Quiz 1 Solutions

Problem 1 [16 points] Let

\[ A = \{ w \in \{0,1\}^* : \text{Either } w \text{ begins with a 0 and contains at least one 1,} \]
\[ \text{or } w \text{ ends with a 1 and contains at least one 0} \} \]

In the box below, write a regular expression describing the language \( A \):

\[ 0(0 \cup 1)^*1(0 \cup 1)^* \cup (0 \cup 1)^*0(0 \cup 1)^*1 \]

In the first case, \( w \) is a string beginning with a 0, followed by any string consisting of zero or more ones and zeros, followed by a 1, followed again by any string consisting of zero or more ones and zeros. In the second case, \( w \) is a string consisting of any string of zero or more ones and zeros, followed by a 0, followed by any string of zero or more ones and zeros, followed by a 1.

Problem 2 [30 points] Let

\[ A = \{ w \in \{0,1\}^* : w \text{ contains either } 111 \text{ or } 101 \text{ as a substring} \} \]

1. [15 points] Draw the state diagram of a DFA with \textit{at most five states} that recognizes \( A \).

2. [15 points] Draw the state diagram of a NFA with \textit{at most four states} that recognizes \( A \).
Problem 3 [32 points] If $w \in \{0,1\}^*$ is a string then $\text{flip}(w)$ is the string obtained by flipping each bit of $w$. (For example, $\text{flip}(01101) = 10010$). If $A \subseteq \{0,1\}^*$ is a language, we let

$$B = \{ w \in \{0,1\}^* : \text{flip}(w) \in A \}$$
$$C = \{ w \in A : \text{flip}(w) \not\in A \}.$$

Assuming $A$ is regular, prove the following:

1. [16 points] $B$ is regular.

   We use the template for proofs of closure properties that we have used many times in class.

   **Given:** $A$ is regular. So there is a DFA $M = (Q,\{0,1\},\delta,q_0,F)$ that accepts $A$.

   **Want:** To show that $B$ is regular. We will do this by constructing a DFA $N$ that recognizes $B$. (It would suffice to construct an NFA, but in this case it is just as easy to construct a DFA, so we do.)

   **Construction:** We simply flip the labels on the arrows of $M$, turning ones into zeros and zeros into ones. Formally, our DFA is $N = (Q,\{0,1\},\delta',q_0,F)$, meaning all components are the same as in $M$ except for the transition function. The new transition function is defined for all $q \in Q$ and all $\sigma \in \{0,1\}$ by

   $$\delta'(q,\sigma) = \begin{cases} 
   \delta(q,0) & \text{if } \sigma = 1 \\
   \delta(q,1) & \text{if } \sigma = 0 .
   \end{cases}$$

   **Correctness of construction:** $M$ goes from $q_0$ to a state $f$ on an input $w$ if and only if $N$ goes from $q_0$ to $f$ on input $\text{flip}(w)$. So $M$ accepts $w$ if and only if $N$ accepts $\text{flip}(w)$.

2. [16 points] $C$ is regular. (Hint: Use the fact that $B$ is regular. You may do so even if you did not prove this.)

   A string $w$ is in $C$ exactly when the following two conditions are both met: (1) $w$ is in $A$, and (2) $\text{flip}(w) \not\in A$, meaning $w$ is not in $B$. This means that $C = A \cap \overline{B}$. Now, we can show that $C$ is regular by using known closure properties of the class of regular languages:
<table>
<thead>
<tr>
<th>Claim</th>
<th>Justification</th>
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<tbody>
<tr>
<td>$B$ is regular</td>
<td>$B$ is regular, and we know that $L$ regular implies $\overline{L}$ regular for any language $L$.</td>
</tr>
<tr>
<td>$A$ is regular</td>
<td>by assumption</td>
</tr>
<tr>
<td>$A \cap B$ is regular</td>
<td>$A, \overline{B}$ are both regular, and we know that $L_1, L_2$ regular implies $L_1 \cap L_2$ regular for any languages $L_1, L_2$.</td>
</tr>
<tr>
<td>$C$ is regular</td>
<td>$C = A \cap \overline{B}$</td>
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**Problem 4 [22 points]** Recall that $|x|$ denotes the length of a string $x$, and let

$$A = \{ x0^{2|x|} : x \in \{0, 1\}^* \}.$$  

Prove that $A$ is not regular.

We use the usual template. The proof is by contradiction.

**Assume:** $A = \{ x0^{2|x|} : x \in \{0, 1\}^* \}$ is regular.

The assumption means that the pumping lemma (Theorem 1.37, page 78 of the text) applies to $A$. We imagine ourselves “interacting” with the lemma as follows:

We give it $A$, and it returns a pumping length $p$. Now, we choose a string $s \in A$ of length greater than $p$, and return it to the lemma. The choice of string is important for the rest of the argument, and we set it to $s = 1^p0^{2p}$. This is in $A$, because $w = 1^p$ is a string of length $p$, and thus $s$ is $w0^{2|w|}$. Because $s$ has a length greater than $p$, the pumping lemma says that $s$ can be split into $xyz$ which obey the three conditions of the pumping lemma. The lemma returns $x, y, z$ to us. We then choose $i = 2$ and return it to the lemma. At this point, the lemma guarantees that
(1) \( xy^2z \in A \)
(2) \( |y| > 0 \)
(3) \( |xy| \leq p \)

Because the first \( p \) symbols of \( s \) are all 1, we know by condition (3) above that \( x \) and \( y \) must contain only ones. Condition (2) states that \( y \) must have length greater than zero, so we know \( y \) contains at least one one. So there exist \( a, b, c \) such that \( x = 1^a \), \( y = 1^b \) and \( z = 1^{p-a-b}0^{2p} \) and \( b \geq 1 \) and \( a + b \leq p \). So

\[
xy^2z = 1^a1^b1^b1^{p-a-b}0^{2p} = 1^{a+2b+p-a-b}0^{2p} = 1^{p+b}0^{2p} .
\]

However, there is no string \( w \) such that \( 1^{p+b}0^{2p} = w0^{|w|} \), because the only choice for \( w \) would be \( 1^{p+b} \) which has length \( p+b \geq p+1 \). So the definition of \( A \) tells us that \( xy^2z \not\in A \). But condition (1) says \( xy^2z \in A \). They can’t both be true, so we have a contradiction. This means our assumption that \( A \) was regular is false.