# Lecture 13

- Connectedness in graphs
- Spanning trees in graphs
- Finding a minimal spanning tree
- Time costs of graph problems and NP-completeness
- Finding a minimal spanning tree: Prim's and Kruskal's algorithms
- Intro to disjoint subsets and union/find

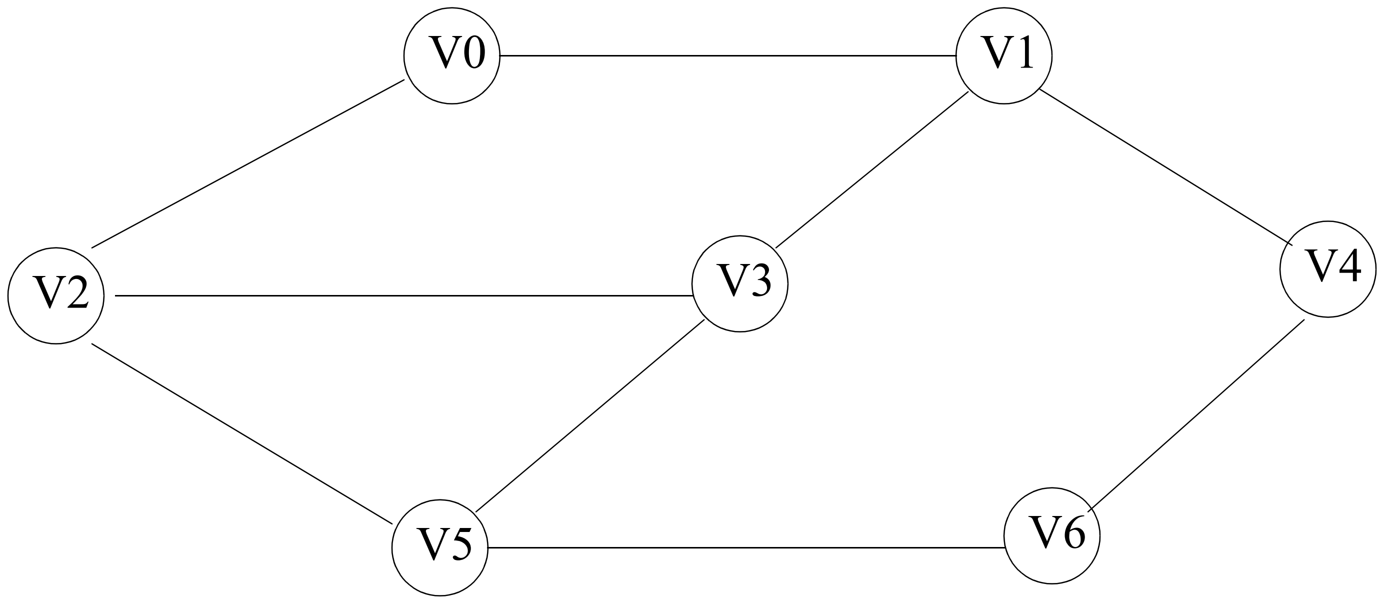
Reading: Weiss, Ch. 9, Ch 8

# Connectedness of graphs

- Some definitions:
- An undirected graph is *connected* if
  - For every vertex $v$ in the graph, there is a path from $v$ to every other vertex
- A directed graph is *strongly connected* if
  - For every vertex $v$ in the graph, there is a path from $v$ to every other vertex
- A directed graph is *weakly connected* if
  - The graph is not strongly connected, but the underlying undirected graph (i.e., considering all edges as undirected) is connected
- A graph is *completely connected* if for every pair of distinct vertices $v1$, $v2$, there is an edge from $v1$ to $v2$

# Connected graphs: an example

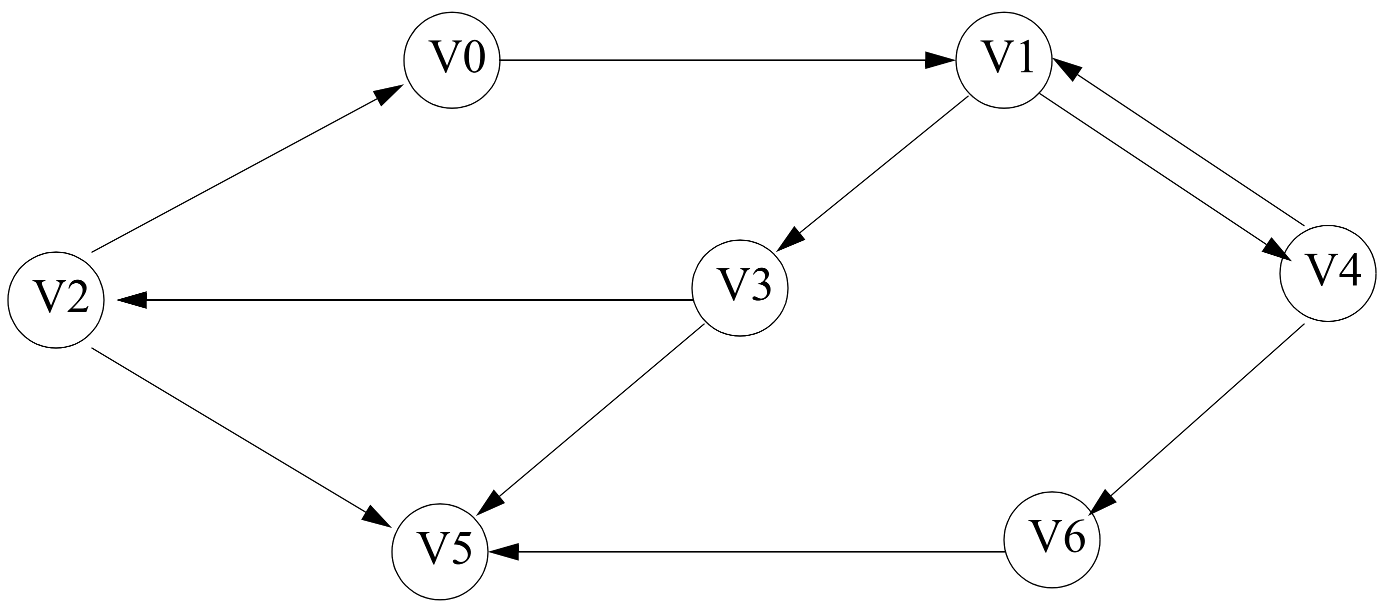
- Consider this undirected graph:




