Lecture 12

- Algorithms on graphs
- Breadth first, depth first searches
- Shortest path in unweighted graphs
- Greedy algorithms
- Djikstra's algorithm for shortest path in weighted graphs

Reading: Weiss, Chapter 9, 10

Shortest path problems

- Suppose graph vertices represent computers, and graph edges represent network links between computers, and edge weights represent communications times...
  - ... then a shortest-path algorithm can find the fastest route to send email between one computer and another
- Suppose graph vertices represent cities, and graph edges represent airline routes between cities, and edge weights represent travel costs ...
  - ... then a shortest-path algorithm can find the cheapest route to travel by air between one city and another
- Many, many other examples...
- We will look at shortest-path algorithms in unweighted and weighted graphs
- These algorithms will find the shortest path from a “source” or “start” vertex to every other vertex in the graph
- (Often you may want only the shortest path from a source vertex to one particular destination vertex... but there is no known algorithm to do that with better worst-case time cost than a good algorithm to find shortest path to *every* other vertex!)

The unweighted shortest path problem

- Input: an unweighted directed graph G = (V,E) and a “source vertex” *s* in V
- Output: for each vertex *v* in V, a representation of the shortest path in G that starts at *s* and ends at *v*
- What is the best approach to this problem?

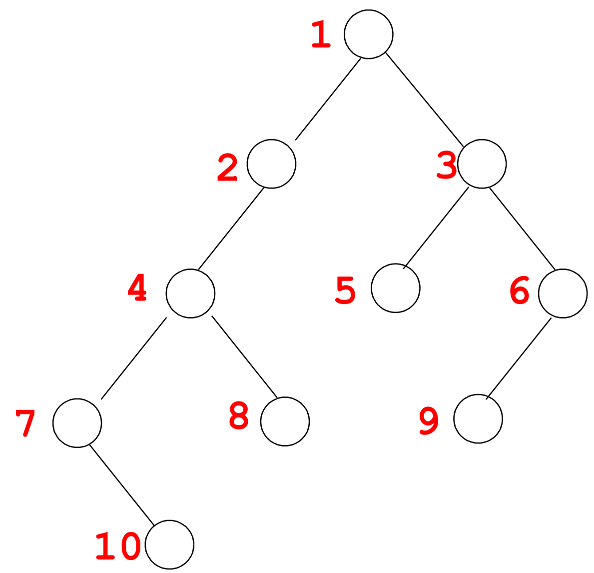
Search in graphs

- The shortest-path problem is a search problem in a graph: starting at a vertex s, you are searching for the shortest path to another vertex v
- When searching in a graph, three important approaches are:
  - depth-first search
  - breadth-first search
  - best-first search
- These approaches can be applied to many different search problems
- We'll consider applying depth-first and breadth-first to the unweighted shortest-path problem. (Best-first will arise in the weighted shortest-path case)

Breadth-first search

- Breadth-first search in a graph visits a node; and then all the nodes adjacent to that node; then all the nodes adjacent to those nodes; etc.

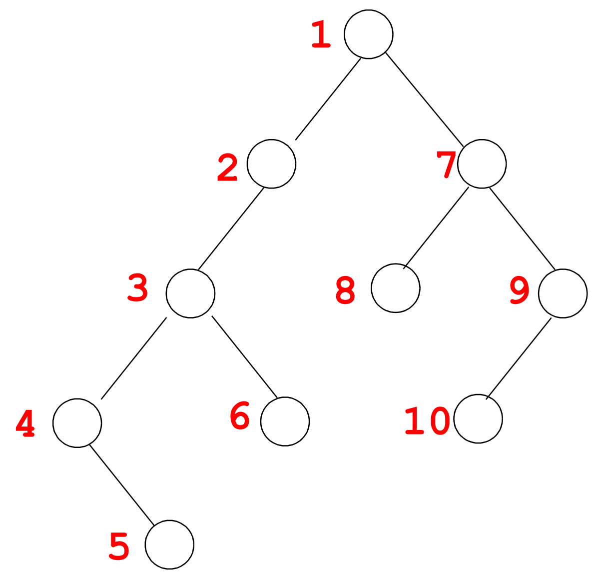
- A level-order traversal of a tree is a breadth-first search:



Depth first search

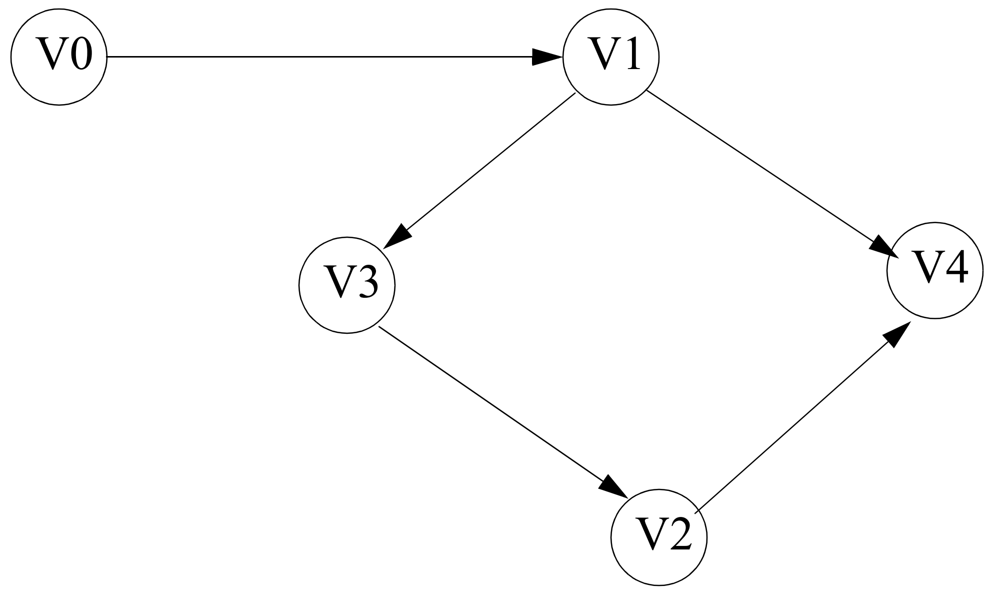
- Depth first search in a graph visits a node; and then recursively does depth-first search from each of the nodes adjacent to that node

