

8/9/2015 Russell's lemma 4

Lemma: if  $F$  is depth size  $S$ ,  $d \geq 2 \log_2 S$ , circ,  $p = 1/D^{d-1}$

Then  $\text{Prob} [F|_p \text{ is } \mathcal{O}(D^d)] \leq 2^{-D}$

Cor  $\text{Prob} [F|_p \text{ has } d_s \neq 0 \text{ for } d_s > D] \leq 2^{-D}$

Cor  $E[\sum_{s, |s| > D} (\hat{d}_s)^2] \leq 2^{-D}$  (w prob.  $1 - \frac{1}{2^D}$   $d_s = 0$ )

Cor  $\sum_{T, |T| > O(D^d)} (\hat{d}_T)^2 \leq O(2^{-D})$

Lemma let  $f, g$  be multilinear polynomials over reals,  
 $f = \sum_S \beta_S \prod_{i \in S} x_i$ ,  $g = \sum_S \gamma_S \prod_{i \in S} x_i$ ; then  $\sum_x (f(x) - g(x))^2$

$$= \sum_S (\beta_S - \gamma_S)^2 = \|f - g\|^2$$

Set  $D = O(\log_2 S + \log_2 1/\epsilon)$ ,  $m = \binom{n}{0(D^d)}$ ,  $\delta = (\epsilon/2m)^{1/2}$

1) set  $\approx 1/d^2 \log_2(m/\epsilon)$  random samples of form  $(\vec{x}, f(\vec{x}))$

For each monomial  $T$ ,  $|T| \leq O(D^d)$ , use samples to approx

~~total~~  $\text{Prob} [F|_D \text{ is } \mathcal{O}(D^d) - \text{Prob} \neq] \approx \delta_T$

$$g = \sum_{|T| > O(D^d)} \prod_{i \in T} x_i$$

$$\sum_{|S| > O(D^d)} (\hat{d}_S - \gamma_S)^2 + \sum_{|S| > O(D^d)} (\hat{d}_S - \gamma_S)^2 \leq \sum_{|S| > O(D^d)} \delta^2 + 2^{-D} \leq \epsilon$$

since  $m \delta^2 + 2^{-D} \leq \epsilon$

$$\frac{1}{2^n} \sum_{\vec{x}} (f(\vec{x}) - g(\vec{x}))^2 \geq 1, \quad \text{Prob}_x [f(x) \neq h(x)] \leq \epsilon \text{ where}$$

$$x, h(x) \neq f(x) \quad c(\log_2 S/\epsilon)^d$$

$$\text{Time} = \text{poly}(n/\epsilon) = (n/\epsilon)^{c(\log_2 S/\epsilon)^d}$$

Open & proper reading.

Next topic: representation of matrices as low-degree over finite field

$\mathbb{R}^2$ :  $\theta_1$ , every point can be written as  $f(x) = \sum_S \beta_S \prod_{i \in S} x_i$

$\mathbb{R}^2$ :  $f \in \mathbb{R}_1$  or  $f|_{x_i=0} + (1+x_i)f|_{x_i=1}$

$A \subset \mathbb{R}^2$ : depth  $d$  constant circ w/  $1, x_i$  &  $\mathbb{O}$  nodes

$\alpha_i, 1 \leq i \leq n = \prod_{i=1}^n \alpha_i$ ; no small deg. DR, as well.  
 - no small Fourier coeffs, since partition.

$$\oplus \alpha_i = \sum \alpha_i$$

A probability distribution over polynomials

$$\text{Approx } \# \text{ vs } \text{Prob} [f(x) \neq \text{PGI}] \leq \epsilon$$

$\alpha_i, 1 \leq i \leq n$ . Let  $\epsilon = 1/2$ . Pick a subset of  $\alpha_i, 1 \leq i \leq n$  at random, then  $\sum_{i \in S} \alpha_i$

9/9/2015 Ch. 12, "PPAD"

NP: given  $f, f: [n] \rightarrow [n] \exists ? \bar{x} \in [n]^k \Phi(\bar{x}, \bar{y})$

TFNP: complex problems?  $\exists \bar{x} \Phi(\bar{x}, \bar{y})$  is valid for all  $\bar{y}$ .

Idea: express problems in TFNP in terms of proof strategies (combinatorial lemmas). 5 such lemmas known.

1) Pigeonhole principle: given  $f: [n] \rightarrow [n]$ , find  $x, f(x) = 0$  or  $f(x) = f(y), x \neq y$  PPD

2) Parity: given odd deg. hypergraph and coloring PPA

3) PPA  $\exists$ : given  $v_i, v_j$  and  $u$  to be reached node, find  $x$  such that  $\Phi(x, u)$  PPA

4) PPADS;

5) PLS: given dag w/ nodes solved, find sink PPAD

6) CLS:  $\text{PLS} \cup \text{continuous optimization}$  CLS

2) Combinatorial nullstellensatz?

Bertrand postulate?

LLL?

unique solutions?

Ramsey? (given  $G$ , find chromatic number)

is not hard complete problems for PPA  $\exists$ , PLS (not meta-problems)

PPP: Minkowski's theorem

PPA: Smith (if of homogeneous system every edge), Chevalley mod 2