- Is it connected?
- Is it completely connected?

# Strongly/weakly connected graphs: an example

- Consider this directed graph:



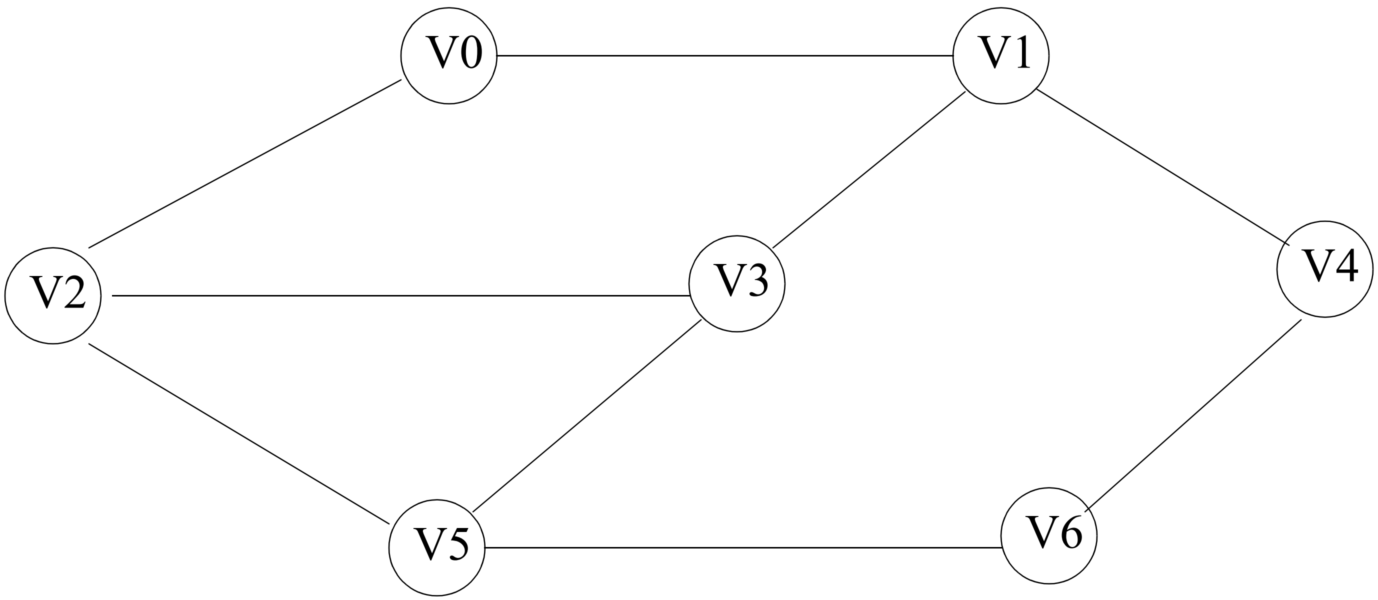

- Is it strongly connected?
- Is it weakly connected?
- Is it completely connected?

# Spanning trees

- We will consider spanning trees for *un*directed graphs
- A spanning tree of an undirected graph G is an undirected graph that...
  - contains all the vertices of G
  - contains only edges of G
  - has no cycles
  - is connected
- So, only connected graphs have spanning trees
- A spanning tree is called "spanning" because it connects all the graph's vertices
- A spanning tree is called a "tree" because it has no cycles (recall the definition of *cycle* for undirected graphs)
- What is the root of the spanning tree?
  - you could pick any vertex as the root; the vertices adjacent to that one are then the children of the root; etc.

