbf{u} = \frac{\mathbf{b} \times \mathbf{w}}{||\mathbf{b} \times \mathbf{w}||}
\]

\[
\mathbf{v} = \mathbf{w} \times \mathbf{u}
\]

Matrices

- Can be used to transform points (vectors)
  - Translation, rotation, shear, scale
  - (more detail next lecture)

What is a matrix

- Array of numbers (m\times n = m \text{ rows}, n \text{ columns})

\[
\begin{pmatrix}
1 & 3 \\
5 & 2 \\
0 & 4
\end{pmatrix}
\]

- Addition, multiplication by a scalar simple: element by element

Matrix-matrix multiplication

- Number of columns in first must = rows in second

\[
\begin{pmatrix}
1 & 3 \\
5 & 2 \\
0 & 4
\end{pmatrix}
\begin{pmatrix}
3 & 6 & 9 & 4 \\
2 & 7 & 8 & 3
\end{pmatrix}
= 
\begin{pmatrix}
27 & 54 & 13 \\
10 & 44 & 81 & 26 \\
8 & 28 & 32 & 12
\end{pmatrix}
\]

- Element \((i,j)\) in product is dot product of row \(i\) of first matrix and column \(j\) of second matrix
Matrix-matrix multiplication

- Number of columns in first must = rows in second

\[
\begin{pmatrix}
1 & 3 \\
5 & 2 \\
0 & 4
\end{pmatrix}
\begin{pmatrix}
1 & 6 & 9 & 4 \\
2 & 7 & 8 & 3
\end{pmatrix}
= 
\begin{pmatrix}
2 & 27 & 33 & 13 \\
5 & 44 & 61 & 26
\end{pmatrix}
\]

- Element \((i,j)\) in product is dot product of row \(i\) of first matrix and column \(j\) of second matrix

\[
\begin{pmatrix}
3 & 6 & 9 & 4 \\
2 & 7 & 8 & 3
\end{pmatrix}
\begin{pmatrix}
1 & 3 \\
5 & 2 \\
0 & 4
\end{pmatrix}
= 
\begin{pmatrix}
0 & 4 & 14 & 6 \\
2 & 7 & 8 & 3
\end{pmatrix}
\]

- Non-commutative \((AB\) and \(BA\) are different in general)
- Associative and distributive
  - \(A(B+C) = AB + AC\)
  - \((A+B)C = AC + BC\)

Matrix-Vector Multiplication

- Key for transforming points (next lecture)
- Treat vector as a column matrix \((m \times 1)\)

E.g. 2D reflection about y-axis

\[
\begin{pmatrix}
-1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix} = 
\begin{pmatrix}
-x \\
y
\end{pmatrix}
\]

Transpose of a Matrix (or vector?)

\[
\begin{pmatrix}
1 & 2 \\
3 & 4 \\
5 & 6
\end{pmatrix}^T
= 
\begin{pmatrix}
1 & 3 & 5 \\
2 & 4 & 6
\end{pmatrix}
\]

\((AB)^T = B^T A^T\)

Identity Matrix and Inverses

\[
I_{3 \times 3} = 
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

\(AA^{-1} = A^{-1}A = I\)

\((AB)^{-1} = B^{-1}A^{-1}\)
Vector multiplication in Matrix form

- Dot product?
  \[ \mathbf{a} \cdot \mathbf{b} = \mathbf{a}^T \mathbf{b} \]

\[
\begin{pmatrix}
  x_a & y_a & z_a \\
  x_b & y_b & z_b
\end{pmatrix}
\begin{pmatrix}
  x_b \\
  y_b \\
  z_b
\end{pmatrix}
= (x_a x_b + y_a y_b + z_a z_b)
\]

- Cross product?
  \[ \mathbf{a} \times \mathbf{b} = \mathbf{A}^T \mathbf{b} = 0 \]

\[
\begin{pmatrix}
  0 & -z_a & y_a \\
  z_a & 0 & -x_a \\
  -y_a & x_a & 0
\end{pmatrix}
\begin{pmatrix}
  x_b \\
  y_b \\
  z_b
\end{pmatrix}
= (x_a y_b - y_a x_b)
\]

Dual matrix of vector \( \mathbf{a} \)