CSE 291: Fourier analysis
Homework 2

1. **Total influence and noise sensitivity.** Compute (at least approximately) the total influence and noise sensitivity for the following functions:
   (a) **PARITY:** \( f(x) = x_1 \oplus \ldots \oplus x_n \).
   (b) **MAJORITY:** \( f(x) = \text{sign}(x_1 + \ldots + x_n) \).
   (c) **DICTATOR:** \( f(x) = x_1 \).
   (d) **TRIBES:** Assume \( n = w2^w \) and take \( f(x) = \text{TRIBES}_{w,2^w} = \text{AND}(\text{OR}(x_1,\ldots,x_w),\ldots,\text{OR}(x_{n-w+1},\ldots,x_n)) \).

2. **Robust Arrow’s theorem.** Consider a 3-candidate Condorcet election using a boolean function \( f : \{-1,1\}^n \rightarrow \{-1,1\} \). Assume that \( f \) is far from a dictator, in the sense that \( |\hat{f}(|i|)| \leq 1 - \varepsilon \) for all \( i \in [n] \). Prove that the probability of a Condorcet winner is at most \( 1 - \delta \), where \( \delta = \delta(\varepsilon) \). What is the best bound you can give on \( \delta \)?

3. **Fourier granularity.** Let \( f : \{-1,1\}^n \rightarrow \mathbb{Z} \).
   (a) Prove that for any \( S \), \( \hat{f}(S) = \frac{a_S}{2^n} \) for some \( a_S \in \mathbb{Z} \).
   (b) Assume furthermore that \( \deg(f) = d \). Let \( S \) be a maximal monomial, i.e. \( |S| = d \).
       Let \( g : \{-1,1\}^d \rightarrow \mathbb{Z} \) be obtained by restricting \( f \) to the variables in \( S \), and fixing the rest arbitrarily. Argue that \( \hat{g}([d]) = \hat{f}(S) \). Conclude that \( \hat{f}(S) = \frac{a_S}{2^n} \) for some \( a_S \in \mathbb{Z} \).
   (c) Use this iteratively to show that if \( \deg(f) = d \) then for all \( S \) it holds that \( \hat{f}(S) = \frac{a_S}{2^n} \) for some \( a_S \in \mathbb{Z} \).
   (d) Assume now that \( f : \{-1,1\}^n \rightarrow \{-1,1\} \) has degree \( d \). Prove that \( f \) has at most \( 2^{2d} \) nonzero Fourier coefficients.

4. **Exact learning.** Let \( f : \{-1,1\}^n \rightarrow \{-1,1\} \). Assume that \( \deg(f) = d \) (for example, \( f \) can be computed by a decision tree of depth \( d \)). We consider learning algorithms which need to compute \( f \) exactly (that is, with no error), and succeed with high probability (say, 99%).
   (a) Show that \( f \) can be learned exactly from \( \text{poly}(2^d,n) \) uniform samples in time \( n^d \cdot \text{poly}(2^d,n) \).
(b) Show that $f$ can be learned exactly using $\text{poly}(2^d, n)$ membership queries in time $\text{poly}(2^d, n)$ (hint: use Q2).

(c) Prove that learning a $d$-junta exactly requires at least $2^d$ membership queries.

**Open Problem.** Consider the problem of learning $d$-juntas (or more generally, depth $d$ decision trees) for $d = O(\log n)$, say. Both types of learning algorithms (using uniform samples, or using membership queries) both require near optimal number of samples / queries, but they differ in their time complexity. It is unknown whether learning from uniform samples really requires $n^d$ time, or whether it can be improved to $\text{poly}(2^d, n)$ time.