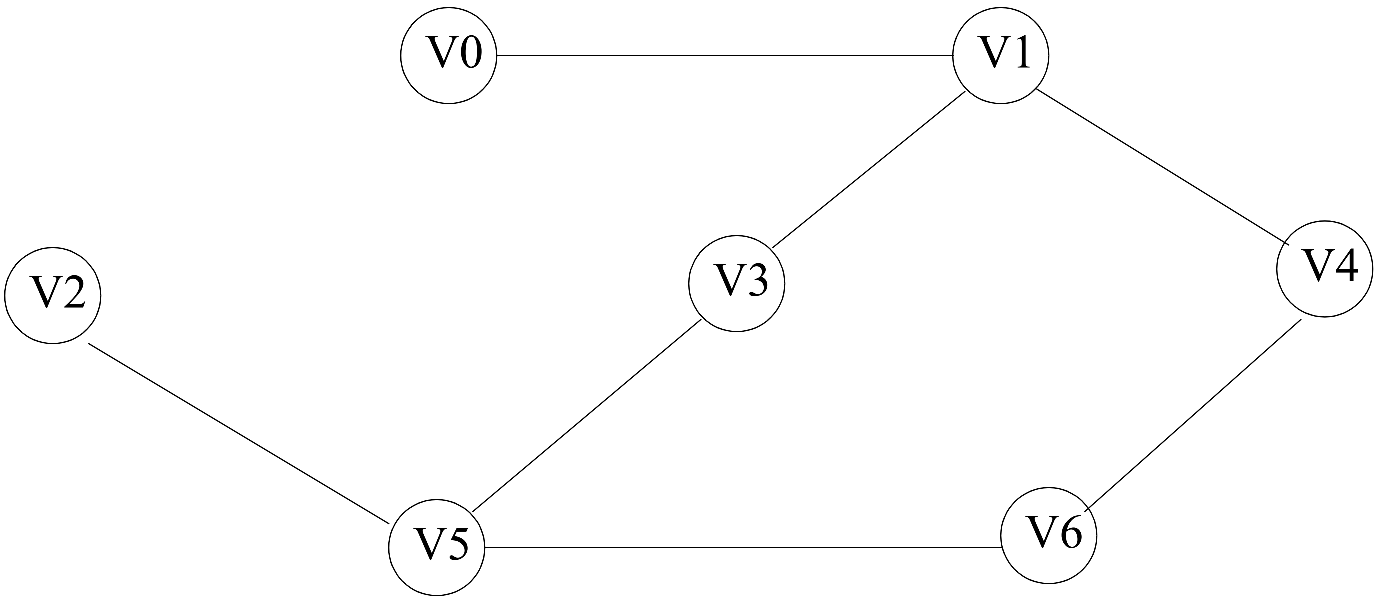
# Spanning trees:  examples

- Consider this undirected graph G:



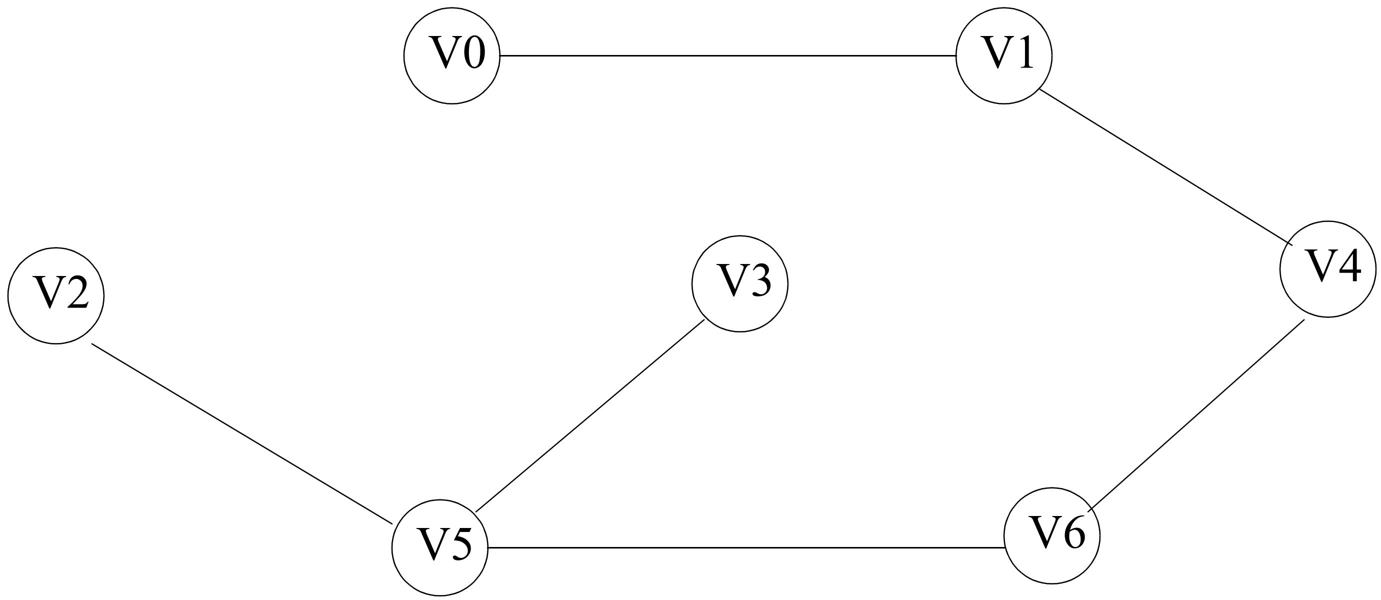
# Spanning tree?  Ex. 1

- Is this graph a spanning tree of G?



