There are 3 questions for a total of 10 points.

1. Consider the network shown in the figure. Consider running the Ford-Fulkerson algorithm on this network.

(a) \( \frac{1}{2} \) point We start with a zero \( s-t \) flow \( f \). The algorithm then finds an augmenting path in \( G_f \). Suppose the augmenting path is \( s \rightarrow b \rightarrow c \rightarrow t \). Give the flow \( f' \) after augmenting flow along this path.

(b) \( \frac{1}{2} \) point Show the graph \( G_{f'} \). That is, the residual graph with respect to \( s-t \) flow \( f' \).

(c) (1 point) The algorithm then sets \( f \) as \( f' \) and \( G_f \) as \( G_{f'} \) and repeats. Suppose the augmenting path chosen in the next iteration of the while loop is \( s \rightarrow a \rightarrow d \rightarrow t \). Give \( f' \) after augmenting flow along this path and show \( G_{f'} \).

(d) (1 point) Let \( f \) be the flow when the algorithm terminates. Give the flow \( f \) and draw the residual graph \( G_f \).

(e) (1 point) Give the value of the flow \( f \) when the algorithm terminates. Let \( A^* \) be the vertices reachable (using edges of positive weight) from \( s \) in \( G_f \) and let \( B^* \) be the remaining vertices. Give \( A^* \) and \( B^* \). Give the capacity of the cut \( (A^*, B^*) \).

2. Consider the network shown below. Consider running the Ford-Fulkerson algorithm on this network. Suppose you use the DFS algorithm to find augmenting paths.

(a) (1 point) What is the number of times the while loop runs in the worst case? Give a short description of this scenario.

(b) (1 point) What is the number of times the while loop runs in the best case? Give a short description of this scenario.

3. Suppose you are given a bipartite graph \( (L, R, E) \), where \( L \) denotes the vertices on the left, \( R \) denotes the vertices on the right and \( E \) denote the set of edges. Furthermore it is given that degree of every vertex is exactly \( d \). We will construct a flow network \( G \) using this bipartite graph in the following manner: \( G \) has \( |L| + |R| + 2 \) vertices. There is a vertex corresponding to every vertex in \( L \) and \( R \). There is also a source vertex \( s \) and a sink vertex \( t \). There are directed edges with weight 1 from \( s \) to all vertices in \( L \)
and directed edges of weight 1 from all vertices in $R$ to $t$. For each edge $(u, v) \in E$, there is a directed edge from $u$ to $v$ with weight 1 in $G$.

(The figure below shows an example of a bipartite graph and the construction of the network.)

![Figure 1: An example bipartite graph (with $d = 2$) and network construction.](image)

(a) (1 point) Argue that for any such bipartite graph where the degree of every vertex is equal to $d$, $|L|$ is equal to $|R|$.

(b) (3 points) Argue that for any given bipartite graph where the degree of every vertex is the same, there is an integer $s$-$t$ flow (i.e., flow along any edge is an integer) in the corresponding network with value $|L| = |R|$.