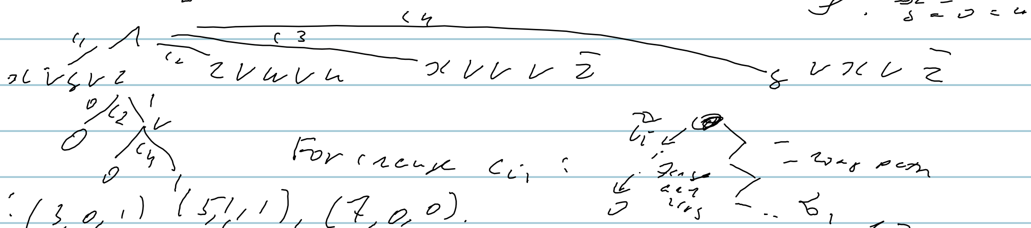
Russell's course, lec 2.

Switching lemma:

Let φ be a k-CNF fn. Pick rand rest ρ leaving pn vars unset. $\exists c>0$ so that $P[CDT[\varphi|_\rho] \ge D] \le (cpk)^D$

Proof idea: we'll code each ρ with $CDT[\varphi|_\rho] \ge D$ by a rest. ρ_{code} leaving $pn-D$ vars unset and a hint, a short message of length $D\lceil \log k + 2\rceil$ bits. There are relatively few (ρ_{code}, hint) pairs possible, so few ρ's have $CDT[\varphi|_\rho] \ge D$



h_1: (3,0,1) (5,1,1), (7,0,0).

x set to 0 in more ... i'th clause. $(i, b_1, b_2) \in [k]$, 6.75 requires 2 bits

On b_1, long path. in clause C_{i_1}, then same for C_{i_2}

h_2: same for second unset clause C_{i_2}. Visit $D' \le D$ clauses.

$\rho_{code} = \rho \circ t_1 \circ \dots \circ t_{d'}$ has $pn - D$ vars unset. $|hint| = D(\log k + 2)$

decoding: look at $\varphi|_{\rho_{code}}$. Let C_{i_1} be first clause falsified. Look at hint to see what vars in C_{i_1} were unset, and how they were set in b_1. Temporarily set them to values in b_1. Repeat until D vars unset have been recovered

ρ_{random}: ρ set D vars in random, h_r: random sequence of same size, $n - pn + D$ bits, $n - D$ of them are "impostors". Prob. of guessing ~~each one~~ them: $\frac{D}{n-pn+D} \cdot \frac{D-1}{n-pn+D-1} \cdots \le \left(\frac{D}{n-pn}\right)^D$

Given ρ, there are $\binom{pn}{D} \cdot 2^D$ ρ_{random}: prob of ρ_{rand} correct 1/this

D prob. that hint correct: $(1/4k)^D$

$\binom{D}{pn}$ prob[decode[ρ_{rand}, h_r] $= \rho$] $\ge$ Prob[$CDT[\varphi|_\rho] \ge D$] $\cdot$ Prob[$\rho_r = \rho_c$] $\cdot$ prob[$h_r = h$] $=$ Prob[$CDT[\varphi|_\rho] \ge D$] $\cdot 1/\left(\binom{pn}{D} \cdot 2^D \cdot (4k)^D\right) \le$

$\le \left(\frac{epn}{D}\right)^D \cdot 2^D \cdot (4k)^d \cdot D^D / (n-pn)^D = (8epk/(1-p))^D = (cpk)^D$

Applications of switching lemma:

1) lb on AC^0 circs for parity

Lemma: let $\mathcal{C}$ be depth d circ using $\le$ gates at top $d-1$ levels $\le S$, and fan-in at bottom $\le \log S$

If $\mathcal{C}$ computes parity of n vars, $S \geq 2^{\Omega(n^{1/(d-1)})}$

Proof: ind on d. Base: $d=2$: last class exercise

Ind. step: Restr. $\mathcal{C}$ by ρ that leaves small const $c'n/\log S$ vars unset. $\mathcal{C}$ has $\le S$ CNF/DNF that are $\log S$-CNF. Prob that any of these $\mathcal{C}_i$ has $P[CDT[\mathcal{C}_i|_\rho] \geq \log S] \le \left(\frac{c \cdot c'}{\log S} \cdot \log S\right)^{\log S} < \frac{1}{S}$

So $\exists \rho$ s.t. $\forall \mathcal{C}_i$, $CDT[\mathcal{C}_i|_\rho] \le \log S$

So replacing each CDT w/ a DNF and breaking at next higher level set $\mathcal{C}|_\rho$ has $\le S$ gates at levels $d-1..1$, and $\log S$ fan-in at bot

This should compute parity of pn vars. Apply induction w/ $d-1$ and $n' = cn/\log S$. So $n \le (\log S)^{d-1}$, $n' \le c(\log S)^{d-2}$

$n \le O((\log S)^{d-1})$

2) k-CNF counting: count # of x s.t. $\mathcal{C}(x)=1$, given k-CNF $\mathcal{C}$.

Alg: partition vars into groups $|X| = (1 - c'/k)n$, $|Y| = \frac{c'n}{k}$

For each setting of vars in X, look at corresp. rest. ρ.

Create $CDT[\mathcal{C}|_\rho]$ and count # of sols.

Exp time of the alg: $\sum_{\substack{\text{sets }\rho \\ \text{to } X}} O(|CDT(\mathcal{C}|_\rho)|) \le 2^{(n-pn)}\left(E_{\rho}\, CDT[\mathcal{C}|_\rho]\right)$

$\le \sum 2^{\ell}\, Prob[CDT(\mathcal{C}|_\rho) \geq \ell] \le O(2^{n-pn})$

here $p \approx \frac{c'}{k}$

Next class: learning from rand samples.