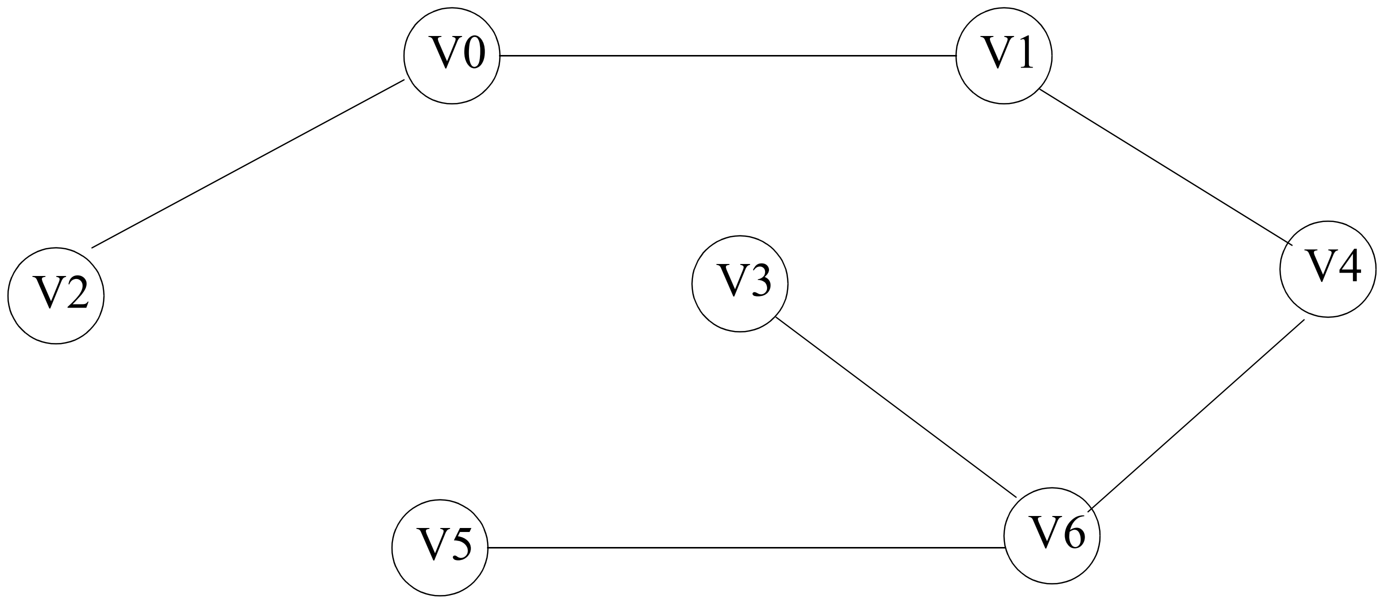
# Spanning tree? Ex. 2

- Is this graph a spanning tree of G?



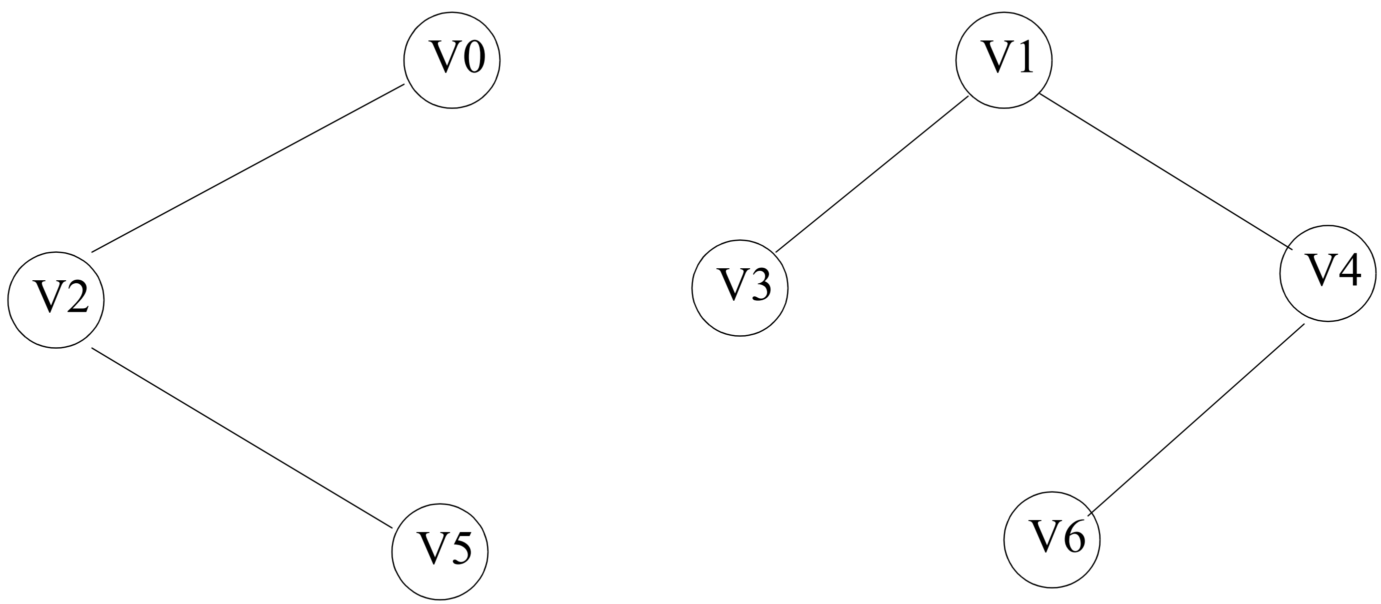
# Spanning tree?  Ex. 3

- Is this graph a spanning tree of G?



