Spectral Algorithms for Graph Partitioning

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Based on work with Tsz Chiu Kwok, Lap Chi Lau, James Lee, Yin Tat Lee, and Shayan Oveis Gharan
spectral graph theory

Use *linear algebra* to

- understand/characterize *graph properties*
- design efficient *graph algorithms*
linear algebra and graphs

Take an undirected graph $G = (V, E)$, consider matrix

$$A[i, j] := \text{weight of edge } (i, j)$$
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$$\begin{bmatrix}
0., & 1., & 0., & 0., & 1., & 1., & 0., & 0., & 0., & 0. \\
1., & 0., & 1., & 0., & 0., & 0., & 1., & 0., & 0., & 0. \\
0., & 1., & 0., & 1., & 0., & 0., & 0., & 1., & 0., & 0. \\
0., & 0., & 1., & 0., & 1., & 0., & 0., & 0., & 1., & 0. \\
1., & 0., & 0., & 1., & 0., & 0., & 0., & 0., & 0., & 1. \\
1., & 0., & 0., & 0., & 0., & 0., & 1., & 1., & 0., & 0. \\
0., & 1., & 0., & 0., & 0., & 0., & 1., & 1., & 0., & 0. \\
0., & 0., & 1., & 0., & 0., & 0., & 0., & 1., & 1., & 0. \\
0., & 0., & 0., & 1., & 0., & 0., & 0., & 0., & 0., & 1. \\
0., & 0., & 0., & 0., & 1., & 1., & 0., & 0., & 0., & 0. \\
0., & 0., & 0., & 0., & 1., & 1., & 0., & 0., & 0., & 0. \\
\end{bmatrix}$$
Take an undirected graph $G = (V, E)$, consider matrix $A[i, j] := \text{weight of edge } (i, j)$

\[
\begin{bmatrix}
0., 1., 0., 0., 1., 1., 0., 0., 0., 0. \\
1., 0., 1., 0., 0., 0., 1., 0., 0., 0. \\
0., 1., 0., 0., 0., 0., 0., 1., 0., 0. \\
0., 0., 1., 0., 1., 0., 0., 0., 1., 0. \\
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1., 0., 0., 0., 0., 0., 0., 0., 1., 0. \\
0., 1., 0., 0., 0., 0., 0., 0., 1., 1. \\
0., 0., 1., 0., 0., 0., 0., 0., 1., 1. \\
0., 0., 1., 0., 0., 0., 0., 0., 0., 1. \\
0., 0., 0., 1., 0., 0., 0., 0., 0., 0. \\
0., 0., 0., 0., 1., 0., 1., 0., 0., 0. \\
0., 0., 0., 0., 0., 1., 0., 1., 0., 0. \\
0., 0., 0., 0., 0., 0., 1., 0., 0., 0. \\
0., 0., 0., 0., 0., 0., 0., 1., 0., 0. \\
0., 0., 0., 0., 0., 0., 0., 0., 1., 0. \\
0., 0., 0., 0., 0., 0., 0., 0., 0., 1.
\end{bmatrix}
\]

eigenvalues $\rightarrow$ combinatorial properties

eigenvenctors $\rightarrow$ algorithms for combinatorial problems
fundamental thm of spectral graph theory

\[ G = (V, E) \text{ undirected, } A \text{ adjacency matrix, } d_v \text{ degree of } v, \ D \text{ diagonal matrix of degrees} \]

\[ L := I - D^{-\frac{1}{2}} AD^{-\frac{1}{2}} \]
fundamental thm of spectral graph theory

\( G = (V, E) \) undirected, \( A \) adjacency matrix, \( d_v \) degree of \( v \), \( D \) diagonal matrix of degrees

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\( L \) is symmetric, all its eigenvalues are real and
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- \( 0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq 2 \)
fundamental thm of spectral graph theory

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\[
L := I - D^{-\frac{1}{2}} AD^{-\frac{1}{2}}
\]

