Lecture 6: Epipolar Geometry
Recap
Structure from Motion

Several images observe a scene from different viewpoints
Feature detection

Detect features using, for example, SIFT [Lowe, IJCV 2004]
Feature matching

Match features between each pair of images
Feature matching
Feature matching

Refine matching using RANSAC to estimate fundamental matrix between each pair
Fundamental Matrix

\[ x_1 \leftrightarrow x_2 \]

\[ x_1^T F x_2 = 0 \]
8-point algorithm

Given $n$ point correspondences, set up a system of equations:

$$
\begin{bmatrix}
    u_1 u'_1 & v_1 u'_1 & u'_1 & u_1 v'_1 & v_1 v'_1 & v'_1 & u_1 & v_1 & 1 \\
    u_2 u'_2 & v_2 u'_2 & u'_2 & u_2 v'_2 & v_2 v'_2 & v'_2 & u_2 & v_2 & 1 \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
    u_n u'_n & v_n u'_n & u'_n & u_n v'_n & v_n v'_n & v'_n & u_n & v_n & 1
\end{bmatrix}
\begin{bmatrix}
    f_{11} \\
    f_{12} \\
    f_{13} \\
    f_{21} \\
    f_{22} \\
    f_{23} \\
    f_{31} \\
    f_{32} \\
    f_{33}
\end{bmatrix} = 0
$$

- In reality, instead of solving $Af = 0$, we seek $f$ to minimize $\|Af\|$.
8-point algorithm – Problem?

• \( \mathbf{F} \) should have rank 2
• To enforce that \( \mathbf{F} \) is of rank 2, \( \mathbf{F} \) is replaced by \( \mathbf{F}' \) that minimizes \( \| \mathbf{F}^\top \mathbf{F}' \| \) subject to the rank constraint.

• This is achieved by SVD. Let \( \mathbf{F} = \mathbf{U} \Sigma \mathbf{V} \), where

\[
\Sigma = \begin{pmatrix}
    s_1 & 0 & 0 \\
    0 & s_2 & 0 \\
    0 & 0 & s_3
\end{pmatrix}
\]

Let \( \Sigma' = \begin{pmatrix}
    s_1 & 0 & 0 \\
    0 & s_2 & 0 \\
    0 & 0 & 0
\end{pmatrix} \)

then \( \mathbf{F}' = \mathbf{U} \Sigma' \mathbf{V} \) is the solution.
RANSAC

1. Randomly choose $s$ samples (correspondences)
   – For fundamental matrix, what is the size of $s$?

2. Fit the model (fundamental matrix) to those samples
   – How would you fit the model to $s$ correspondences?

3. Count the number of inliers among all other correspondences
   – How can you determine which points are inliers?

4. Repeat $N$ times
   – How do you choose $N$?

5. Choose the model with the largest set of inliers
Homogeneous coordinates

- Converting to homogeneous coordinates
  
  \[
  (x, y) \Rightarrow \begin{bmatrix} x \\
  y \\
  1 \end{bmatrix} \quad \text{Homogeneous 2D point}
  \]

  \[
  (x, y, z) \Rightarrow \begin{bmatrix} x \\
  y \\
  z \\
  1 \end{bmatrix} \quad \text{Homogeneous 3D point}
  \]

- Converting from homogeneous coordinates
  
  \[
  \begin{bmatrix} x \\
  y \\
  w \end{bmatrix} \Rightarrow (x/w, y/w)
  \]

  \[
  \begin{bmatrix} x \\
  y \\
  z \\
  w \end{bmatrix} \Rightarrow (x/w, y/w, z/w)
  \]

- \((x, y, w)^T\) and \((kx, ky, kw)^T\) are the same point.
Homogeneous points and lines

- Consider a line in 2D image:
  \[ ax + by + c = 0 \]

- Condition for 2D point \((x, y, 1)^\top\) to lie on the line is:
  \[(a, b, c)^\top \cdot (x, y, 1)^\top = 0\]

- Equation of the line in homogeneous coordinates:
  \[ 1 = (a, b, c)^\top \]

- Condition for point \(\mathbf{x}\) to lie on line \(1\) is:
  \[ 1^\top \mathbf{x} = 0 \]

Note: by default, we will denote any vector as a column vector.
Homogeneous points and lines

• Consider two homogeneous 2D points \( x \) and \( x' \).
• What is the line passing through these two points?
• Consider the vector \( l = x \times x' \)
• We have:
  \[
  l^T x = 0 \quad \text{and} \quad l^T x' = 0
  \]
• Both \( x \) and \( x' \) lie on the line \( l \) and there is a unique line between two points!

• Consider two lines \( l \) and \( l' \) in the 2D image
• What is the point of intersection of these two lines?
  \[
  x = l \times l'
  \]
• Do you notice a duality?
We have:

- Define:
  \[ E = [t] \times R \]

Then, we have:

\[ q^T E p = 0 \]
Essential matrix constraint in pixel space: \((K_2^{-1}q')^\top E(K_1^{-1}p') = 0\).
Rearranging:
\[ q'^\top (K_2^{-\top} E K_1^{-1}) p' = 0 \]
Define: \(F = K_2^{-\top} E K_1^{-1}\)
Then, we have:\[ q'^\top F p' = 0 \]How many degrees of freedom does \(F\) have?
For corresponding points $x$ and $x'$, we have $x'^{T}Fx = 0$
Define $l' = Fx$, then we have $x'^{T}l' = 0$
Then, for point $x$, the line $Fx$ contains corresponding point $x'$
So, $l' = Fx$ is the epipolar line in the second image
Epipole in the second image

- For corresponding points $x$ and $x'$, we have $x'^{T}Fx = 0$
- We saw that $l' = Fx$ is the epipolar line in the second image
- **Epipole**: the point that lies on all epipolar lines $l'$ for any $x$
- Thus, for any $x$, we need point $e'$, such that $e'^{T}Fx = 0$
- Rewrite as $(F^{T}e')^{T}x = 0$
- So, epipole is given by $e' = \text{null}(F^{T})$
For corresponding points $x$ and $x'$, we have $x'^T F x = 0$

- Taking transpose, it is the same as $x^T F^T x' = 0$
- Define $l = F^T x'$, then we have $x^T l = 0$
- Then, for point $x'$, the line $F^T x'$ contains corresponding point $x$
- So, $l = F^T x'$ is the epipolar line in the first image
• For corresponding points $x$ and $x'$, we have $x'^T F x = 0$
• We saw that $l = F^T x'$ is the epipolar line in the second image
• **Epipole**: the point that lies on *all* epipolar lines $l$ for *any* $x'$
• Thus, for any $x'$, we need point $e$, such that $x'^T F e = 0$
• Group the elements as $x'^T (F e) = 0$
• So, epipole is given by $e = \text{null}(F)$
Properties of the fundamental matrix

- $Fx$ is the epipolar line associated with $x$
- $F^Tx'$ is the epipolar line associated with $x'$
- Epipoles given by $Fe = 0$ and $F^Te' = 0$
- $F$ is rank 2.
Results