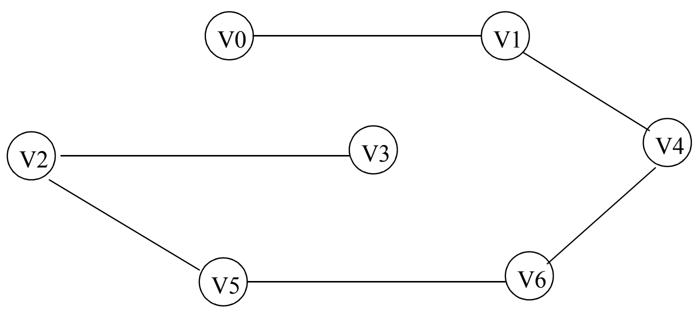
# Spanning tree? Ex. 4

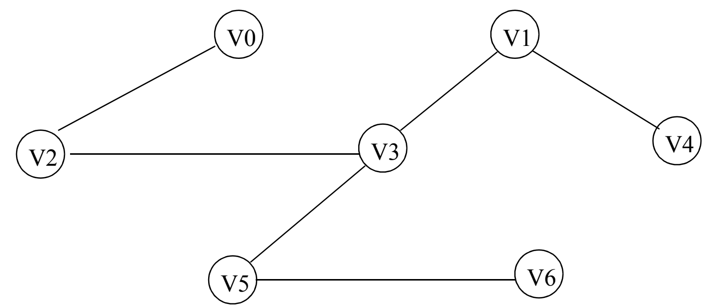
- Is this graph a spanning tree of G?

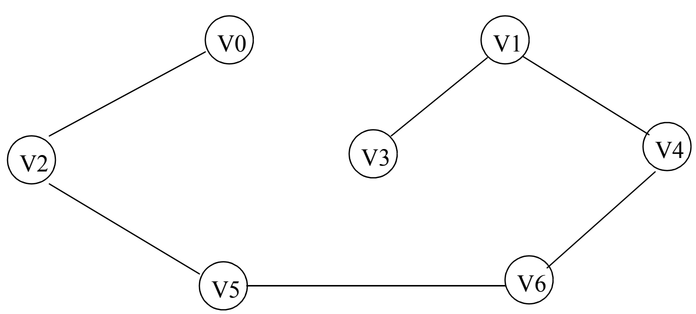


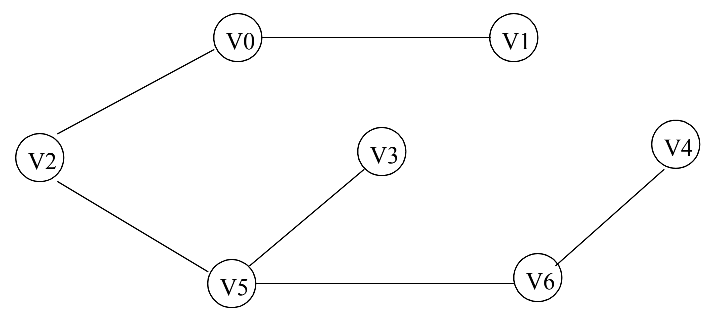
# Multiple spanning trees

- In general a graph can have more than one spanning tree.  All these are spanning trees of that graph G (and there are more):




- Note:  The spanning tree for a graph with N vertices always has N-1 edges (like a tree!)

# Finding a spanning tree in an unweighted graph

- A spanning tree in an unweighted graph is easy to construct...
- Use the basic unweighted single-source shortest-path algorithm (breadth-first search):
  - (That algorithm is designed for directed graphs. Convert an undirected graph to a directed one by treating each undirected edge as two antiparallel directed edges)
- Pick any vertex as the start vertex *s*. (Think of it as the root of the spanning tree.)
- When done, the "prev" indices in the table will give, for each vertex in the spanning tree, the index of its parent
- This represents a spanning tree because (if the graph is connected) each vertex except the start vertex will have exactly one parent, and each vertex appears in the table
- So, a spanning tree can be found in an unweighted graph in time $O(|V| + |E|)$
- ...What about weighted graphs?

# Minimum spanning trees in a weighted graph

- A single graph can have many different spanning trees
- They all must have the same *number* of edges, but if it is a weighted graph, they may differ in the *total weight* of their edges
- Of all spanning trees in a weighted graph, one with the least total weight is a *minimum* spanning tree (MST)
- It can be useful to find a minimum spanning tree for a graph: this is the least-cost version of the graph that is still connected, i.e. that has a path between every pair of vertices
- How to do it?

