The Inverse Conjecture for the Gowers Norm is False

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Informal overview

- **Problem**: Test if a function $f : \mathbb{F}_2^n \rightarrow \mathbb{F}_2$ is somewhat close to a degree $d$ polynomial, using few queries

- The Gowers Norm of $f$ is noticeable

- The test works in one direction: if $f$ is close to a degree $d$ polynomial, the test accepts with noticeable probability

- Main question: Does the inverse direction work? If the test accepts with noticeable probability, is $f$ close to a degree $d$ polynomial?

  - $d = 1, 2$: Yes ([BLR93], [BCHKS96] / [Sam05], [GT05])
  - $d \geq 3$: No (this work)

Lovett, Meshulam, Samorodnitsky

The ICGN is False
**Problem:** Test if a function $f : \mathbb{F}_2^n \rightarrow \mathbb{F}_2$ is somewhat close to a degree $d$ polynomial, using few queries

**Natural test:** Take $d + 1$ derivatives of $f$ in random directions. Accept if zero
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Derivatives

- $f(x) : \mathbb{F}_2^n \rightarrow \mathbb{F}_2$
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- \( f(x) : \mathbb{F}_2^n \rightarrow \mathbb{F}_2 \)

- Derivative in direction \( y \in \mathbb{F}_2^n \): \( f_y(x) = f(x + y) - f(x) \)
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  - Derivation reduces degree
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  - Derivation reduces degree

- Multiple derivatives:
  $$f_{y_1,\ldots,y_d}(x) = \sum_{S \subset \{1,\ldots,d\}} (-1)^{|S|} f(x + \sum_{i \in S} y_i)$$
Choose random $x, y_1, \ldots, y_d \in \mathbb{F}_2^n$
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Test if:

$$f_{y_1,\ldots,y_d}(x) = \sum_{S \subseteq \{1,\ldots,d\}} (-1)^{d-|S|} f(x + \sum_{i \in S} y_i) = 0$$
Random derivatives test

- Choose random $x, y_1, \ldots, y_d \in \mathbb{F}_2^n$
- Test if:
  \[
  f_{y_1,\ldots,y_d}(x) = \sum_{S \subseteq \{1,\ldots,d\}} (-1)^{|S|} f(x + \sum_{i \in S} y_i) = 0
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- The test uses $2^d$ queries
Random derivatives test

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Random derivatives test

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The test accepts always \( \iff \) \( f \) is a degree \( d - 1 \) polynomial

\( f \) is random \( \Rightarrow \) the test accepts with probability \( 1/2 \)

Main problem: If the test accepts with probability \( \frac{1+\delta}{2} \), is \( f \) somewhat close to a degree \( d - 1 \) polynomial?
$d$-th Gowers Norm of $f$ - average bias of random $d$ derivatives
**Gowers Norm**

- **d-th Gowers Norm** of \( f \) - average bias of random \( d \) derivatives

\[
\|f\|_{U^d} = \left( \mathbb{E}_{x,y_1,...,y_d} \left[ (-1)^{f_{y_1,...,y_d}(x)} \right] \right)^{1/2}
\]

**Facts:**

- \( \|\cdot\|_{U^d} \) is a norm
- \( 0 \leq \|f\|_{U^d} \leq 1 \)
- \( f \) is a degree \( d-1 \) polynomial \( \iff \|f\|_{U^d} = 1 \)
- \( f \) is random \( \Rightarrow \|f\|_{U^d} \approx 0 \)
**Gowers Norm**

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\[ \|f\|_{U^d} = \left( \mathbb{E}_{x, y_1, \ldots, y_d} \left[ (-1)^{f_{y_1, \ldots, y_d}(x)} \right] \right)^{1/2^d} \]
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- Facts:
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  - $\| \cdot \|_{U^d}$ is a norm
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Closeness to low degree poly. $\Rightarrow$ large Gowers Norm

**Definition:** $f$ is $\delta$-close to $g$ if $\Pr_x[f(x) = g(x)] \geq \frac{1+\delta}{2}$
Closeness to low degree poly. $\Rightarrow$ large Gowers Norm

