Problem Set 2 Solutions

Problem 1. [30 points] Prove that the following language is neither r.e. nor co-r.e.:

\[ L = \{ \langle M_1, M_2 \rangle : M_1, M_2 \text{ are TMs and } M_1(\varepsilon) \text{ halts and } M_2(\varepsilon) \text{ loops} \} \].

The following shows that \( L \) is not co-r.e..

Claim 1: \( \text{BTH} \leq_m L \).

Proof: We need to specify a computable function \( f \) that takes as input \( \langle M \rangle \) and returns \( \langle M_1, M_2 \rangle \) such that:

1. If \( M(\varepsilon) \) halts then \( M_1(\varepsilon) \) halts and \( M_2(\varepsilon) \) loops
2. If \( M(\varepsilon) \) loops then either \( M_1(\varepsilon) \) loops or \( M_2(\varepsilon) \) halts.

Let \( N \) be a TM that loops on all inputs, for example the one whose code is simply “On input \( x \), loop”. Let \( f \) given \( \langle M \rangle \) output \( \langle M, N \rangle \). Checking that the two properties above are true is quite easy.

The following shows that \( L \) is not r.e..

Claim 2: \( \text{BTH} \leq_m \overline{L} \).

Proof: We need to specify a computable function \( f \) that takes as input \( \langle M \rangle \) and returns \( \langle M_1, M_2 \rangle \) such that:

1. If \( M(\varepsilon) \) halts then either \( M_1(\varepsilon) \) loops or \( M_2(\varepsilon) \) halts
2. If \( M(\varepsilon) \) loops then \( M_1(\varepsilon) \) halts and \( M_2(\varepsilon) \) loops.

Let \( N \) be a TM that halts on all inputs, for example the one whose code is simply “On input \( x \), accept”. Let \( f \) given \( \langle M \rangle \) output \( \langle N, M \rangle \). Checking that the two properties above are true is quite easy.

Problem 2. [30 points] Let \( A, B, D \) be languages. We say that \( D \) separates \( A \) from \( B \) if \( A \subseteq D \) and \( D \cap B = \emptyset \). We say that \( A, B \) are separable if there exists a decidable language \( D \) such that \( D \) separates \( A \) from \( B \). Now let

\[
A = \{ \langle M \rangle : M(\langle M \rangle) \text{ rejects} \}
\]

\[
B = \{ \langle M \rangle : M(\langle M \rangle) \text{ accepts} \}
\]

Show that \( A, B \) are not separable.
**Hint:** Assume to the contrary that they are separable by decidable language $D$ and derive a contradiction (thereby showing $D$ could not exist after all) using ideas similar to those in the proof of undecidability of the halting problem.

Assume towards a contradiction that there is a decidable $D$ that separates $A$ from $B$. This means:

1. There is a TM $M_D$ that decides $D$
2. $A \subseteq D$
3. $B \cap D = \emptyset$

To derive our contradiction, we consider the execution of $M_D$ on input $\langle M_D \rangle$. We know that $M_D(\langle M_D \rangle)$ either accepts or rejects (i.e. it does not loop) because $M_D$ decides a language and thus halts on all inputs. We now consider these two possibilities in turn:

- **$M_D(\langle M_D \rangle)$ accepts**
  \[
  \Rightarrow \langle M_D \rangle \in D \quad \text{(because } M_D \text{ decides } D )
  \]
  \[
  \Rightarrow \langle M_D \rangle \notin B \quad \text{(because } B \cap D = \emptyset )
  \]
  \[
  \Rightarrow M_D(\langle M_D \rangle) \text{ does not accept} \quad \text{(by def. of } B. )
  \]

- **$M_D(\langle M_D \rangle)$ rejects**
  \[
  \Rightarrow \langle M_D \rangle \notin D \quad \text{(because } M_D \text{ decides } D )
  \]
  \[
  \Rightarrow \langle M_D \rangle \notin A \quad \text{(because } A \subseteq D )
  \]
  \[
  \Rightarrow M_D(\langle M_D \rangle) \text{ does not reject} \quad \text{(by def. of } A. )
  \]