- A pre-order traversal of a tree is a depth-first search:



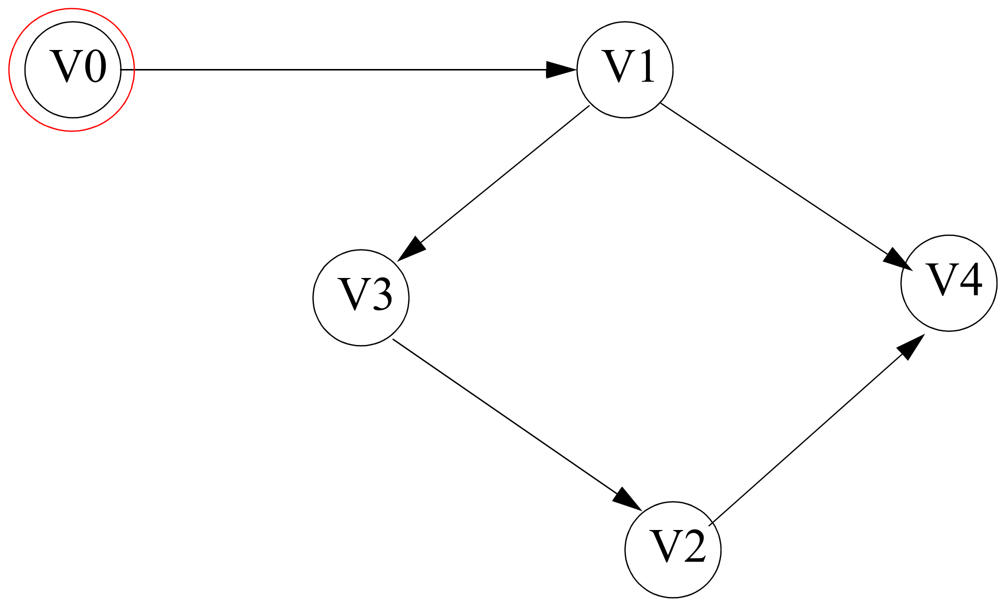
Depth-first search for shortest paths

- Consider searching for shortest paths with start vertex V0 in this unweighted graph, using depth-first search:



Depth-first search for shortest paths: frame 1

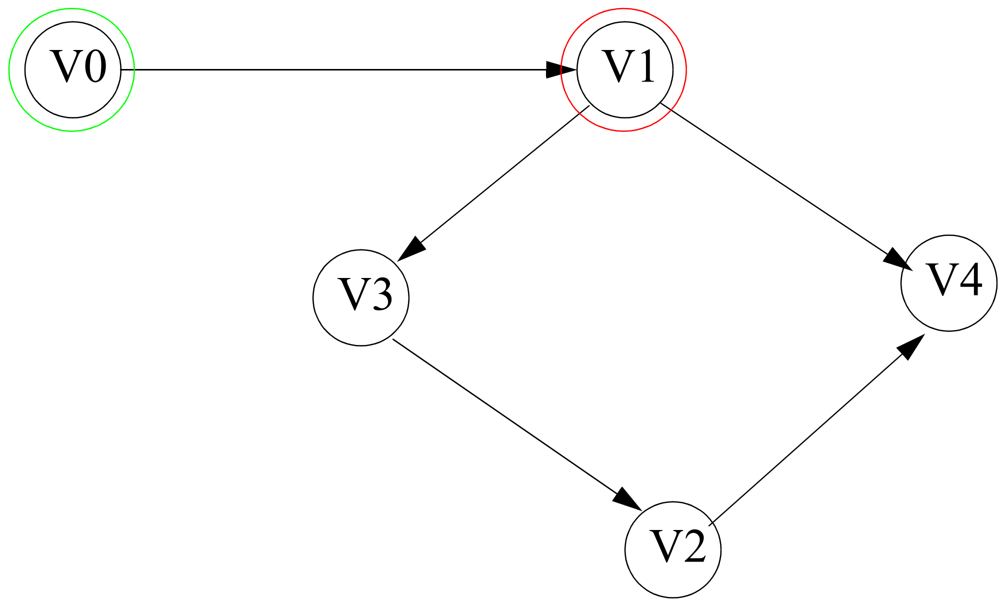
- Immediately we have the shortest path from V0 to V0, following 0 edges.