# Finding a minimum spanning tree: Prim's algorithm

- As you know, minimum weight paths from a start vertex can be found using Djikstra's algorithm
- At each stage, Djikstra's algorithm extends the best path from the start vertex (priority queue ordered by total path cost) by adding edges to it
- To build a minimum spanning tree, you can modify Djikstra's algorithm slightly to get Prim's algorithm
- At each stage, Prim's algorithm adds the edge that has the least cost from any vertex in the spanning tree being built so far (priority queue ordered by single edge cost)
- Like Djikstra's algorithm, Prim's algorithm has worst-case time cost O(|E| log |V|)
- We will look at another algorithm: Kruskal's algorithm, which also is a simple greedy algorithm
- Kruskal's has the same big-O worst case time cost as Prim's, but in practice it can be made to run faster than Prim's, if efficient supporting data structures are used

# A note about graph algorithm time costs

- So far we have mentioned these graph problems:
  - Find shortest path in unweighted graphs
    - Solved by basic breadth-first search: $O(|V|+|E|)$ worst case
  - Find shortest path in weighted graphs
    - Solved by Dijkstra's algorithm: $O(|E| \log|V|)$ worst case
  - Find minimum-cost spanning tree in weighted graphs
    - Solved by Prim's or Kruskal's algorithm: $O(|E| \log|V|)$ worst case

- The "greedy" algorithms used for solving these problems have polynomial time cost functions in the worst case
  - since $|E|<=|V|^2$, Dijkstra's, Prim's and Kruskal's algorithms are $O(|V|^3)$

- As a result, these problems can be solved in a reasonable amount of time, even for large graphs; they are considered to be 'tractable' problems

- However, many graph problems do not have any known polynomial time solutions...!

# Intractable graph problems

- For many interesting graph problems, the best known algorithms to solve them have exponential time costs $O(2^{|V|})$
- In the worst case, these intractable problems simply cannot be solved exactly, except for quite small graphs (say, 50 or at most 100 vertices, even on the world's fastest computers); the best known algorithms for these problems take too long to run
- For these problems, simple greedy best-first algorithms do not work... Essentially the best approach known for solving them exactly is basically to try all the possibilities, and there can be exponentially many possibilities to try
- These intractable graph problems are often members of the class called "NP-complete" problems, which includes many non-graph problems as well...

# NP-complete problems

- A problem that can be exactly *solved* in time that is a polynomial function of the size of the problem is in the class "**P**" (for ***Polynomial*** time)

- A problem whose solution can be *checked for correctness* in time that is a polynomial function of the size of the problem is in the class "**NP**" (for ***Nondeterministic Polynomial*** time)
  - A "nondeterministic" computer could guess the solution to the problem, and then check if it is a solution in polynomial time, and never give a wrong answer
  - Note that the class P is contained in NP

- A problem that is in NP, and is as hard as any problem in NP (an algorithm for it is also essentially an algorithm for any NP problem) is *NP-complete*

- For all the NP-complete problems, the best known algorithms take exponential time in the worst case...
  - ...However, nobody has yet been able to *prove* that there are no polynomial time algorithms for them! If you find one, you will be instantly very very famous

- What are some of the NP-complete graph problems?...

# Examples of intractable graph problems

- Here are a few examples of the many graph problems that are NP-complete, and so seem to require $O(2^{|V|})$ time worst-case:

  - "Hamiltonian circuit": Given a graph, say whether the graph has a cycle that includes all the vertices of the graph exactly once.

  - "Travelling salesman": Given a weighted graph, find the Hamiltonian circuit that has the smallest total cost.

  - "Longest path": Given a graph and two vertices s and d, find the longest path from s to d that doesn't contain any cycles. (But note that "shortest path" is solvable in polynomial time!)

  - "Shortest total path length spanning tree": Given a graph, find the spanning tree that has the smallest total path lengths between every pair of vertices

  - "Steiner tree": Given a graph (V,E) and a subset S of V, find the minimum-cost spanning tree that spans every vertex in S (and may also span some other vertices) (but note that if S=V, the problem is solvable in polynomial time!)

# The problem with intractable problems

- If a problem is NP-complete, the best known algorithms to solve it requires exponentially many steps in the worst case

- Simple greedy algorithms do not work for these problems
  - backtracking, or some other way of looking at, and checking, possible alternatives is usually required...
  - ... and there are exponentially many alternatives to check!
  - For example:
    - The problem has N boolean variables, and you need to check the $2^N$ possible different assignments of truth values to them
    - The problem has N items, and you need to check each of the $2^N$ different subsets of those items

- Because of the exponential time costs of the best known solutions to these problems, you have to either...
  - restrict yourself to small instances of the problems, or
  - try to find approximate algorithms that are fast, but not always exactly correct

# Finding a minimum spanning tree: Kruskal's algorithm

- Prim's algorithm starts with a single vertex, and grows it by adding edges until the MST is built

- Kruskal's algorithm starts with a forest of single-node trees (one for each vertex in the graph) and joins them together by adding edges until the MST is built

# Kruskal's algorithm

- Pseudocode for Kruskal's MST algorithm, on a weighted undirected graph G = (V,E):

  1. Create a forest of one-node trees, one for each vertex in V

  2. Create a priority queue containing all the edges in E, ordered by edge weight

  3. While fewer than |V|-1 edges have been added to the forest:

     3a. Delete the smallest-weight edge, ($v_i$, $v_j$), from the priority queue.