\( L \) is symmetric, all its eigenvalues are real and

- \( 0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq 2 \)
- \( \lambda_k = 0 \) iff \( G \) has \( \geq k \) connected components
- \( \lambda_n = 2 \) iff \( G \) has a bipartite connected component
0, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{5}{3}, \frac{5}{3}, \frac{5}{3}, \frac{5}{3}, \frac{5}{3}, \frac{5}{3}, \frac{5}{3}, \frac{5}{3}, \frac{5}{3}, \frac{5}{3}, \frac{5}{3}, \frac{5}{3}, \frac{5}{3}, \frac{5}{3}, \frac{5}{3}, \frac{5}{3}, \frac{5}{3}, \frac{5}{3}, \frac{5}{3}, \frac{5}{3}, \frac{5}{3}, \frac{5}{3}, \frac{5}{3}, \frac{5}{3}, \frac{5}{3}, \frac{5}{3}, \frac{5}{3}, \frac{5}{3}, \frac{5}{3}, \frac{5}{3}, \frac{5}{3}, \frac{5}{3}, \frac{5}{3}, \frac{5}{3}, \frac{5}{3}, \frac{5}{3}, \frac{5}{3}, \frac{5}{3}, \frac{5}{3}, \frac{5}{3}, \frac{5}{3}, \frac{5}{3}, \frac{5}{3}, \frac{5}{3}, \frac{5}{3}, \frac{5}{3}, \frac{5}{3},
0, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{5}{3}, \frac{5}{3}, \frac{5}{3}, \frac{5}{3}
0, 0, .69, .69, 1.5, 1.5, 1.8, 1.8
0, 0, .69, .69, 1.5, 1.5, 1.8, 1.8
0, $\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{4}{3}, \frac{4}{3}, \frac{4}{3}, 2$
0, $\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{4}{3}, \frac{4}{3}, \frac{4}{3}, 2$
If \( M \in \mathbb{R}^{n \times n} \) is symmetric, and

\[ \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \]

are eigenvalues counted with multiplicities, then

\[ \lambda_k = \min_{A \text{ a } k-\text{dim subspace of } \mathbb{R}^V} \max_{x \in A} \frac{x^T M x}{x^T x} \]

and eigenvectors of \( \lambda_1, \ldots, \lambda_k \) are basis of \( \text{opt } A \)
why eigenvalues relate to connected components

If $G = (V, E)$ is undirected and $L$ is Laplacian, then

$$\frac{x^T L x}{x^T x} = \frac{\sum_{(u,v) \in E} |x_u - x_v|^2}{\sum_v d_v \cdot x_v^2}$$

Note: $\sum_{(u,v) \in E} |x_u - x_v|^2 = 0$ iff $x$ constant on connected components
why eigenvalues relate to connected components

If \( G = (V, E) \) is undirected and \( L \) is Laplacian, then

\[
\frac{x^T L x}{x^T x} = \frac{\sum_{(u,v) \in E} |x_u - x_v|^2}{\sum_v d_v \cdot x_v^2}
\]

If \( \lambda_1 \leq \lambda_2 \leq \cdots \lambda_n \) are eigenvalues of \( L \) with multiplicities,

\[
\lambda_k = \min_A \max_{x \in A} R(x)
\]

where

\[
R(x) := \frac{\sum_{(u,v) \in E} |x_u - x_v|^2}{\sum_v d_v \cdot x_v^2}
\]

Note: \( \sum_{(u,v) \in E} |x_u - x_v|^2 = 0 \) iff \( x \) constant on connected components
why eigenvalues relate to connected components

If $G = (V, E)$ is undirected and $L$ is Laplacian, then

$$x^T L x \over x^T x = \sum_{(u,v) \in E} |x_u - x_v|^2 \over \sum_v d_v \cdot x_v^2$$