- Continue the depth-first search, following the edge to V1

Depth-first search for shortest paths: frame 2

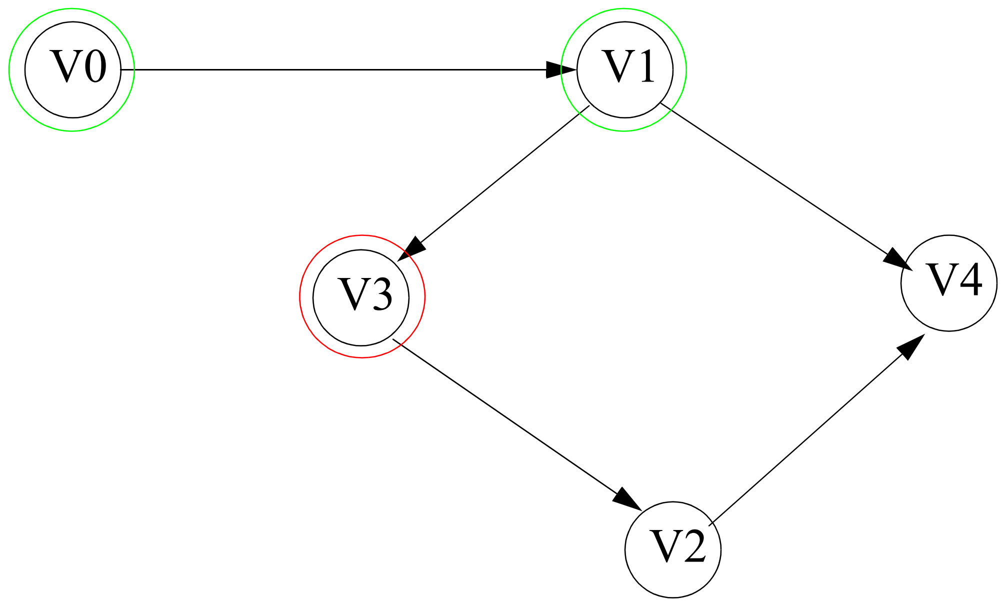
- We have found a path to V1, of length 1.



- Now continuing the depth-first search, there are two choices. Suppose we follow the edge to V3...

Depth-first search for shortest paths: frame 3

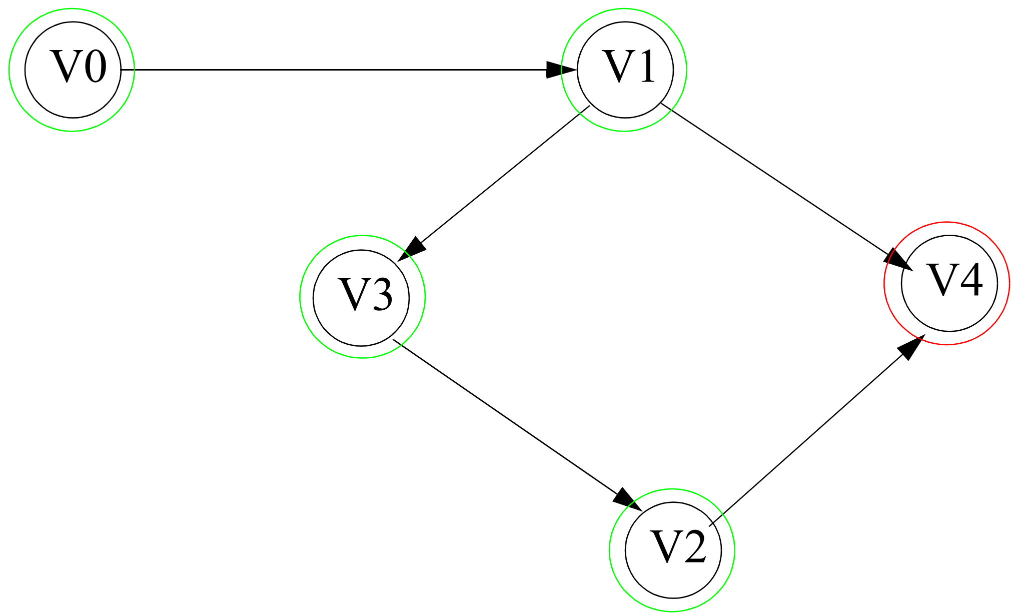
- We have found a path to V3, of length 2.



- Continuing the depth-first search, we follow edges to V2, and then to V4...

Depth-first search for shortest paths: frame 4

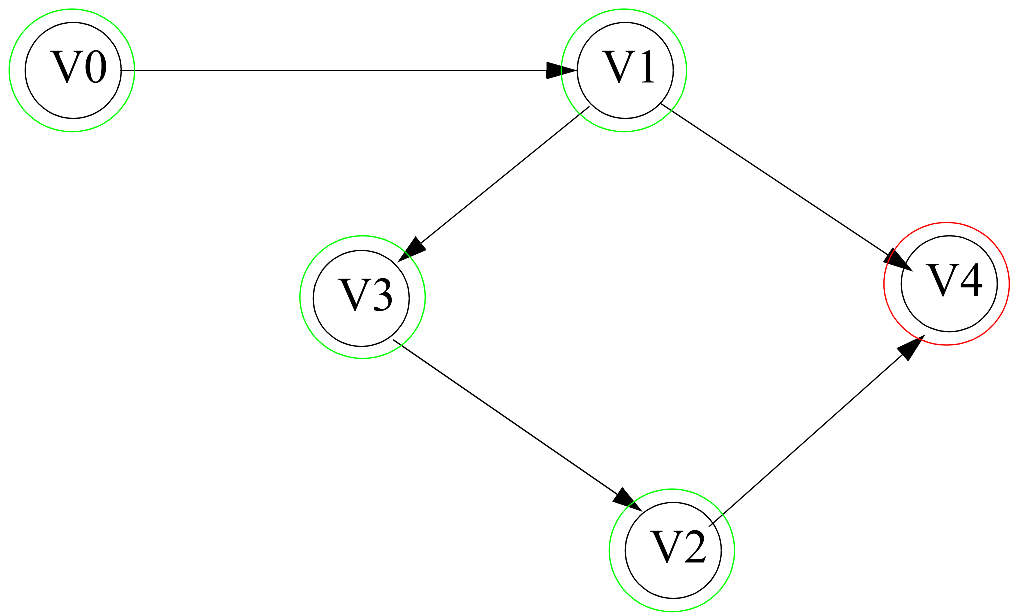
- We have found a path to V2, of length 3, and a path to V4 of length 4.



