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How much can we save over exhaustive search?

Generic problem:  $n$ : # of vars,  $\Sigma$ : set of possible values,  $m$ : # of clauses

Exhaustive search:  $|\Sigma|^n \cdot \text{poly}(n)$

Ex:  $\Sigma$ :  $\{0,1\}$ ,  $n$  bool vars, 2 values,  $\{0,1\}$   $m$  clauses,  $2^n \cdot \text{poly}(n)$

3-col:  $n = |V|$ ,  $\Sigma = \{R, G, B\}$ ,  $m = |E|$   $3^{|V|} \cdot |E|$

Ind. set:  $n = |V|$ ,  $\Sigma = \{\text{in ind. set}, \text{out}\}$ ,  $m = |E|$ ,  $2^{|V|} \cdot |E|$

An alg is subexponential if it is in time  $2^{o(n)} \cdot \text{poly}(n)$

Moderate improvement:  $|\Sigma|^{(1-d)n} \cdot \text{poly}(n)$ ,  $d > 0$ .

Non-trivial:  $|\Sigma|^n \cdot \text{poly}(m) / n^{o(1)}$

Most reductions:  $(n, m) \rightarrow (n', m')$ , unless  $n' = O(n + m)$ ,  $m' = O(n + m)$

Given  $m$  small, or  $n' = f(n)$ ,  $n' = O(n)$

$m = O(n)$ : Sparsification.

Strongly subexp:  $2^{n^{o(1)}}$  - standard reductions work

Sparsification & trade  $n$  vs  $m$  time or describing parameters

Schoning's alg for  $k$ -SAT.

almost Schoning's alg: Sch A: Guess assignment  $\mathcal{A}$  or random.

(Note that  $\mathcal{A}$  is close to a SAT assign)

Let  $d = \text{Ham}(\mathcal{A}, \mathcal{C}_i) = \#\{j: \mathcal{A}_j \neq \mathcal{C}_{i,j}\}$

until  $\exists$  unsat clause, or until  $\mathcal{A}$  is  $\mathcal{A}^*$  (in time  $2^{o(n)}$ ) not in original Schoning's alg.

Pick such clause  $(\ell_1, \dots, \ell_d)$ : in  $\mathcal{A}$ , all  $\ell_i = 0$

pick  $i$  at random, and flip the bit in  $\mathcal{A}$  corresponding to  $\mathcal{A}_i$ . quit if fixed.

If  $\mathcal{A}$  gets to SAT assign, great. If fixed vars force a clause to unsat, restart.

worst case 1 uniform  $\mathcal{C}_i$ , and each clause just has one positive var.

Let  $d_t = \text{Ham}(\mathcal{A}_t, \mathcal{C}_i)$   $i$  dist  $\mathcal{A}$  at time step  $t$ . w/ prob  $\geq 1/k$ ,  $d_{t+1} = d_t - 1$

$$\text{Prob}(\text{recur } \mathcal{C}_i \text{ or some other } \mathcal{C}_i \text{ exists from } \mathcal{A}_0) \leq \sum_{d=0}^n \binom{n}{d} 2^{-n} \cdot (1/k)^d = 2^{-n} \sum_{d=0}^n \binom{n}{d} k^{-d} = 2^{-n} (1 + 1/k)^n = 2^{-n} \left(\frac{k+1}{k}\right)^n$$

- gives moderate improvement for each fixed  $\mathcal{C}_i$ . Exp. time is

$$2^n \cdot \left(\frac{k}{k+1}\right)^n = 2^n \left(1 - \frac{1}{k+1}\right)^n = 2^{n(1 - O(1/k))}$$

Sch B (actual Schoning's alg): w/o fixing, repeat main loop  $\leq n$  times

how prob.  $\frac{k-1}{k}$  of  $\mathcal{A}$  increasing  $d$  by 1.

$$\binom{n}{d} \cdot 2^{-n} \sum_{i=0}^{d-1} \binom{d+i-1}{i} \cdot \left(\frac{1}{k}\right)^{d-i} \left(\frac{k-1}{k}\right)^i, \text{ where } i: \# \text{ of steps in wrong direction.}$$

$$= 2^n \left(1 - \frac{1}{k}\right)^n \text{ time } (k \text{ instead of } k+1)$$

Between this and PPSZ, best exact alg for  $k$ -SAT (worst case).

For ind. set, a different approach: backtracking in this next prog each case is

Tarjan & Trojanovski:

$TT(G)$  "T(G,  $N_1, \dots, N_i, a_i$ )"

If  $V = \emptyset$ , return  $\emptyset$ . If  $|E| = \emptyset$ , return  $V$ .

Pick  $x \in V$  "candidate"

$$S_1 = \{x\} \cup TT(G - \{x\} - N(x))$$

$$S_0 = TT(G - \{x\})$$

Return  $S_1$  unless  $S_0$  is larger; if  $S_0$ , return  $S_0$

works well on high-degree graphs. On a chain, either choose 1 or 2 nodes

for  $S_0$  to be larger, need to pick at least two neighbors of  $x$

$N_1, \dots, N_i$ : # of nodes to pick from each set (??)

- with picking carefully ~~enumeration~~  $\downarrow^{n^{13}}$ .

Robson, w/ memoization & other tricks, helps improve  $2^{2n}$  to  $2^{2\sqrt{n}}$

Sparsification Johnson & Stepanov  $|E| \leq O(|V|)$ .

$TTS(G)$ :

if  $G$  is sparse, return  $G$

pick  $x \in V$  "candidate" of degree  $\geq 2d$

$$G_1 = \{x\} \cup TTS(G - \{x\} - N(x))$$

$$G_0 = TTS(G - \{x\})$$

Return  $\{G_1\} \cup \{G_0\}$ . // max ind set  $\in G$  is max ind set in  $G_0$  or  $G_1$

$$\begin{aligned} \text{if } |E| \leq \epsilon \text{ degrees: } T(n) &= T(n - 2d - 1) + T(n - 1) \leq \binom{n}{n - 2d - 1} = \\ & \leq 2^n \frac{2^{2d+1}}{2^{2d+1}} \end{aligned}$$

So  $\forall \epsilon \exists d$  s.t. can reduce general ind. set to  $d$ -sparse  $d$ -ind set in  $2^{2n}$  time

~ map one instance to many instances, but each instance is sparse.

Cor If  $\exists$  S s.t. time  $T_{MS} \geq 2^{2n}$ , then  $\exists d$  s.t.  $T_{\text{sparse}}(n) \geq 2^{2n}$

For  $k$ -SAT: same kind of sparsification equivalent to CIM.

[IPZ] Sparsification lemma for  $k$ -SAT:  $\forall k, \epsilon \exists D = O(k/\epsilon)^k$  s.t. every

[IPZ] formula  $\mathcal{U}$  can be written as  $\mathcal{U} = \bigvee_{i=1}^{2^{2n}} \mathcal{U}_i$  where each  $\mathcal{U}_i$  is a  $k$ -CNF with some vars and  $\leq Dn$  clauses. (in time  $2^{2n}$ ).

(so  $\mathcal{U}$  is SAT  $\Leftrightarrow \mathcal{U}_i$  is SAT)