- **Definition:** $f$ is $\delta$-close to $g$ if $\mathbb{P}_x[f(x) = g(x)] \geq \frac{1+\delta}{2}$

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- **Inverse Conjecture:**
  $\|f\|_{U^d} \geq \delta$ $\Rightarrow$ $f$ is $\delta'$-close to a degree $d - 1$ polynomial $g$
Closeness to low degree poly. \(\Rightarrow\) large Gowers Norm

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- **Inverse Conjecture:**
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- The Inverse Conjecture is known to hold when:
  - \(d = 1:\) \(\delta' = \delta\) [BLR93, BCHKS96]
  - \(d = 2:\) \(\delta' = e^{-\left(1 - (1 - \delta)^2\right)}O(1)\) [Sam05, GT05]
  - \(\delta \gtrsim 1 - 2^{-2^d}:\) \(\delta' \approx \delta\) [AKKLR03]

We show:
The Inverse Conjecture is already false for \(d = 3\)

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Closeness to low degree poly. \implies large Gowers Norm

- **Definition:** \( f \) is \( \delta \)-close to \( g \) if \( \Pr_x [f(x) = g(x)] \geq \frac{1+\delta}{2} \)

- \( f \) is \( \delta \)-close to a degree \( d - 1 \) polynomial \( g \) \implies \( \|f\|_{U^d} \geq \delta \)

- **Inverse Conjecture:**
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- **We show:** The Inverse Conjecture is already false for $d = 3$
The counterexample (over $\mathbb{F}_2$)

- The symmetric polynomial $S_4(x_1, \ldots, x_n)$:

\[
S_4(x_1, \ldots, x_n) = \sum_{i<j<k<l} x_i x_j x_k x_l
\]
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$$S_4(x_1, \ldots, x_n) = \sum_{i<j<k<l} x_ix_jx_kx_l$$

- We prove:

There is no cubic polynomial which is $2^{-\Omega(n)}$-close to $S_4$.

Independently proven by Green and Tao.

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- We prove:
  - $\|S_4\|_{U^4} \approx \left(\frac{1}{8}\right)^{1/16} (= 0.88\ldots)$

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- We prove:
  - $\|S_4\|_{U^4} \approx (\frac{1}{8})^{1/16}$ ($\approx 0.88\ldots$)
  - There is no cubic polynomial which is $2^{-\Omega(n)}$-close to $S_4$
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- The symmetric polynomial $S_4(x_1, ..., x_n)$:
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We prove:
- $\|S_4\|_{U^4} \approx (\frac{1}{8})^{1/16}$ (≈ 0.88...)
- There is no cubic polynomial which is $2^{-\Omega(n)}$-close to $S_4$

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  There is no cubic polynomial which is $2^{-\Omega(n)}$-close to $S_4$

- Independently proven by Green and Tao
  There is no cubic polynomial which is $\frac{1}{\log \log n}$-close to $S_4$
Counterexamples over general fields

- The **Gowers Norm** can be defined over general prime fields $\mathbb{F}_p$
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Counterexamples over general fields

• The Gowers Norm can be defined over general prime fields $\mathbb{F}_p$

• Counterexamples - the symmetric polynomials $S_p^k(x_1, \ldots, x_n)$

• $\|S_p^k\|_{U_p^k} > \epsilon_{p,k} \ (k > 1)$
Counterexamples over general fields

- The **Gowers Norm** can be defined over general prime fields \( \mathbb{F}_p \)

- Counterexamples - the symmetric polynomials \( S_{p^k}(x_1, \ldots, x_n) \)

\[
\|S_{p^k}\|_{U^{p^k}} > \epsilon_{p,k} \quad (k > 1)
\]

- Using a variant of the Alon-Beigel argument (used in [GT07]), \( S_{p^k} \) is not \( \delta(n) \)-close to lower degree polynomials, for

\[
\delta(n) = \frac{1}{\log \ldots \log_n} = o(1)
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Counterexamples over general fields