- There are no untravelled edges to follow from V4. So we backtrack to the last node visited with an untravelled edge: V1, and follow the edge to V4

Depth-first search for shortest paths: frame 5

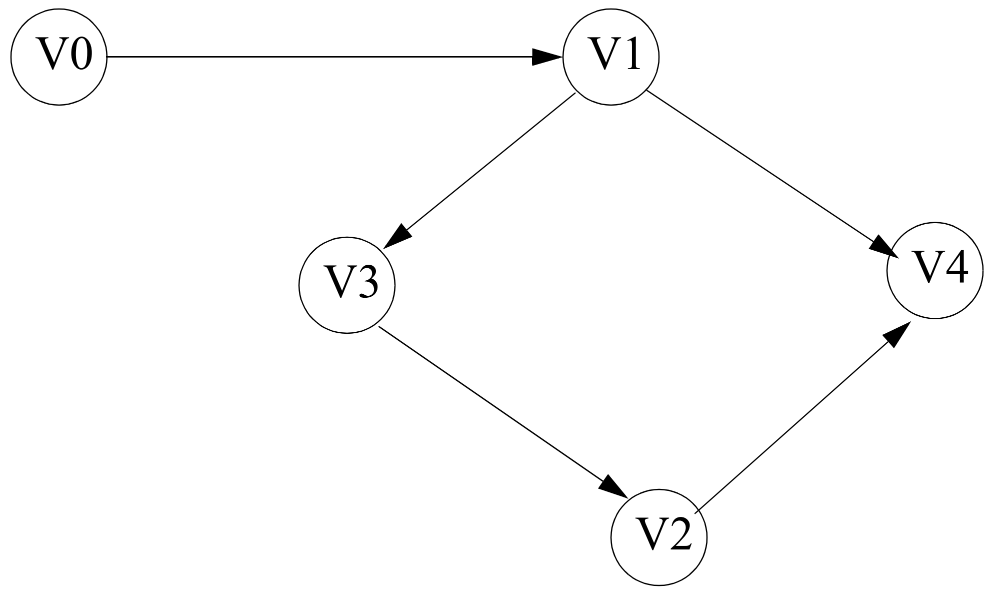
- Following the other edge from V1 to V4, we find a path to V4 of length 2, which is shorter than the one found previously (going through V3 and V2)



- Now all edges have been traversed, and we have visited all the nodes.
- However, note that if there were nodes reachable from V4, we would have to search them again, now having found a shorter path to V4!
- We can eventually find all shortest paths this way, but any simple implementation of this idea could be *very* inefficient

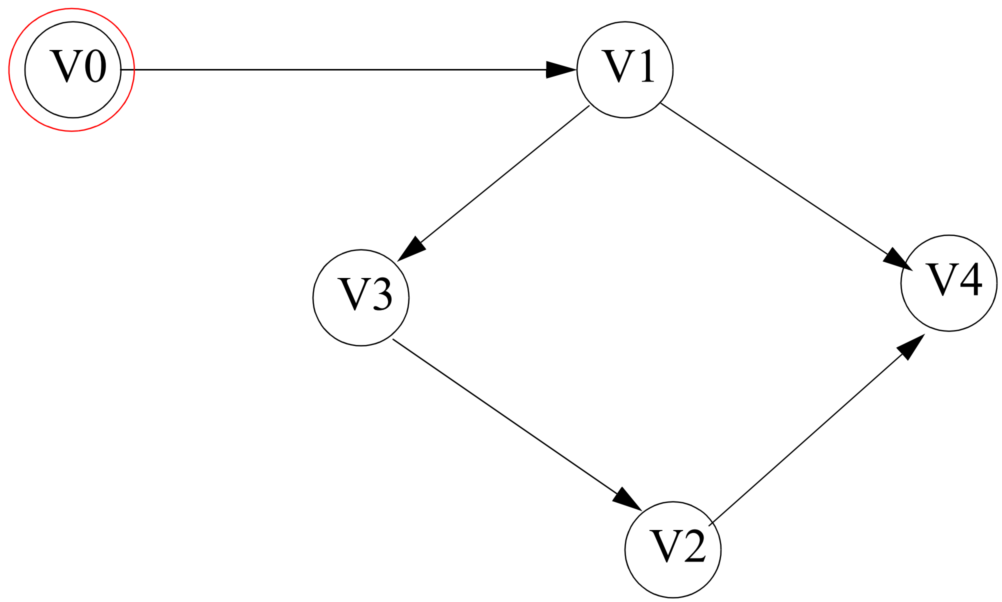
Breadth-first search for shortest paths

- Consider searching for shortest paths with start vertex V0 in this unweighted graph, using breadth-first search:



Breadth-first search for shortest paths: frame 1

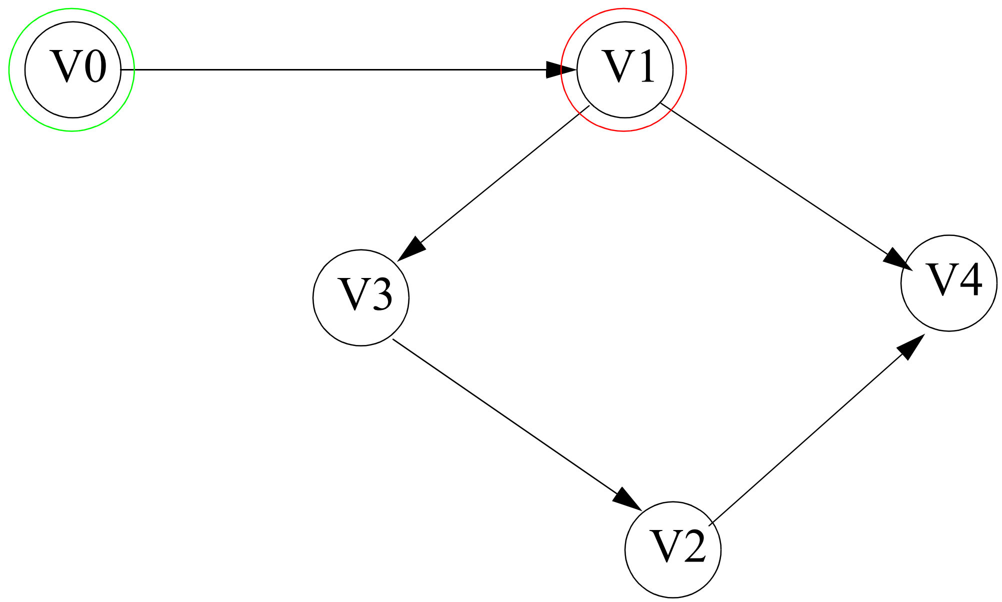
- First visit V0, giving the shortest path from V0 to V0, following 0 edges.



- Continue the breadth-first search, visiting all the unvisited nodes adjacent to V0. (There is only one: V1)

Breadth-first search for shortest paths: frame 2

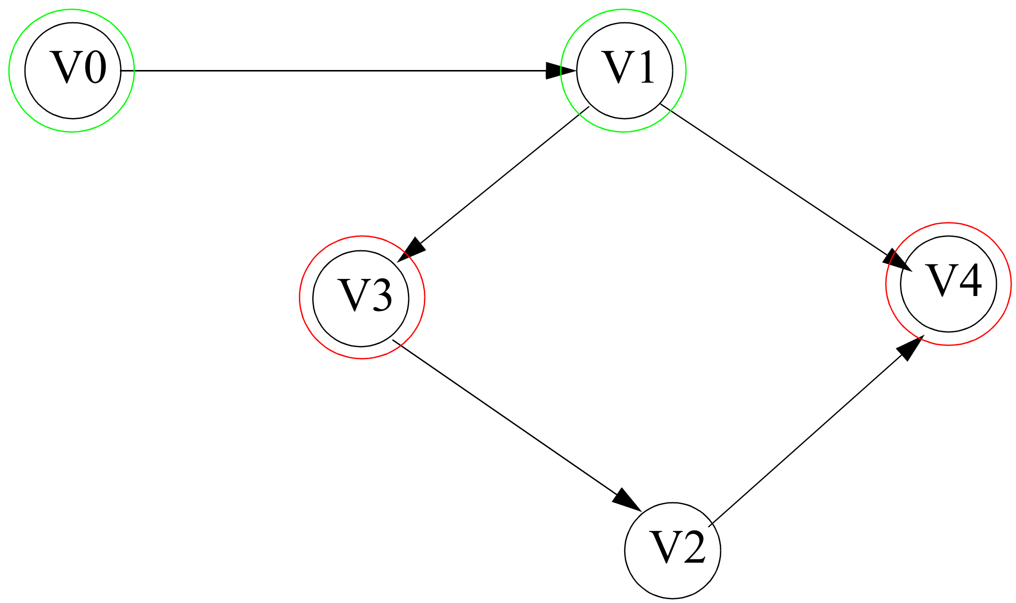
- We have found a path to V1, of length 1.   We can be sure this is the shortest path to V1.



- Now visit continuing the breadth-first search, there are two unvisited vertices adjacent to V1. They can be visited in any order, as long as they are visited before any other vertices

Breadth-first search for shortest paths: frame 3

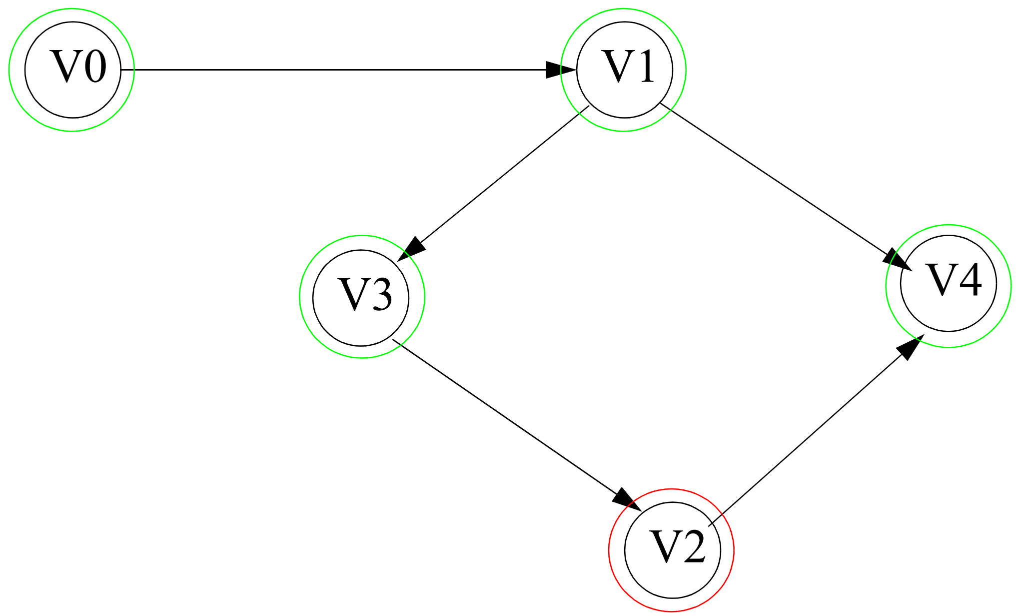
- We have found a paths to V3 and V4, of length 2. We can be sure these are the shortest paths to these vertices