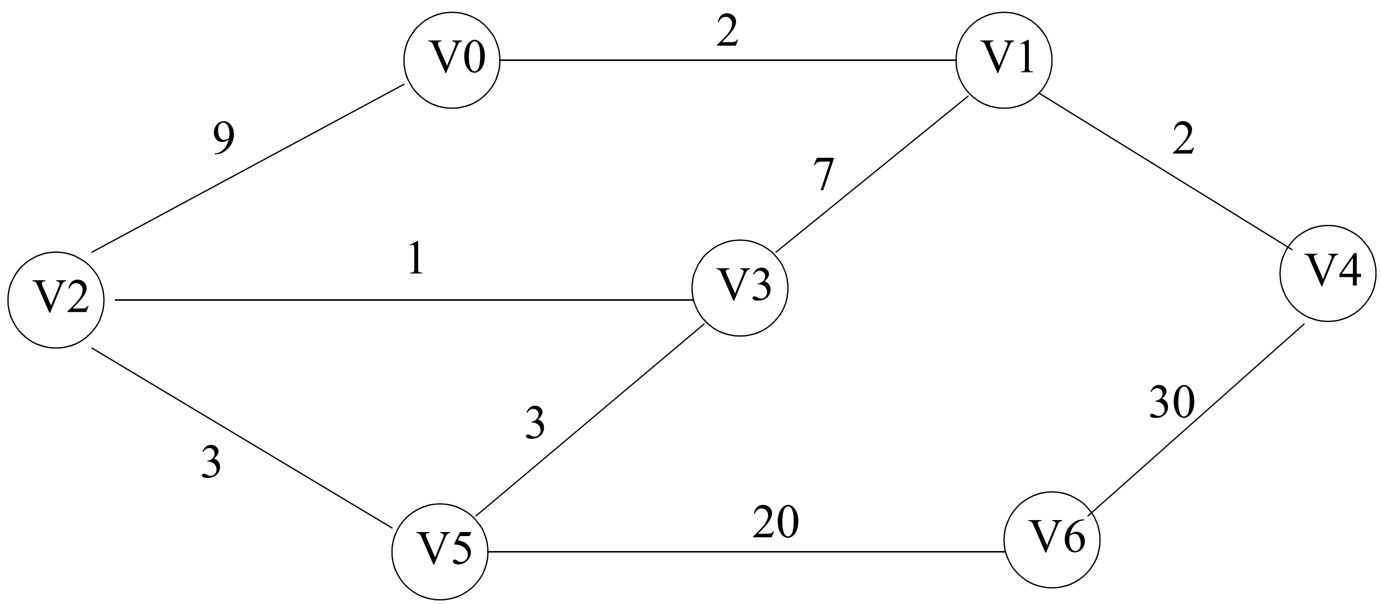
     3b. If $v_i$ and $v_j$ already belong to the same tree in the forest, go to 3a. (Adding this edge would create a cycle.)

     3c. Else, $v_i$ and $v_j$ are in different trees. Join those vertices with that edge (this joins their trees, reducing the number of trees in the forest by 1), and continue.

- When run on a connected graph, the forest will finally contain one tree, which is a minimum spanning tree