If $\lambda_1 \leq \lambda_2 \leq \cdots \lambda_n$ are eigenvalues of $L$ with multiplicities, 

$$\lambda_k = \min_{A \ k- \text{dim subspace of } \mathbb{R}^V} \max_{x \in A} R(x)$$

where

$$R(x) := \sum_{(u,v) \in E} |x_u - x_v|^2 \over \sum_v d_v \cdot x_v^2$$

Note: $\sum_{(u,v) \in E} |x_u - x_v|^2 = 0$ iff $x$ constant on connected components
if $G$ has $\geq k$ connected components

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Consider the space of all vectors that are constant in each connected component; it has dimension at least $k$ and so it witnesses $\lambda_k = 0$. 
If \( \lambda_k = 0 \) there is a \( k \)-dimensional space \( A \) of vectors, each constant on connected components. The graph must have \( \geq k \) connected components.
If $\lambda_k = 0$ there is a $k$-dimensional space $A$ of vectors, each constant on connected components.
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The graph must have $\geq k$ connected components.
\[ \text{vol}(S) := \sum_{v \in S} d_v \]

\[ \phi(S) := \frac{E(S, \bar{S})}{\text{vol}(S)} \]

\[ \phi(G) := \min_{S \subseteq V: 0 < \text{vol}(S) \leq \frac{1}{2} \text{vol}(V)} \phi(S) \]

In regular graphs, same as \textit{edge expansion} and, up to factor of 2 \textit{non-uniform sparsest cut}
conductance

\[0, 0, .69, .69, 1.5, 1.5, 1.8, 1.8\]

\[\phi(G) = \phi(\{0, 1, 2\}) = \frac{0}{6} = 0\]
conductance

$0, .055, .055, .211, \ldots$

$\phi(G) = \frac{2}{18}$
conductance

\[ 0, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{5}{3}, \frac{5}{3}, \frac{5}{3}, \frac{5}{3} \]

\[ \phi(G) = \frac{5}{15} \]
finding $S$ of small conductance

- $G$ is a social network
finding $S$ of small conductance

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  - $S$ is a set of users more likely to be friends with each other than anyone else
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- $G$ is a similarity graph on items of a data sets
finding $S$ of small conductance

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  - $S$ are items more similar to each other than to other items
  - [Shi, Malik]: image segmentation
finding $S$ of small conductance

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- $G$ is a similarity graph on items of a data sets
  - $S$ are items more similar to each other than to other items
  - [Shi, Malik]: image segmentation

- $G$ is an input of a NP-hard optimization problem
  - Recurse on $S$, $V - S$, use dynamic programming to combine in time $2^{O(|E(S,\bar{S})|)}$
conductance versus $\lambda_2$

Recall:

$\lambda_2 = 0 \iff G \text{ disconnected}$
Recall:
\[
\lambda_2 = 0 \iff G \text{ disconnected} \iff \phi(G) = 0
\]
conductance versus $\lambda_2$

Recall:

$$\lambda_2 = 0 \iff G \text{ disconnected} \iff \phi(G) = 0$$

Cheeger inequality (Alon, Milman, Dodziuk, mid-1980s):

$$\frac{\lambda_2}{2} \leq \phi(G) \leq \sqrt{2\lambda_2}$$
Cheeger inequality

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\frac{\lambda_2}{2} \leq \phi(G) \leq \sqrt{2\lambda_2}
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Cheeger inequality

\[ \frac{\lambda_2}{2} \leq \phi(G) \leq \sqrt{2\lambda_2} \]

Constructive proof of \( \phi(G) \leq \sqrt{2\lambda_2} \):
- given eigenvector \( x \) of \( \lambda_2 \).
- sort vertices \( v \) according to \( x_v \)
- a set of vertices \( S \) occurring as initial segment or final segment of sorted order has

\[ \phi(S) \leq \sqrt{2\lambda_2} \]
Cheeger inequality

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\[ \phi(S) \leq \sqrt{2\lambda_2} \leq 2\sqrt{\phi(G)} \]