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- Counterexamples - the symmetric polynomials $S_{p^k}(x_1, \ldots, x_n)$

$$\|S_{p^k}\|_{U_{p^k}} > \epsilon_{p,k} \ (k > 1)$$

Using a variant of the Alon-Beigel argument (used in [GT07]), $S_{p^k}$ is not $\delta(n)$-close to lower degree polynomials, for

$$\delta(n) = \frac{1}{\log \ldots \log_{p^k} n} = o(1)$$

- If $f$ is a polynomial, $\deg(f) < p \Rightarrow$ The ICGN is true [GT07]
Proof sketch

- We prove:

\[ \| S_4 \|_{U^4} \approx \left( \frac{1}{8} \right)^{1/16} \approx 0.88 \ldots \]

There is no cubic polynomial which is $2^{-\Omega(n)}$-close to $S_4$. 

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Proof sketch

- We prove:
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\[ \| S_4 \|_{U^4} \approx \left( \frac{1}{8} \right)^{1/16} \approx 0.88 \ldots \]

There is no cubic polynomial which is \( 2^{-\Omega(n)} \)-close to \( S_4 \).
4-th Gowers Norm of $S_4$ is large

$$\|S_4\|_4^2 = \mathbb{E}_{x,y_1,y_2,y_3,y_4} \left[ (-1)^{(S_4)_{y_1,y_2,y_3,y_4}(x)} \right]$$
4-th Gowers Norm of $S_4$ is large

$\|S_4\|_4^2 = \mathbb{E}_{x,y_1,y_2,y_3,y_4} \left[ (-1)^{(S_4)_{y_1,y_2,y_3,y_4}(x)} \right]$

$(S_4)_{y_1,y_2,y_3,y_4} =$
4-th Gowers Norm of $S_4$ is large

$$\| S_4 \|_4^2 = \mathbb{E}_{x, y_1, y_2, y_3, y_4} \left[ (-1)^{ (S_4)_{y_1,y_2,y_3,y_4} (x) } \right]$$

$$(S_4)_{y_1,y_2,y_3,y_4} = \sum_{i \neq j \neq k \neq l} (y_1)_i (y_2)_j (y_3)_k (y_4)_l = \det (\langle y_i, y_j \rangle)_{i,j=1}^4 \approx \det (\text{random } 4 \times 4 \text{ symmetric matrix}) = \frac{9}{16} \left( > \frac{1}{2} \right)$$
4-th Gowers Norm of $S_4$ is large

\[ \| S_4 \|_4^2 = \mathbb{E}_{x, y_1, y_2, y_3, y_4} \left[ (-1)^{(S_4)_{y_1, y_2, y_3, y_4}(x)} \right] \]

\[ (S_4)_{y_1, y_2, y_3, y_4} = \sum_{i \neq j \neq k \neq l} (y_1)_i (y_2)_j (y_3)_k (y_4)_l = \]

\[ \text{det} \left( \langle y_i, y_j \rangle \right)_{i, j = 1}^4 \approx 9/16 \]
4-th Gowers Norm of $S_4$ is large

\[ \|S_4\|_4^2 = \mathbb{E}_{x,y_1,y_2,y_3,y_4} \left[ (-1)^{(S_4)_{y_1,y_2,y_3,y_4}(x)} \right] \]

\[ (S_4)_{y_1,y_2,y_3,y_4} = \sum_{i \neq j \neq k \neq l} (y_1)_i(y_2)_j(y_3)_k(y_4)_l = \]

\[ \det \left( \langle y_i, y_j \rangle \right)_{i,j=1}^4 \]

\[ (\langle y_i, y_j \rangle)_{i,j=1}^4 \sim \text{random } 4 \times 4 \text{ symmetric matrix} \]
4-th Gowers Norm of $S_4$ is large

$\|S_4\|_4^2 = E_{x,y_1,y_2,y_3,y_4} \left[ (-1)^{(S_4)_{y_1,y_2,y_3,y_4}(x)} \right]$ 

$(S_4)_{y_1,y_2,y_3,y_4} = \sum_{i \neq j \neq k \neq l} (y_1)_i(y_2)_j(y_3)_k(y_4)_l = det \left( \langle y_i, y_j \rangle \right)_{i,j=1}^4$ 