- Continuing the breadth-first search, we follow edges to vertices adjacent to V3 and to V4. There is only one, V2

Breadth-first search for shortest paths: frame 4

- We have found a path to V2, of length 3



- Now we have visited all the vertices in the graph, and have found the shortest path from V0 to each of them

Comparing depth- and breadth-first search for shortest path

- Either depth-first or breadth-first search will traverse the graph, visiting all nodes. But breadth-first search is better for the shortest-path problem in graphs
- In depth-first search:
  - you need to keep track of which edges have been traversed
  - you need to know which node to backtrack to when you reach a “dead end” (this can be done with a stack)
  - when you visit a node, you may not be sure if you have found the shortest path or not; you may find a shorter path later. If you do, you will have to repeat the depth-first search from that node! These repetitions can lead to exponentially many steps
- In breadth-first search:
  - you need to keep track of which nodes have been visited
  - you need to make sure you visit all nodes adjacent to a node before visiting any others (this can be done with a queue)
  - the first time you visit a node, you can be sure you have found the shortest path to it, so that node does not need to be visited again

Unweighted shortest path

- Input: an unweighted directed graph G = (V,E) ; and a source vertex *s* in V
- Output: for each vertex *v* in V, a representation of the shortest path in G that starts at *s* and ends at *v*
- Ordinary breadth-first search solves this problem efficiently: worst-case time cost $O(|E| + |V|)$

Breadth-first search for unweighted shortest path: basic idea

- By **distance** between two nodes u,v we mean the number of edges on the shortest path between u and v. Now:
- Start at the start vertex s. It is at distance 0 from itself, and there are no other nodes at distance 0
- Consider all the nodes adjacent to s. These all are at distance at most 1 from s (maybe less than 1, if s has an edge to itself; but then we would have found a shorter path already) and there are no other nodes at distance 1
- Consider all the nodes adjacent to the nodes adjacent to s. These are all at distance at most 2 from s (maybe less than 2; but then we would have found a shorter path already) and there are no other nodes at distance 2
- ... and so on. In this breadth-first search, as soon as we visit a node in the graph, we know the shortest path from s to it; and so by the time we have visited all the nodes in the graph, we know the shortest path from s to each of them

Unweighted shortest path: sketch of algorithm

- The basic idea is a breadth-first search of the graph, starting at source vertex *s*
- Initially, give all vertices in the graph a distance of INFINITY
- Start at *s*; give *s* distance = 0
- Consider the vertices in the adjacency list of *s*
  - these all have a distance 1 from *s*, so give each of them distance = 1 unless a shorter distance (i.e. 0) has already been found
  - there is no shorter path from *s* to any of these vertices!
- Now consider the vertices in the adjacency lists of vertices at distance 1 from *s*
  - these all have a distance 2 from *s*, so give each of them distance = 2 unless a shorter distance (i.e. 1 or 0) has already been found
  - there is no shorter path from *s* to any of these vertices!
- Continue in this fashion until all vertices have been visited
- This finds the lengths of the shortest paths. Representing the actual sequence of vertices on the paths is easy to do, as we will see momentarily

Unweighted shortest path: auxilliary data structures

- Maintain a sequence (e.g. an array) of vertex objects, indexed by vertex number
  - Vertex objects contain these 2 fields (and others):
    - “dist”: the cost of the best (least-cost) path discovered so far from the start vertex to this vertex (initially INFINITY)
    - “prev”: the vertex number (index) of the previous node on that best path
- Maintain a queue, containing vertices, or vertex numbers (similar to use of queue in level-order traversal of a tree)
  - Initially the queue contains only the start vertex *s*, with dist field 0
  - A vertex is “visited” when it is enqueued: then shortest path to it is known
  - When a vertex *v* is dequeued, vertices on *v*’s adjacency list that have not yet been visited are enqueued, and given a distance = 1 + *v*’s distance
  - When the queue is empty, the algorithm terminates

Unweighted shortest path: more algorithm details

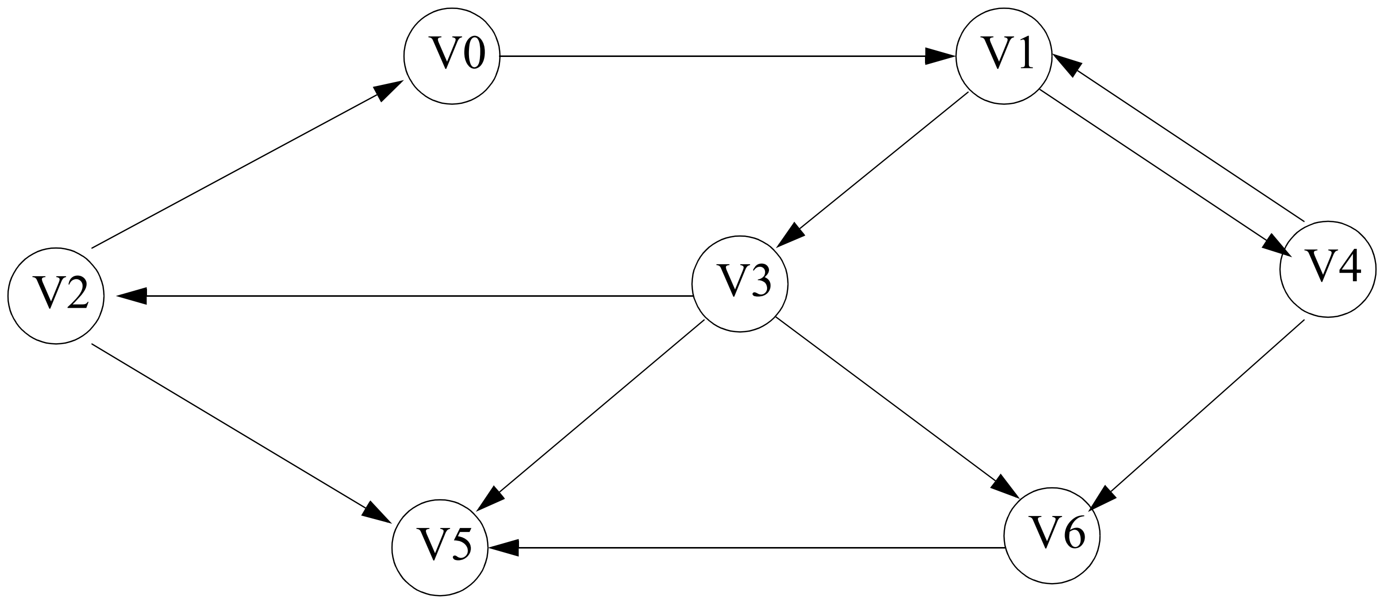
- How do you tell if a vertex has been visited before or not?
  - Check to see if its distance is INFINITY. If it is, it has not been visited yet

