Motivation and Outline

- Many graphics concepts need basic math like linear algebra
  - Vectors (dot products, cross products, …)
  - Matrices (matrix-matrix, matrix-vector mult., …)
  - E.g: a point is a vector, and an operation like translating or rotating points on object can be matrix-vector multiply
- Should be refresher on very basic material for most of you
  - Only basic high school math required

Vectors

- Usually written as $\vec{a}$ or in bold. Magnitude written as $||\vec{a}||$
- Length and direction. Absolute position not important
- Use to store offsets, displacements, locations
  - But strictly speaking, positions are not vectors and cannot be added: a location implicitly involves an origin, while an offset does not.

Vector Addition

- Geometrically: Parallelogram rule
- In cartesian coordinates (next), simply add coords

Cartesian Coordinates

- $A = 4 \ x + 3 \ y$
- $A^T = \begin{pmatrix} x & y \end{pmatrix}$
- $||A|| = \sqrt{x^2 + y^2}$
- $X$ and $Y$ can be any (usually orthogonal unit) vectors

Course: Next Steps

- Complete HW 0
  - Sets up basic compilation issues
  - Verifies you can work with feedback/grading servers
- First few lectures core math ideas in graphics
  - This lecture is a revision of basic math concepts
- HW 1 has few lines of code (but start early)
  - Use some ideas discussed in lecture, create images
- Textbooks: None required
  - OpenGL/GLSL reference helpful (but not required)
Vector Multiplication

- Dot product
- Cross product
- Orthonormal bases and coordinate frames

Note: We use right-handed (standard) coordinates

Dot (scalar) product

\[ a \cdot b = b \cdot a = ? \]
\[ a \cdot b = |a| |b| \cos \phi \]
\[ \phi = \cos^{-1} \left( \frac{a \cdot b}{|a||b|} \right) \]

Dot product in Cartesian components

\[ a \cdot b = \begin{pmatrix} x_a \\ y_a \end{pmatrix} \cdot \begin{pmatrix} x_b \\ y_b \end{pmatrix} = ? \]

Dot (scalar) product

\[ a \cdot b = b \cdot a = ? \]

Dot (scalar) product

\[ a \cdot b = b \cdot a = ? \]
\[ a \cdot (b + c) = a \cdot b + a \cdot c \]
\[ (ka) \cdot b = a \cdot (kb) = k(a \cdot b) \]
\[ \phi = \cos^{-1} \left( \frac{a \cdot b}{|a||b|} \right) \]

Dot product in Cartesian components

\[ a \cdot b = \begin{pmatrix} x_a \\ y_a \end{pmatrix} \cdot \begin{pmatrix} x_b \\ y_b \end{pmatrix} = ? \]
\[ a \cdot b = \begin{pmatrix} x_a \\ y_a \end{pmatrix} \cdot \begin{pmatrix} x_b \\ y_b \end{pmatrix} = x_a x_b + y_a y_b \]
**Dot product: some applications in CG**

- Find angle between two vectors (e.g. cosine of angle between light source and surface for shading)

- Finding projection of one vector on another (e.g. coordinates of point in arbitrary coordinate system)

- Advantage: computed easily in cartesian components

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**Foundations of Computer Graphics**

*Online Lecture 2: Review of Basic Math*

*Vectors: Cross Products*

Ravi Ramamoorthi

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**Cross (vector) product**

- Cross product orthogonal to two initial vectors
- Direction determined by right-hand rule
- Useful in constructing coordinate systems (later)

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**Projections (of b on a)**

\[
\|b \rightarrow a\| = ? \quad \|b \rightarrow a\| = \|b\| \cos \phi = \frac{a \cdot b}{\|a\|} \\
b \rightarrow a = ? \quad b \rightarrow a = \frac{a \cdot b}{\|a\|^2} a
\]

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**Vector Multiplication**

- **Dot product**
- **Cross product**
- Orthonormal bases and coordinate frames

*Note: We use right-handed (standard) coordinates*

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**Cross product: Properties**

- \(x \times y = +z\)
- \(y \times x = -z\)
- \(y \times z = +x\)
- \(z \times y = -x\)
- \(z \times x = +y\)
- \(x \times z = -y\)

- \(a \times b = -b \times a\)
- \(a \times a = 0\)
- \(a \times (b + c) = a \times b + a \times c\)
- \(a \times (kb) = k(a \times b)\)
Cross product: Cartesian formula?

\[
a \times b = \begin{pmatrix} x_a & y_a & z_a \\ x_b & y_b & z_b \end{pmatrix} = \begin{pmatrix} y_bz_a - y_az_b \\ z_ax_b - z_bx_a \\ x_by_a - x_ay_b \end{pmatrix} = \begin{pmatrix} y_a \\ z_a \\ x_a \end{pmatrix} \times \begin{pmatrix} y_b \\ z_b \\ x_b \end{pmatrix} = \begin{pmatrix} 0 & -z_a & y_a \\ z_a & 0 & -x_a \\ -y_a & x_a & 0 \end{pmatrix} \begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix}
\]

Dual matrix of vector a

Vector Multiplication

- Dot product
- Cross product
- Orthonormal bases and coordinate frames

Note: We use right-handed (standard) coordinates

Orthonormal bases/coordinate frames

- Important for representing points, positions, locations
- Often, many sets of coordinate systems (not just X, Y, Z)
  - Global, local, world, model, parts of model (head, hands, …)
- Critical issue is transforming between these systems/bases
  - Topic of next 3 lectures

Coordinate Frames

- Any set of 3 vectors (in 3D) so that
  \[
  \left| \begin{array}{c} u \\ v \\ w \end{array} \right| = 1
  \]
  \[
  u \cdot v = v \cdot w = u \cdot w = 0
  \]
  \[
  w = u \times v
  \]
  \[
  p = (p \cdot u)u + (p \cdot v)v + (p \cdot w)w
  \]
Constructing a coordinate frame

- Often, given a vector \( \mathbf{a} \) (viewing direction in HW1), want to construct an orthonormal basis
- Need a second vector \( \mathbf{b} \) (up direction of camera in HW1)
- Construct an orthonormal basis (for instance, camera coordinate frame to transform world objects into in HW1)

Constructing a coordinate frame?