Fiedler’s sweep algorithm can be implemented in \( \tilde{O}(|V| + |E|) \) time
\[
\frac{\lambda_2}{2} \leq \phi(G) \leq \sqrt{2\lambda_2}
\]

applications
\[
\frac{\lambda_2}{2} \leq \phi(G) \leq \sqrt{2\lambda_2}
\]

- rigorous analysis of *sweep algorithm*  
  (practical performance usually better than worst-case analysis)
applications

\[ \frac{\lambda_2}{2} \leq \phi(G) \leq \sqrt{2}\lambda_2 \]

- rigorous analysis of sweep algorithm
  (practical performance usually better than worst-case analysis)

- to certify \( \phi \geq \Omega(1) \), enough to certify \( \lambda_2 \geq \Omega(1) \)
\[
\frac{\lambda_2}{2} \leq \phi(G) \leq \sqrt{2\lambda_2}
\]

- rigorous analysis of sweep algorithm
  (practical performance usually better than worst-case analysis)

- to certify \( \phi \geq \Omega(1) \), enough to certify \( \lambda_2 \geq \Omega(1) \)

- mixing time of random walk is \( O\left(\frac{1}{\lambda_2} \log |V|\right) \) and so also

\[
O\left(\frac{1}{\phi^2(G)} \log |V|\right)
\]
A similar theory for other eigenvectors, with algorithmic applications

One intended application: Like Cheeger’s inequality is worst case analysis of Fiedler’s algorithm, develop worst-case analysis of spectral clustering
spectral clustering

The following algorithm works well in practice. (But to solve what problem?)

Compute $k$ eigenvectors $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(k)}$ of $k$ smallest eigenvalues of $L$

Define mapping $F : V \rightarrow \mathbb{R}^k$

$$F(v) := (x_v^{(1)}, x_v^{(2)}, \ldots, x_v^{(k)})$$

Apply $k$-means to the points that vertices are mapped to
spectral embedding of a random graph

$k = 3$
When $\lambda_k$ is small
From fundamental theorem of spectral graph theory:

- $\lambda_k = 0 \iff G \text{ has } \geq k \text{ connected components}$
From fundamental theorem of spectral graph theory:

- $\lambda_k = 0 \iff G$ has $\geq k$ connected components

We prove:

- $\lambda_k$ small $\iff G$ has $\geq k$ disjoint sets of small conductance
Define order-\(k\) conductance

\[ \phi_k(G) := \min_{S_1, \ldots, S_k \text{ disjoint}} \max_{i=1, \ldots, k} \phi(S_i) \]

Note:

- \(\phi(G) = \phi_2(G)\)
- \(\phi_k(G) = 0 \iff G \text{ has } \geq k \text{ connected components}\)
0, 0, .69, .69, 1.5, 1.5, 1.8, 1.8
\[ \phi_3(G) = \max \left\{ 0, \frac{2}{6}, \frac{2}{4} \right\} = \frac{1}{2} \]
\[ \phi_3(G) = \max \left\{ \frac{2}{12}, \frac{2}{12}, \frac{2}{14} \right\} = \frac{1}{6} \]
\[ \lambda_k = 0 \iff \phi_k = 0 \]
\( \lambda_k = 0 \iff \phi_k = 0 \)

\[
\frac{\lambda_k}{2} \leq \phi_k(G) \leq O(k^2) \cdot \sqrt{\lambda_k}
\]

(Lee, Oveis Gharan, T, 2012)

Upper bound is constructive: in nearly-linear time we can find \( k \) disjoint sets each of expansion at most \( O(k^2) \sqrt{\lambda_k} \).
\[ \frac{\lambda_k}{2} \leq \phi_k(G) \leq O(k^2) \cdot \sqrt{\lambda_k} \]
\[
\frac{\lambda_k}{2} \leq \phi_k(G) \leq O(k^2) \cdot \sqrt{\lambda_k}
\]

\[
\phi_k(G) \leq O(\sqrt{\log k} \cdot \lambda_{1.1k})
\]

(Lee, Oveis Gharan, T, 2012)

\[
\phi_k(G) \leq O(\sqrt{\log k} \cdot \lambda_{O(k)})
\]