- How do you figure out the actual paths, not just their costs?
  - When you enqueue a vertex *w*, it is because it is on the adjacency list of a vertex *v*, which is the previous vertex on the shortest path from the source vertex to *w*
  - You give *w* a distance equal to the distance to *v*, plus 1; and you also set the previous- node field of *w* to be *v*
  - So, the entire shortest path from source to *w* can be found by tracing back these previous node indices to the source vertex

- What is the time complexity of this algorithm?
  - Each vertex is visited exactly once: so there are O(|V|) enqueue and dequeue operations
  - When you dequeue a vertex, you traverse the adjacency list for that vertex; these traversals total O(|E|) steps
  - So, the algorithm has total worst-case time cost O(|V| + |E|)

Unweighted shortest path: an example

- We will find shortest paths in this graph, with source vertex V0



Do it!

- The array of vertices, which include dist and prev fields (initialize dist to 'INFINITY'):

V0:	dist=	prev=	adj: V1
V1:	dist=	prev=	adj: V3, V4
V2:	dist=	prev=	adj: V0, V5
V3:	dist=	prev=	adj: V2, V5, V6
V4:	dist=	prev=	adj: V1, V6
V5:	dist=	prev=	adj:
V6:	dist=	prev=	adj: V5

- The queue (give source vertex dist=0 and prev=-1 and enqueue to start):

HEAD **TAIL**

Unweighted shortest path, C++ code

```
/** Compute the unweighted shortest path. */
void unweightedShortestPath( int startNode ){
   queue<Vertex*> q;
   initData( );  // sets all Vertex dists to INFINITY, prevs to -1
   Vertex* s = vertexVec[startNode];
   s->dist = 0;
   q.push( s ) ;
   while( !q.empty( ) )       {
      Vertex* v = q.front( );  // get a Vertex* from the queue
      q.pop();                 // and remove it from the queue
      std::list<Edge*>::iterator it = v->adj.begin();
      for( ; it != v->adj.end() ; ++it ) { // go thru v's adj list
         Vertex* w = vertexVec[ it->dest ];
          if( w->dist == INFINITY )  { // not yet visited
              w->dist = v->dist + 1;
              w->prev = v->indx;
              q.push( w );
          }
      }
   }
```

Weighted shortest path

- Input: a weighted directed graph G = (V,E) with no negative edge weights; and a source vertex *s* in V

- Output: for each vertex *v* in V, a representation of the shortest weighted path in G that starts at *s* and ends at *v*

- The best general algorithm for this problem is Djikstra's algorithm [Djikstra, 1959]

- Djikstra's algorithm uses 'greedy', 'best-first' search and runs in worst-case time O(|E| log|V|)

Greedy algorithms

- A greedy algorithm is one that, at each stage of the algorithm, commits to a move that seems to be the best, considering just the state of computation at that stage
- Greedy algorithms tend to be fast: they don't require any backtracking or second-guessing
  - However not every problem can be solved with a greedy algorithm!
  - And not every greedy algorithm is fast (may still have exponentially many moves)
- Example: Huffman's algorithm for constructing a coding tree is a greedy algorithm
  - At each step you choose two trees to join together, and you never second-guess the choice; but if symbols are not independent the result will not be optimal
- Example: the usual algorithm for making change (in the U.S.) is a greedy algorithm:
  - Give back as many quarters as possible, then as many dimes as possible, then as many nickels as possible, then the remaining in pennies
  - This algorithm is optimal: it always makes change with the fewest coins possible
  - But suppose there were also a 12-cent coin; then the algorithm is not optimal (consider making 20 cents change...)
- Djikstra's algorithm is fast, but it requires the precondition: no edge weights are negative

Weighted vs. unweighted shortest path algorithms

- The basic idea is similar to the unweighted case

- A major difference is this:
  - In an unweighted graph, breadth-first search guarantees that when we first make it to a node *v*, we can be sure we have found the shortest path to it; more searching will never find a path to *v* with fewer edges
  - In a weighted graph, when we first make it to a node *v*, we can't be sure we have found the best path to *v*: there could be a path with more edges, but less overall cost, that we would find later

