3.1 Noisy-OR

Nodes: $X \in \{0, 1\}, \ Y \in \{0, 1\}, \ Z \in \{0, 1\}$

Noisy-OR CPT: $\ P(Z = 1|X, Y) = 1 - (1 - p_x)^X (1 - p_y)^Y$

Parameters: $p_x \in [0, 1], \ p_y \in [0, 1], \ p_y > p_x$

Suppose that the nodes in this network represent binary random variables and that the CPT for $P(Z|X, Y)$ is parameterized by a noisy-OR model, as shown above. Suppose also that

\[
0 < P(X = 1) < 1, \\
0 < P(Y = 1) < 1, \\
\]

while the parameters of the noisy-OR model satisfy:

\[
0 < p_x < p_y < 1. \\
\]

Consider the following pairs of probabilities. In each case, indicate whether the probability on the left is equal (=), greater than (>), or less than (<) the probability on the right.

(a) $P(Z = 1|X = 0, Y = 0)$ \hspace{1cm} $P(Z = 1|X = 0, Y = 1)$
(b) $P(Z = 1|X = 0, Y = 1)$ \hspace{1cm} $P(Z = 1|X = 1, Y = 0)$
(c) $P(Z = 1|X = 1, Y = 0)$ \hspace{1cm} $P(Z = 1|X = 1, Y = 1)$
(d) $P(X = 1)$ \hspace{1cm} $P(X = 1|Y = 1)$
(e) $P(X = 1)$ \hspace{1cm} $P(X = 1|Z = 1)$
(f) $P(X = 1|Y = 1, Z = 1)$ \hspace{1cm} $P(X = 1|Z = 1)$
(g) $P(X = 1, Y = 1, Z = 1)$ \hspace{1cm} $P(X = 1) P(Y = 1) P(Z = 1)$
3.2 Conditional independence

For the belief network shown below, indicate whether the following statements of conditional independence are true (T) or false (F).

\[
\begin{align*}
P(B|G, C) &= P(B|G) \\
P(F, G|D) &= P(F|D) P(G|D) \\
P(D|B, F, G) &= P(D|B, F, G, A) \\
P(A, C) &= P(A) P(C) \\
P(F, H) &= P(F) P(H) \\
P(F, C|G) &= P(F|G) P(C|G) \\
P(D, E|F, H) &= P(D|F) P(E|H) \\
P(D, E, G) &= P(D) P(E) P(G|D, E) \\
P(H|C) &= P(H|A, B, C, D, F) \\
P(H|A, C) &= P(H|A, C, G)
\end{align*}
\]
3.3 Subsets

For the belief network shown above, consider the following statements of conditional independence. Indicate the largest subset of nodes $S \subset \{A, B, C, D, E, F\}$ for which each statement is true. Note that one possible answer is the empty set $S = \emptyset$ or $S = \{\}$ (whichever notation you prefer). The first one is done as an example.

\[
\begin{align*}
P(A) &= P(A|S) \quad S = \{B, C, E\} \\
P(A|D) &= P(A|S) \\
P(A|B, D) &= P(A|S) \\
P(B|D, E) &= P(B|S) \\
P(D) &= P(D|S) \\
P(D|F) &= P(D|S) \\
P(D|E, F) &= P(D|S) \\
P(D|A, B) &= P(D|S) \\
P(F) &= P(F|S) \\
P(F|D) &= P(F|S) \\
P(F|D, E) &= P(F|S)
\end{align*}
\]
3.4 Node clustering

Consider the belief network shown below over binary variables $X$, $Y_1$, $Y_2$, $Y_3$, $Z_1$, and $Z_2$. The network can be transformed into a polytree by clustering the nodes $Y_1$, $Y_2$, and $Y_3$ into a single node $Y$. From the CPTs in the original belief network, fill in the missing elements of the CPTs for the polytree.

\[
\begin{array}{ccc}
X & P(Y_1 = 1|X) & P(Y_2 = 1|X) & P(Y_3 = 1|X) \\
0 & 0.9 & 0.8 & 0.3 \\
1 & 0.5 & 0.6 & 0.7 \\
\end{array}
\]

\[
\begin{array}{cccc}
Y_1 & Y_2 & Y_3 & P(Z_1 = 1|Y_1, Y_2, Y_3) & P(Z_2 = 1|Y_1, Y_2, Y_3) \\
0 & 0 & 0 & 0.1 & 0.8 \\
1 & 0 & 0 & 0.2 & 0.7 \\
0 & 1 & 0 & 0.3 & 0.6 \\
0 & 0 & 1 & 0.4 & 0.5 \\
1 & 1 & 0 & 0.5 & 0.4 \\
1 & 0 & 1 & 0.6 & 0.3 \\
0 & 1 & 1 & 0.7 & 0.2 \\
1 & 1 & 1 & 0.8 & 0.1 \\
\end{array}
\]

\[
\begin{array}{cccc}
Y_1 & Y_2 & Y_3 & Y & P(Y|X = 0) & P(Y|X = 1) & P(Z_1 = 1|Y) & P(Z_2 = 1|Y) \\
0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 \\
1 & 0 & 0 & 1 & 2 & 3 & 4 & 5 \\
0 & 1 & 0 & 1 & 2 & 3 & 4 & 5 \\
0 & 0 & 1 & 1 & 2 & 3 & 4 & 5 \\
1 & 1 & 0 & 1 & 2 & 3 & 4 & 5 \\
1 & 0 & 1 & 1 & 2 & 3 & 4 & 5 \\
0 & 1 & 1 & 1 & 2 & 3 & 4 & 5 \\
1 & 1 & 1 & 1 & 2 & 3 & 4 & 5 \\
\end{array}
\]