(Louis, Raghavendra, Tetali, Vempala, 2012)

Factor $O(\sqrt{\log k})$ is tight
\[ \phi_k(G) \leq O(\sqrt{\log k} \cdot \lambda_O(k)) \]

(Louis, Raghavendra, Tetali, Vempala, 2012)
(Lee, Oveis Gharan, T, 2012)

Miclo 2013:
- Analog for infinite-dimensional Markov process.
- Application: every hyper-bounded operator has a spectral gap.
- Solves 40+ year old open problem
Recall spectral clustering algorithm

1. Compute $k$ eigenvectors $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(k)}$ of $k$ smallest eigenvalues of $L$
2. Define mapping $F : V \rightarrow \mathbb{R}^k$
   
   $$F(v) := (x_v^{(1)}, x_v^{(2)}, \ldots, x_v^{(k)})$$
3. Apply $k$-means to the points that vertices are mapped to
Recall spectral clustering algorithm

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2. Define mapping $F : V \rightarrow \mathbb{R}^k$
   $$F(v) := (x^{(1)}_v, x^{(2)}_v, \ldots, x^{(k)}_v)$$
3. Apply $k$-means to the points that vertices are mapped to

LOT algorithms and LRTV algorithm find $k$ low-conductance sets if they exist

First step is spectral embedding

Then geometric partitioning but not $k$-means
When $\lambda_k$ is large
The Cheeger inequality states:

\[
\frac{\lambda_2}{2} \leq \phi(G) \leq \sqrt{2\lambda_2}
\]

Nearly-linear time sweep algorithm finds \( S \) such that, for every \( k \),

\[
\phi(S) \leq \phi(G) \cdot O(k) \sqrt{\lambda_k}
\]

(Kwok, Lau, Lee, Oveis Gharan, 2013)
Cheeger inequality

\[ \frac{\lambda_2}{2} \leq \phi(G) \leq \sqrt{2\lambda_2} \]

\[ \phi(G) \leq O(k) \cdot \frac{\lambda_2}{\sqrt{\lambda_k}} \]

(Kwok, Lau, Lee, Oveis Gharan, T, 2013)

and the upper bound holds for the sweep algorithm
Cheeger inequality

\[ \frac{\lambda_2}{2} \leq \phi(G) \leq \sqrt{2\lambda_2} \]

\[ \phi(G) \leq O(k) \cdot \frac{\lambda_2}{\sqrt{\lambda_k}} \]

(Kwok, Lau, Lee, Oveis Gharan, T, 2013)

and the upper bound holds for the sweep algorithm

Nearly-linear time sweep algorithm finds \( S \) such that, for every \( k \),

\[ \phi(S) \leq \phi(G) \cdot O \left( \frac{k}{\sqrt{\lambda_k}} \right) \]
Cheeger inequality

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Cheeger inequality

$$\frac{\lambda_2}{2} \leq \phi(G) \leq \sqrt{2\lambda_2}$$

$$\phi(G) \leq O(k) \cdot \frac{\lambda_2}{\sqrt{\lambda_k}}$$

(Kwok, Lau, Lee, Oveis Gharan, T, 2013)

Liu 2014: analog for Riemann manifolds. Applications to bounding eigenvalue gaps
We find a $2k$-valued vector $y$ such that

$$\|x - y\|^2 \leq O\left(\frac{\lambda_2}{\lambda_k}\right)$$

where $x$ is eigenvector of $\lambda_2$
proof structure

1. We find a \( 2k \)-valued vector \( y \) such that

\[
||x - y||^2 \leq O \left( \frac{\lambda_2}{\lambda_k} \right)
\]

where \( x \) is eigenvector of \( \lambda_2 \)