$(\langle y_i, y_j \rangle)_{i,j=1}^4 \sim$ random $4 \times 4$ symmetric matrix 

$\mathbb{P} \left[ (S_4)_{y_1,y_2,y_3,y_4} = 0 \right] \approx$
4-th Gowers Norm of $S_4$ is large

- $\|S_4\|_4^2 = \mathbb{E}_{x,y_1,y_2,y_3,y_4} \left[ (-1)^{(S_4)_{y_1,y_2,y_3,y_4}(x)} \right]$ 

$$(S_4)_{y_1,y_2,y_3,y_4} = \sum_{i\neq j \neq k \neq l} (y_1)_i (y_2)_j (y_3)_k (y_4)_l =$$ 

$$\det \left( \langle y_i, y_j \rangle \right)_{i,j=1}^4$$

- $\langle y_i, y_j \rangle_{i,j=1}^4 \sim \text{random } 4 \times 4 \text{ symmetric matrix}$

$$\mathbb{P} \left[ (S_4)_{y_1,y_2,y_3,y_4} = 0 \right] \approx$$

$$\mathbb{P} \left[ \det (\text{random } 4 \times 4 \text{ symmetric matrix}) \right] =$$
4-th Gowers Norm of $S_4$ is large

\[ \|S_4\|_4^2 = \mathbb{E}_{x, y_1, y_2, y_3, y_4} \left[ (-1)^{(S_4)_{y_1, y_2, y_3, y_4}(x)} \right] \]

\[ (S_4)_{y_1, y_2, y_3, y_4} = \sum_{i \neq j \neq k \neq l} (y_1)_i (y_2)_j (y_3)_k (y_4)_l = \]

\[ \det (\langle y_i, y_j \rangle)_{i, j = 1}^4 \]

\[ (\langle y_i, y_j \rangle)_{i, j = 1}^4 \sim \text{random } 4 \times 4 \text{ symmetric matrix} \]

\[ \mathbb{P} [(S_4)_{y_1, y_2, y_3, y_4} = 0] \approx \]

\[ \mathbb{P} [\det (\text{random } 4 \times 4 \text{ symmetric matrix})] = 9/16 \]
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\[9/16 \quad (> 1/2)\]
$S_4$ is not close to cubics

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It's enough to bound derivatives $P_{x,y,z}[S_4(x) = g(x)]$ since $S_4$ is quadratic, $g$ is linear - well understood.
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- **Hard case:** For the other half of $(y, z)$, $(S_4)_{y,z}$ has just 8 non-zero Fourier coefficients

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The ICGN is False
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We need to show that for any cubic \(g:\)

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P_{y,z}[g_{y,z}(x) \equiv \langle (y \cdot z), x \rangle + c_{y,z}] = 2^{-\Omega(n)}
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Equivalently - the number of common roots to the $n$ polynomials is exponentially small

Proof uses the special structure of the polynomials set
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Equivalently - the number of common roots to the $n$ polynomials is exponentially small.

Proof uses the special structure of the polynomials set.
Let $f_1, \ldots, f_m$ be polynomials in $\mathbb{F}_2[x_1, \ldots, x_n]$
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- Let:
  - $M = \mathbb{F}_2[x_1, ..., x_n]/\langle x_1^2 - x_1, ..., x_n^2 - x_n \rangle$
  - $I = \langle f_1, ..., f_m \rangle \subset M$
  - $R = \{ u \in \mathbb{F}_2^n : f_1(u) = ... = f_m(u) = 0 \}$
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- **Lemma:**
  \[ R \leq \dim(M/I) \]
Summary

- A natural test for low degree polynomials

Equivalent to the Gowers Norm

It is correct in special cases ($d = 1, 2/\delta \sim 1/\deg(f) < p$)

We showed it is generally false

Counterexamples for $d = pk$ over $\mathbb{F}_p$

Over $\mathbb{F}_2$, $S_4$ is at most exponentially-small close to cubics

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The ICGN is False
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Open problems

- Can you test proximity to low degree polynomials over small fields?
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  - Using constant number of queries
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Thank you