- Still, Dijkstra's is a greedy algorithm:
  - At each stage of the algorithm, we will extend the *best* path we have found so far: this guarantees we will know when we have found the shortest (least-cost) path from the source vertex, if there are no negative-cost edges
  - Keeping track of paths in terms of which is best will require a priority queue -- a very common data structure in greedy algorithms

- (Note that Djikstra's algorithm will work for an unweighted graph: treat it as a weighted graph with all edge weights the same, e.g. 1)

Weighted shortest path: auxilliary data structures

- Maintain a sequence (e.g. an array) of vertex objects, indexed by vertex number
  - Vertex objects contain these 3 fields (and others):
    - “dist”: the cost of the best (least-cost) path discovered so far from the start vertex to this vertex
    - “prev”: the vertex number (index) of the previous node on that best path
    - “done”: a boolean indicating whether the “dist” and “prev” fields contain the final best values for this vertex, or not

- Maintain a priority queue
  - The priority queue will contain (*pointer-to-vertex*, *path cost*) pairs
  - *Path cost* is priority, in the sense that low cost means high priority
  - Note: multiple pairs with the same “*pointer-to-vertex*” part can exist in the priority queue at the same time. These will usually differ in the “*path cost*” part

Weighted shortest path: Dijkstra's algorithm

0. Initialize the vertex vector for the graph. Set all "dist" fields to INFINITY, and all "done" fields to false. Locate the start vertex *s*. Set its "dist" field to 0, and its "prev" field to -1. Put the pair (*s*,0) on the priority queue.

1. Is the priority queue empty? Done!

2. Remove from the priority queue the pair (*v*, cost) with the smallest cost.

3. Is the "done" field of the vertex *v* marked true? Go to 1.

4. Mark the "done" field of vertex *v* true. The shortest path from *s* to *v* is now known, and the "cost" and "prev" fields in *v* are correct

5. Traverse *v*'s adjacency list. For each vertex *w* adjacent to *v*:

   Compute the path cost *c* from *s* to *w* going through *v* (this is just the sum of the true cost from *s* to *v*, which is now known, plus the cost of the edge from *v* to *w*).

   If *c* is less than the best cost from *s* to *w* known so far (this value is stored in the "dist" field of *w*), we have found a better path to *w*. Update the "dist" field in *w* with this value, and insert the pair (*w,c*) into the priority queue.

6. Go to 1.

Weighted shortest path: more algorithm details

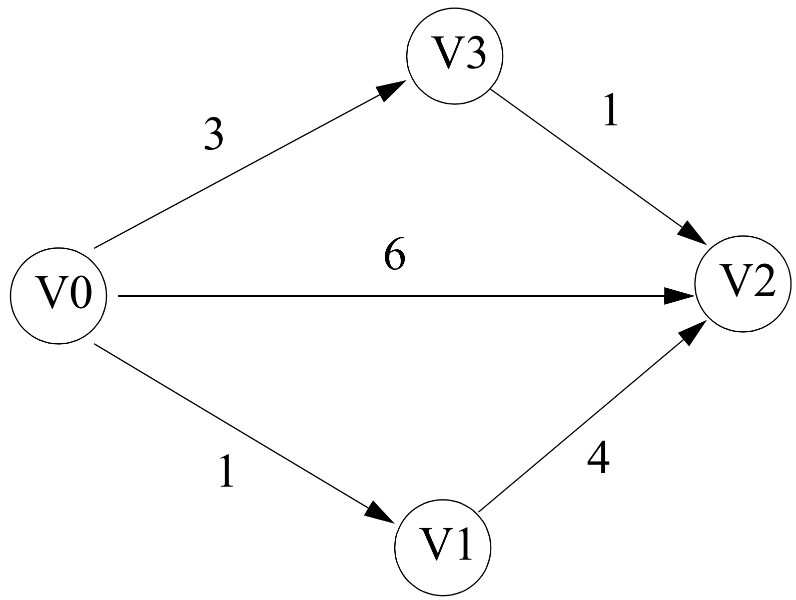
- How do you figure out the actual paths, not just their costs?
  - Same as for the unweighted path algorithm

- How do you know that when you delete-min vertex *v* from the priority queue, that we have searched enough to have found the shortest weighted path to *v*?
  - All the other vertices in the priority queue have at least as great a path cost to them, so we can't do better by extending those paths to *v*
  - (If there were negative weights, continuing to add edges to a path can make its cost less, and this "greedy" approach does not work)

- What is the time complexity of Djikstra's algorithm?
  - Each element of each adjacency list can be inserted and deleted from the priority queue; there are $|E|$ such elements
  - An insertion or delete-min in a binary heap implementation of a priority queue is $O(\log N)$; here $N = |E|$ worst-case
  - So, the algorithm has total worst-case time cost $O(|E| \log |E|)$
  - Since $|E| \le |V|^2$, this is $O(|V|^2 \log |V|)$ and also $O(|E| \log |V|)$

Weighted shortest path: an example

- We will find shortest weighted paths in this graph, with start vertex V0



Do it!

- The array of vertices, which include dist, prev, and done fields (initialize dist to 'INFINITY' and done to 'false'):

```
V0:   dist=        prev=     done=          adj: (V1,1), (V2,6), (V3,3)

V1:   dist=        prev=     done=          adj: (V2,4)

V2:   dist=        prev=     done=          adj:

V3:   dist=        prev=     done=          adj: (V2,1)
```

- The priority queue (set start vertex dist=0, prev=-1, and insert it with priority 0 to start)

Next time

- Connectedness in graphs
- Spanning trees in graphs
- Finding a minimal spanning tree
- Time costs of graph problems and NP-completeness
- Finding a minimal spanning tree: Prim's and Kruskal's algorithms
- Intro to disjoint subsets and union/find

Reading: Weiss, Ch. 9, Ch 8