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UNIVERSAL TRAVERSAL SEQUENCES OF LENGTH $n^{O(\log n)}$ FOR CLIQUES

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1. Introduction

The reachability problem for graphs is a key problem in understanding the power of various logarithmic space complexity classes. For example, the reachability problem for directed graphs is logspace-complete for the complexity class NSPACE(log n) [5] and hence the open question DSPACE(log n) = NSPACE(log n) can be settled by answering whether this reachability problem belongs to DSPACE(log n). On the other hand, the reachability problem for undirected graphs seems to be somewhat easier. Aleliunas et al. [1] proved that the reachability problem for undirected graphs can be solved probabilistically in O(log n) space and polynomial time simultaneously; their proof implied the existence of short *universal traversal sequences* for regular undirected graphs. Their result was recently improved to zero error, O(log n) space and polynomial expected time by Borodin et al. [3]. However, it is an open question whether the reachability problem for undirected graphs is in DSPACE(log n).

One method of finding a DSPACE(log n) algorithm is to construct universal traversal sequences in logspace for regular undirected graphs. These sequences in turn can be used to construct universal traversal sequences for the class of *all* undirected graphs, regular or not (see [1] and [2, Lemma 15]). Because of the lack of explicit constructions of universal traversal sequences for the class of undirected graphs, it is interesting to construct universal sequences for special classes of undirected graphs.

We shall first define universal traversal sequences.

1.1. Definition. A d-regular n-node graph is *labeled* if:

(1) its n nodes have distinct labels $1, 2, \ldots, n$;

(2) for each node, the edges incident upon it are labeled with $1, 2, \ldots, d$ such that these labels form a permutation of $\{1, 2, \ldots, d\}$. Each edge has two labels, one corresponding to each endpoint. The two labels on each edge are independent.

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1.2. Definition. Let $D = \{1, 2, \ldots, d\}$. Each element x of D can be interpreted as an instruction to move from the current node u along the edge incident upon u whose label is x. With this interpretation, given a labeled d-regular graph and a starting node, each $w \in D^*$ visits a sequence of nodes. A string w is a *universal traversal sequence* for d-regular, n-node undirected graphs (or simply a *d,n universal sequence*) if, for all d-regular, n-node connected labeled graphs G, and for all nodes v in G, w visits every node of G, if started at v.

Aleliunas et al. have shown that a random string of length $O(d^2n^3 \log n)$ is universal with high probability. It is an open problem to find logspace constructible (and hence only polynomially long) universal sequences.

Bar-Noy et al. [2] and Bridgland [4] considered the explicit construction of universal sequences. They were able to construct $n^{O(\log n)}$-long universal sequences for 2-regular graphs in $\log^2 n$ space. More recently, Istrail [6] exhibited polynomial-length logspace-constructible universal sequences for 2-regular graphs.

Here we study the problem of finding universal sequences for $(n-1)$–regular graphs (cliques), a problem suggested in [2]. Although the connectivity problem for cliques is trivial, the construction of universal sequences is not. Bar-Noy et al. were able to construct only universal sequences of length $n^{\Theta(n^2)}$ for n-node cliques. We exhibit universal sequences of length $n^{O(\log n)}$ for cliques; they can be constructed uniformly in $\log^2 n$ space.

2. The construction

First, we state and prove a few lemmas. Next, we give the construction and prove that it works.

Let $n \geqslant 3$ and $d = n - 1$. Let us use *labeled clique* as shorthand for 'n-node labeled clique' and *string* as shorthand for an element of D^*. Let V be the set of nodes of a labeled graph.

2.1. Definition. A set T of strings has property P if for every labeled clique G there is a string $w \in T$ such that w visits node 1 when started at any node in G.

2.2. Definition. A set T of strings has property Q iff for every labeled clique G, for all $v \in V$, there is a string $w_v \in T$ such that w_v visits node v independent of its starting node.