2. For every \( k \)-valued vector \( y \), applying the sweep algorithm to \( x \) finds a set \( S \) such that

\[
\phi(S) \leq O(k) \cdot \left( \lambda_2 + \sqrt{\lambda_2} \cdot ||x - y|| \right)
\]
1. We find a $2k$-valued vector $y$ such that

$$||x - y||^2 \leq O\left(\frac{\lambda_2}{\lambda_k}\right)$$

where $x$ is eigenvector of $\lambda_2$

2. For every $k$-valued vector $y$, applying the sweep algorithm to $x$ finds a set $S$ such that

$$\phi(S) \leq O(k) \cdot \left(\lambda_2 + \sqrt{\lambda_2} \cdot ||x - y||\right)$$

So the sweep algorithm finds $S$

$$\phi(S) \leq O(k) \cdot \lambda_2 \cdot \frac{1}{\sqrt{\lambda_k}}$$
If $\lambda_k$ is large,

then a cut of approximately minimal conductance can be found in time exponential in $k$

using spectral methods [ABS]

or convex relaxations [BRS, GS]
If $\lambda_k$ is large,

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using spectral methods [ABS]
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[Kwok, Lau, Lee, Oveis-Gharan, T]: the nearly-linear time sweep algorithm finds good approximation if $\lambda_k$ is large
bounded threshold rank

If $\lambda_k$ is large,

[Kwok, Lau, Lee, Oveis-Gharan, T]: the nearly-linear time sweep algorithm finds good approximation
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If $\lambda_k$ is large,

[Kwok, Lau, Lee, Oveis-Gharan, T]: the nearly-linear time sweep algorithm finds good approximation

[Oveis-Gharan, T]: the Goemans-Linial relaxation (solvable in polynomial time independent of $k$) can be rounded better than in the Arora-Rao-Vazirani analysis
If $\lambda_k$ is large,

[Kwok, Lau, Lee, Oveis-Gharan, T]: the nearly-linear time sweep algorithm finds good approximation

[Oveis-Gharan, T]: the Goemans-Linial relaxation (solvable in polynomial time independent of $k$) can be rounded better than in the Arora-Rao-Vazirani analysis

[Oveis-Gharan, T]: the graph satisfies a *weak regularity lemma* in the sense of Frieze and Kannan
If $\lambda_k$ is large, the graph is an easy instance for several algorithms. Two types of results:

- There is a near-optimal combinatorial solution with a simple structure
  
  *Find it by brute force*

- The optimum of a relaxation has a special structure
  
  *Improved rounding algorithm that uses the special structure*
When $\lambda_{k+1}$ is large and $\lambda_k$ is small
• if \( \lambda_k = 0 \) and \( \lambda_{k+1} > 0 \) then \( G \) has exactly \( k \) connected components;

\[ \text{eigenvalue gap} \]
• if $\lambda_k = 0$ and $\lambda_{k+1} > 0$ then $G$ has exactly $k$ connected components;

• [Tanaka 2012] If $\lambda_{k+1} > 5^k \sqrt{\lambda_k}$, then vertices of $G$ can be partitioned into $k$ sets of small conductance, each inducing a subgraph of large conductance. [Non-algorithmic proof]
• if $\lambda_k = 0$ and $\lambda_{k+1} > 0$ then $G$ has exactly $k$ connected components;

• [Tanaka 2012] If $\lambda_{k+1} > 5^k \sqrt{\lambda_k}$, then vertices of $G$ can be partitioned into $k$ sets of small conductance, each inducing a subgraph of large conductance. [Non-algorithmic proof]

• [Oveis-Gharan, T 2013] Sufficient $\phi_{k+1}(G) > (1 + \epsilon)\phi_k(G)$; if $\lambda_{k+1} > O(k^2)\sqrt{\lambda_k}$, then partition can be found algorithmically
In Summary

\(\lambda_k\) is small: There are \(k\) disjoint non-expanding sets; they can be found efficiently.

\(\lambda_k\) is large: Easy instance for various algorithms.

\(\lambda_k\) is small and \(\lambda_{k+1}\) is large: Nodes can be partitioned into \(k\) sets, each with high internal expansion and small external expansion.