We want to associate \( \mathbf{w} \) with \( \mathbf{a} \), and \( \mathbf{v} \) with \( \mathbf{b} \)
- But \( \mathbf{a} \) and \( \mathbf{b} \) are neither orthogonal nor unit norm
- And we also need to find \( \mathbf{u} \)

\[
\mathbf{w} = \frac{\mathbf{a}}{||\mathbf{a}||} \\
\mathbf{u} = \frac{\mathbf{b} \times \mathbf{w}}{||\mathbf{b} \times \mathbf{w}||} \\
\mathbf{v} = \mathbf{w} \times \mathbf{u}
\]

Constructing a coordinate frame?

- But \( \mathbf{a} \) and \( \mathbf{b} \) are neither orthogonal nor unit norm
- And we also need to find \( \mathbf{u} \)

\[
\begin{align*}
\mathbf{w} &= \mathbf{a} \\
\mathbf{u} &= \frac{\mathbf{b} \times \mathbf{w}}{||\mathbf{b} \times \mathbf{w}||} \\
\mathbf{v} &= \mathbf{w} \times \mathbf{u}
\end{align*}
\]

Foundations of Computer Graphics
Online Lecture 2: Review of Basic Math
Matrices
Ravi Ramamoorthi
Matrices

- Can be used to transform points (vectors)
  - Translation, rotation, shear, scale
    (more detail next lecture)

What is a matrix

- Array of numbers (m×n = m rows, n columns)
  \[
  \begin{bmatrix}
  1 & 3 \\
  5 & 2 \\
  0 & 4 
  \end{bmatrix}
  \]
- Addition, multiplication by a scalar simple: element by element

Matrix-matrix multiplication

- Number of columns in first must = rows in second

\[
\begin{bmatrix}
  1 & 3 \\
  5 & 2 \\
  0 & 4 
\end{bmatrix}
\begin{bmatrix}
  3 & 6 & 9 & 4 \\
  2 & 7 & 8 & 3 
\end{bmatrix}
\]

- Element \((i,j)\) in product is dot product of row \(i\) of first matrix and column \(j\) of second matrix

\[
\begin{bmatrix}
  9 & 27 & 33 & 13 \\
  20 & 44 & 61 & 26 \\
  9 & 28 & 32 & 12 
\end{bmatrix}
\]

Matrix-matrix multiplication

- Number of columns in first must = rows in second

\[
\begin{bmatrix}
  1 & 3 \\
  5 & 2 \\
  0 & 4 
\end{bmatrix}
\begin{bmatrix}
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\end{bmatrix}
\]

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\[
\begin{bmatrix}
  17 & 44 & 61 & 26 \\
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Matrix-matrix multiplication

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  9 & 27 & 33 & 13 \\
  19 & 44 & 61 & 26 \\
  28 & 32 & 12 
\end{bmatrix}
\]
Matrix-matrix multiplication

- Number of columns in first must = rows in second
- NOT EVEN LEGAL!!
- Non-commutative (AB and BA are different in general)
- Associative and distributive
  - $A(B+C) = AB + AC$
  - $(A+B)C = AC + BC$

Matrix-Vector Multiplication

- Key for transforming points (next lecture)
- Treat vector as a column matrix ($m \times 1$)
- E.g. 2D reflection about y-axis
  \[
  \begin{pmatrix}
  -1 & 0 \\
  0 & 1 
  \end{pmatrix}
  \begin{pmatrix}
  x \\
  y 
  \end{pmatrix}
  =
  \begin{pmatrix}
  -x \\
  y 
  \end{pmatrix}
  \]

Transpose of a Matrix (or vector?)

\[
\begin{pmatrix}
  1 & 2 \\
  3 & 4 \\
  5 & 6 
\end{pmatrix}
= \begin{pmatrix}
  1 & 3 & 5 \\
  2 & 4 & 6 
\end{pmatrix}
\]

\[(AB)^T = B^T A^T\]

Identity Matrix and Inverses

\[
I_{3 \times 3} = \begin{pmatrix}
  1 & 0 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & 1 
\end{pmatrix}
\]

\[
AA^{-1} = A^{-1} A = I
\]

\[
(AB)^{-1} = B^{-1} A^{-1}
\]

Vector multiplication in Matrix form

- Dot product? $a \cdot b = a^T b$
- Cross product? $a \times b = A^* b$

\[
\begin{pmatrix}
  x_a & y_a & z_a 
\end{pmatrix}
= \begin{pmatrix}
  x_a \\ y_a \\ z_a 
\end{pmatrix} = \begin{pmatrix}
  x_a & y_a & z_a \\
  0 & -z_a & y_a \\
  z_a & 0 & -x_a \\
  -y_a & x_a & 0 
\end{pmatrix}
\]

Dual matrix of vector $a$