2.3. Lemma. *P is equivalent to Q.*

Proof. The idea is that the traversal of a graph is only dependent on the edge labels and oblivious of the node labels. Hence, properties P and Q are equivalent. To see this, assume that a set T of strings has property P. Let G be any labeled clique and v any node of it. Let G' be the labeled clique obtained from G by interchanging the labels of the nodes 1 and v. Since T has property P, there is a string $w \in T$ such that w visits node 1 in G' from any starting node. It is obvious that w, when started at any node, well visit node v in the labeled graph G. String w is the desired w_v. □

2.4. Lemma. *Suppose T is a set of strings satisfying property P. Then, the concatenation x of all the strings in T (in any order) is universal for cliques.*

Proof. T has property P and hence Q as well. Choose a labeled clique G. Let $v \in V$ be arbitrary. Since T has property Q, there exists a string w_v which visits v in G from any starting node. Since x contains w_v as a substring, x also visits node v. Hence, x is universal. □

Let w be a string and let G be a labeled clique. Let R_w^G be the set of 'good starting nodes': R_w^G is the set of nodes v such that w visits node 1 in G if started at node v.

We now construct a sequence T_k of sets of strings such that $\max_{w \in T_k} |R_w^G|$ increases with increasing k.

Let $T_0 = \{\varepsilon\}$, where ε is the empty string.
For $j \geqslant 1$, let $T_j = \{utu \mid u \in T_{j-1},\ t \in D\}$.
Let $K = \lceil 8 \log n \rceil$.

Let z be a concatenation of all the strings in T_K (in any order). We shall now prove that T_K satisfies property P and the length of the string z is

$n^{O(\log n)}$. From Lemma 2.4 we get that z is a universal traversal sequence.

For each $k \geqslant 0$ and labeled graph G, let S_k^G be a largest set in $\{R_w^G \mid w \in T_k\}$. Let $s_k^G = |S_k^G|$.

The following lemma measures the advantage of T_{k+1} over T_k.

2.5. Lemma. *For all $k \geqslant 0$ and for all G,*

$$s_{k+1}^G \geqslant s_k^G + \frac{s_k^G - 1}{n-1}\left(n - s_k^G\right). \tag{1}$$

Proof. Let $k \geqslant 0$ be arbitrary and let G be a labeled clique. Choose $w = w_k^G \in T_k$ such that $R_w^G = S_k^G$. If w_k^G is started at a node which is not in S_k^G, it will end up at some node v. The idea is that if $l \in D$ takes us from v to some node in S_k^G, then $w_k^G l w_k^G$ visits node 1 of G. We shall show below that we can select l in such a way as to satisfy (1).

Consider the strings $w_k^G l$ for $l \in D$. For each of the $n - s_k^G$ nodes v in $V - S_k^G$, at least $s_k^G - 1$ of the strings $w_k^G l$ will be such that they will end up in the set S_k^G when started at node v, since G is a complete graph. Hence, by an averaging argument, there exists a label l_k^G and at least $[(s_k^G - 1)/(n-1)](n - s_k^G)$ nodes of $V - S_k^G$ such that $w_k^G l_k^G$, started at any of these nodes, will end at one of the nodes in S_k^G. Therefore, $w_k^G l_k^G w_k^G$, if started at any of those nodes—or at a node of S_k^G—will visit node 1. Since T_{k+1} is defined to include all the strings $w_k^G l_k^G w_k^G$, (1) is satisfied and the proof of the lemma is complete. □

2.6. Theorem. *String z, a concatenation of all the strings in T_K, is universal for n-node cliques. Its length is $n^{O(\log n)}$.*

Proof. It is easy to see that the length of the strings in T_j is $2^j - 1$ and that $|T_j| = d^j$. Consequently, the length of z is $(2^K - 1)(n-1)^K = n^{O(\log n)}$. Universality of z will follow from Lemma 2.4 if we can prove $S_K^G = V$, for all G. We have that $S_1^G \geqslant 2$ and $S_2^G \geqslant 3$. Moreover, from Lemma 2.5 it follows that

$$3 \leqslant s_k^G \leqslant \tfrac{1}{2}(n+1) \;\Rightarrow\; s_{k+1}^G \geqslant \tfrac{7}{6} s_k^G$$

and

$$s_k^G \geqslant \tfrac{1}{2}(n+1) \;\Rightarrow\; s_{k+1}^G \geqslant s_k^G + \tfrac{1}{2}\left(n - s_k^G\right).$$

These facts imply that, for $K = \lceil 8 \log_2 n \rceil$, $s_K^G = n$. (In fact, the number of steps to finish up once $s_k^G \geqslant \frac{1}{2}n$ is O(log log n), not just O(log n).) Consequently, for all G, $S_K^G = V$ and the concatenation of all strings in T_K is universal. □

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