

15/9/2015 Russell's lec 6. Back to Williams! Faster APSP via circ comp.  
 APSP: given  $n$ -node  $G$ , defined by weight  $w$ .  $w: n \times n \rightarrow \mathbb{R}^+$  or  $\mathbb{Z}$ ?

Constraint: matrix  $S$ , s.t.  $S[i,j] = \min_k w[i,k] + w[k,j]$ . Fastest path from  $i$  to  $j$  starting w/ edge  $(i,k)$ .

-  $S$  always exists and can be described in  $O(n^2 \log n)$  bits

Value: weights are 2  $\log n$  bits each. Other: the real RAM model  
 registers hold words, can have  $+$ ,  $-$ ,  $\leq$ . Instance of APSP:  $n^2$  word reg.

Major open question: faster than  $O(n^3)$  or APSP.  $\Omega(n^2)$

~~Other~~: Open prob: if only  $+$  and compare on words one can word  
 - Fredman's FS:  $O(n^3 / \log^{1/3} n)$  on real RAM

major question: is APSP  $\in n^{3-\epsilon}$ ?

Either all or none of the following in  $n^{3-\epsilon}$ :

- 1) APSP
- 2) min (edge) weight triangle
- 3) connectivity

Th1 (new) word. alg for APSP in  $n^3 / 2^{L(n)}$ , where  $L(n) \gg \Omega(\log n)^{1/2}$

Th2  $\exists \epsilon > 0$  depend on  $L(n) \gg \Omega(\log n)^{1/2}$

Recently: det. alg for  $L(n) \gg \Omega(\log n)^{1/2}$

Cor and problems above have such times

Th3 term:  $\forall n$ ,  $\exists$  non-dets for APSP w/ int weights in  $[0, n^k]$   
 in  $n^3 / 2^{L(n)}$  with  $L(n) \gg \Omega(\log n)^{1/2}$

Intuition: min-plus matrix product

- 1) min-plus are ACO
- 2) Razborov-Smolensky: 2) eff. but on neg. weights of paths w/ representation '82 matrix mult.

2) Razborov-Smolensky '87: lower ACO (0) circ (  $\exists$  eff. standard  $D(c)$   
 of programs of deg  $(\log s)^{O(d)}$  over  $\mathbb{F}_2$  s.t.  $\forall c \in \mathbb{F}_2^d \exists p \in \mathbb{F}_2^d [p \cdot c = D(c)] \gg 3/4$

3) polynomial evaluation & FFT: univariate deg- $n$  poly on  $n$  roots in  $O(n \log n)$   
 In given  $A, B \in \{0,1\}^n, |A| = |B| = n, g(x_1, \dots, x_n, s_1, \dots, s_n)$  over  $\mathbb{F}_2$  w/  $|g| \leq n^c$ , can eval  $g$  on all  $(x,y) \in A \times B$  in  $n^2 \log(\log n)$  time.

Proof: embed in efficient matrix product

APSP alg: given  $A, B$ , compute  $(A \circ B)[i,j] = \min_k (A[i,k] + B[k,j])$   $\text{lex } d=2$

1) partition  $A$  into  $n^d$  matrices  $A_{i_1, \dots, i_d}$ , vertical, and  $B$  horizontally -  
 done with over  $d$  times  $O(n^2 \cdot \frac{n}{\lambda})$  time.

2) given  $A_k, B_k$ ,  $n^d$  and  $d \times n$ , let  $C$  be ACO circ or min-plus, of degree  $n$  words  
 $O(d \log \log n)$  size. Pick nonzero  $p_1, \dots, p_{d-1}$ , eval on rows of  $A_k$ , columns of  $B_k$   
 def  $d=2$   $(\log n)^{O(d)}$ ,  $c_i$  kept (2)

see FSTTCS14 survey, AWS SODA15, & a FOCS15

- efficient eval  $C$  [14] for  $d \geq 2$  min threshold give on column rectangles.  
 $\Rightarrow$  if min-plus in  $d \times d=2$  time, then APSP  $\in n^{3-\epsilon}$