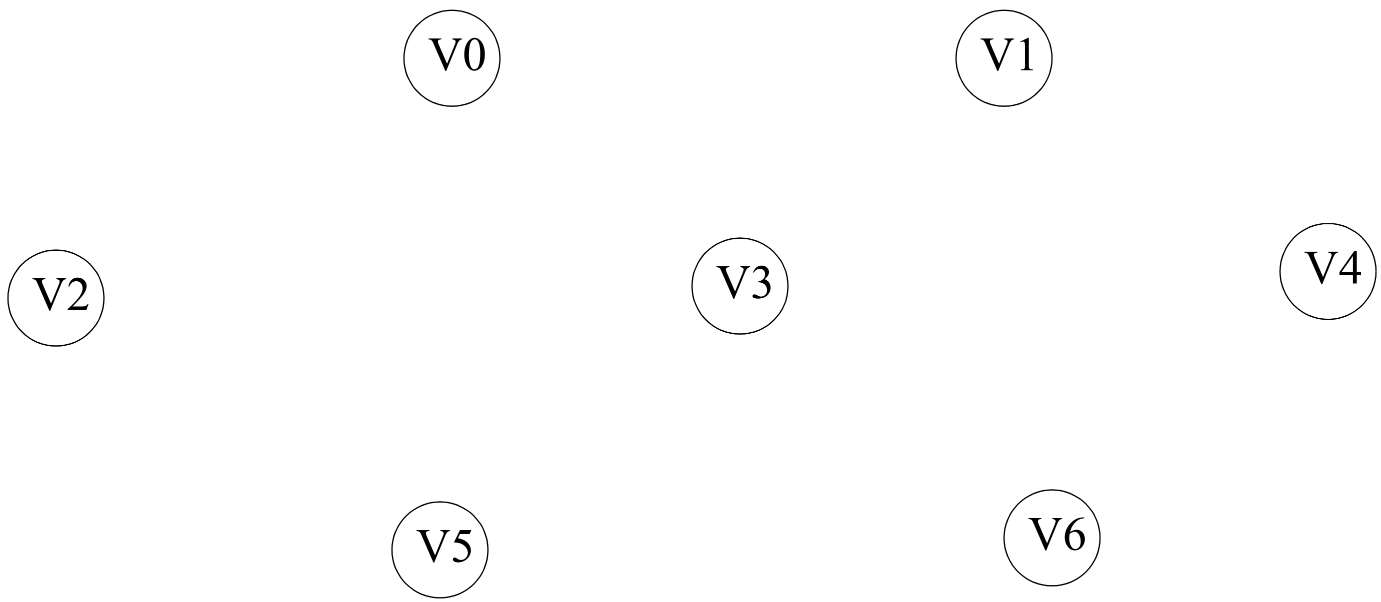
# Kruskal's algorithm: input

- Run Kruskal's algorithm on this weighted undirected graph:



# Kruskal's algorithm:  output

- Show the result here:




- What is the total cost of this spanning tree?

- Is there another spanning tree with lower cost?  With equal cost?

# Implementing Kruskal’s algorithm

- To make Kruskal’s algorithm efficient, all steps in the algorithm must be implemented efficiently:
  - initializing the priority queue must be efficient (2)
  - delete-min in the priority queue must be efficient (3a)
  - testing whether two vertices are already in the same tree in the forest must be efficient (3b)
  - joining two trees in the forest must be efficient (3c)
- Using a binary heap to implement a priority queue leads to efficient steps (2) and (3a):
  - building the heap: $O(|E|)$
  - each delete-min operation: $O(\log|E|)$
- What is an efficient way to do steps 3b and 3c?
  - This requires looking at doing efficient computations with representations of equivalence classes...

# Equivalence relations

- An equivalence relation E(x,y) over a domain S is a boolean function that satisfies these properties for every x,y,z in S:
  - E(x,x) is true (*reflexivity*)
  - If E(x,y) is true, then E(y,x) is true (*symmetry*)
  - If E(x,y) and E(y,z) are true, then E(x,z) is true (*transitivity*)
- For example: Given an undirected graph G. Suppose for any vertices $v_1$, $v_2$ in G, E($v_1$, $v_2$) is true if and only if $v_1$ is connected to $v_2$ (i.e., there is a path from $v_1$ to $v_2$). Then E() is an equivalence relation over the vertices of G:
  - Every vertex is connected to itself (reflexivity)
  - If $v_1$ is connected to $v_2$, then $v_2$ is connected to $v_1$ (it's an undirected graph)
  - If $v_1$ is connected to $v_2$, and $v_2$ is connected to $v_3$, then $v_1$ is connected to $v_3$
- For another example: Suppose E(x,y) is true if and only if x, y are integers and x=y. Then E() is an equivalence relation over the integers

# Equivalence classes

- An equivalence relation E() over a set S defines a system of *equivalence classes* within S
  - The equivalence class of some element x of S is that set of all y in S such that E(x,y) is true
  - Note that every equivalence class defined this way is a subset of S
  - The equivalence classes are disjoint subsets: no element of S is in two different equivalence classes
  - The equivalence classes are exhaustive: every element of S is in some equivalence class
- For example: Given an undirected graph G. Suppose for any vertices $v_1$, $v_2$ in G, E($v_1$, $v_2$) is true if and only if $v_1$ is connected to $v_2$. Then the equivalence classes defined by E() are the connected components of G
- For another example: Suppose E(x,y) is true if and only if x, y are integers and x=y. Then the equivalence classes defined by E() are all singleton sets: each integer is its own equivalence class

# Computing with equivalence classes

- A common problem is this:
  - You are given a set S of items
  - You are given some pairs of items in S that satisfy some equivalence relation E()
  - Given that information, you want to do things like
    - Determine how many equivalence classes there are in S, as defined by the pairs of items satisfying E() that you have seen so far
    - Given an item in S, determine which equivalence class is it in
    - Given two items in S, determine whether they are in the same equivalence class
    - Given a new pair of items in S that satisfy E(), update the system of equivalence classes appropriately

# Computing equivalence classes: an example

- You are manipulating equivalence classes when building a minimum spanning tree using Kruskal's algorithm
  - The initial set of items is the set of vertices in the graph
  - Initially, each vertex is its own equivalence class of one
  - The algorithm considers edges in order of increasing cost.  For each edge:
    - It is accepted if the vertices it connects are not in the same equivalence class
    - If it is accepted, the equivalence classes containing the vertices it connects are joined into one equivalence class
  - When |V|-1 edges have been accepted, the algorithm terminates
  - The edges that have been accepted are the edges of the minimum spanning tree
  - These equivalence-class computations can be done very efficiently using a "disjoint subset", "union/find" structure.   More next time...

# Next time

- An application of disjoint subsets
- Disjoint subset structures and union/find algorithms
- Union-by-size and union-by-height
- Find with path compression
- Amortized cost analysis

Reading: Weiss, Ch. 8