A Network Coloring Game

Kamalika Chaudhuri[1], Fan Chung[2], and Mohammad Shoaib Jamall[2]

[1] Information Theory and Applications Center, UC San Diego
kamalika@soe.ucsd.edu
[2] Department of Mathematics, UC San Diego
{fan,mjamall}@math.ucsd.edu

Abstract. We analyze a network coloring game which was first proposed by Michael Kearns and others in their experimental study of dynamics and behavior in social networks. In each round of the game, each player, as a node in a network G, uses a simple, greedy and selfish strategy by choosing randomly one of the available colors that is different from all colors played by its neighbors in the previous round. We show that the coloring game converges to its Nash equilibrium if the number of colors is at least two more than the maximum degree. Examples are given for which convergence does not happen with one fewer color. We also show that with probability at least $1 - \delta$, the number of rounds required is $O(\log(n/\delta))$.

1 Introduction

We perform a theoretical study of the following coloring game on networks. In the coloring game, there is an associated network G, and each player is associated with a vertex in this network. Each player has a set of available strategies or colors; the payoff of a player is 1 if he plays a color different from the color played by any of his neighbors in G, and 0 otherwise. The goal of each player is to maximize his payoff, and we are interested in the behaviour and strategies of the players as a function of the network structure.

Our study is motivated by the fact that graph-coloring problems arise as natural formalizations of many conflict-resolution problems in practice. An example due to [7], is a scenario where faculty members wish to schedule classes in a limited number of classrooms, and must avoid conflicts with other faculty members. This can be modelled as a coloring problem, where the faculty members represent vertices, classrooms represent colors, and any two faculty members who have classes with overlapping times are connected by an edge. However, in this scenario, typically there is no centralized agency which assigns classrooms to faculty members, players coordinate among themselves to decide on a non-conflicting assignment, and it is also unreasonable to lay down a distributed protocol and expect the players to abide by the rules of this protocol. As a result, a game-theoretic formulation is an appropriate model for this scenario. A second example of a scenario where a game-theoretic formulation is appropriate is when players are employees in an organization, and colors are skills, and employees attempt to perfect skills that are different from the skills possessed by other employees in their department.

An experimental study of various coloring games was initiated by [7], which reports on behavioral experiments on human subjects who are incentivized to play the coloring game on specified networks. Comparisons were made among several different types of networks of moderate sizees. In addition, examples were given to illustrate the difficulties in analyzing the dynamics of large networks in which each node takes simple and selfish steps. The paper [7] has attracted much attention and pointed to the need for theoretical analysis, although there has not been any other prior work in this specific direction to the extent of our knowledge.

In this paper, we model the coloring game as a game played on a network over multiple rounds. We study the dynamics of the game when the players play a very simple, greedy strategy. At each round, each player picks a color uniformly at random from the set of colors unused by any of his neighbors in the previous round, and plays this color. We note that this is the *best response myopic strategy* [1] for the coloring game. We say that the coloring game converges, when the color played by each player is different from the color played by any of its neighbors. This is a *Nash equilibrium* of the coloring game, because no player has an incentive to change his strategy under this configuration. We are interested in the time taken by the players to converge, when each player adopts the greedy strategy.

Our main result in this paper is that for a coloring game played on a network on n vertices with maximum degree Δ, if the number of colors available to each vertex is $\Delta + 2$ or more, and if each player plays the greedy strategy, then the coloring game converges in $O(\log n)$ steps with high probability. Our result is also accompanied by a lower bound, in which we show a graph and a starting assignment of colors, such that, if the number of colors available to each vertex is $\Delta + 1$, and if each player plays the greedy strategy, then, the coloring game does not converge. Our upper bound holds even in the presence of non-participating vertices, which maintain the same color throughout the game. This indicates that if sufficient number of colors are available, then, even in the game-theoretic scenario, even when the players play a simple greedy strategy, convergence to the nash equilibrium is very rapid. In fact, this convergence bound is even comparable to the convergence bound for distributed protocols for graph coloring, which require the nodes to follow a distributed protocol and cooperate with each other (e.g., see, for example, the work of Luby[9]).

Related Work

The problem of graph coloring has a long history, and there is a lot of literature on centralized as well as distributed algorithms for this problem. The network coloring game has been studied experimentally by Kearns *et. al.*; however, to the best of our knowledge, this game has not been analyzed theoretically. On the experimental side, a number of behavior experiments with human subjects were conducted on the coloring game by Kearns, Suri, and Montford [7]. In their experiment, a group of human subjects were assigned to vertices of a graph, and were asked to play the coloring game for some amount of time. Each subject has access to the colors of their neighbors, and did not have any knowledge about the structure of the rest of the graph. Three graphs were studied – a cycle, a cycle with chords, as well as a random preferential attachment graph. In the experiment, the subjects found it more difficult to color the preferential attachment graph in their allotted amount of time.

In general, finding the minimum number of colors required to color a graph, even in a centralized manner, is NP-Hard [3], as well as hard to approximate [8]. Coloring a graph when the number of available colors is more than its maximum degree can be easily done in linear time by a centralized algorithm; however, the same problem becomes more challenging when the algorithm is required to be distributed.

There has also been a line of work on distributed graph-coloring. Luby [9] provides a distributed algorithm which finds a coloring of a graph in $O(\log n)$ rounds, when the number of available colors is $\Delta + 1$ or more. Notice that here Δ is the maximum degree of any vertex in the graph. [2] also provide theoretical as well as experimental results on a simple algorithm for coloring a graph in a distributed manner in $O(\log n)$ rounds when the number of available colors is $\Delta + 1$. Both Luby's algorithm and the algorithm of [2] require each node to communicate to its neighbors its status, which indicates whether this node has any conflicts with its neighbors or not. In contrast, our scenario is purely game-theoretic : our algorithm does not require any cooperation among the vertices, and will succeed even if some nodes do not participate in the game.

Another line of work which is relevant to ours is the literature on Markov Chains for randomly sampling colorings of a graph. In this case, the goal is to bound the mixing time of a Markov Chain on colorings of a graph G. Pioneering work in this field was done by Mark Jerrum [6], who showed a way to randomly sample colorings of a graph G with maximum degree Δ in $O(n \log n)$ time when the number of colors available is at least $2\Delta + 1$. This was later improved by Vigoda [10], who could randomly sample colorings from graphs of degree Δ, when $\frac{11\Delta}{6}$ colors were available. Hayes and Vigoda [5] showed a better bound for triangle-free graphs when the number of colors needed was $\min(\Delta + O(1), O(\log n))$. Finally, it was shown in [4] that colorings from planar graphs can be sampled in $O(n \log n)$ time when the number of colors is at least $\Delta / \log \log \Delta$.

A Summary of Our Results

We consider the coloring game played on a graph G. Before we state our results, we need to define the following concepts.

The Coloring Game In the coloring game, each vertex in G represents a player. Each player has a set of k available strategies or *colors*; the payoff of a player is 1 if he picks a color different from the colors picked by his neighbors in G, and 0 otherwise. The game is played in *rounds*; each round, the players choose their strategies simultaneously. It is assumed that the players only have a *local view* of G, which means that they only know who their neighbors are, and the colors picked by their neighbors, and they do not have any knowledge of the rest of the graph G.

Greedy Strategy In this paper, we study the dynamics of the coloring game when each player has a local view and plays the *greedy strategy*. A player is said to play the *greedy strategy* if, in each round, he picks a color uniformly at random from the set of colors unused by any of his neighbors in the previous round. We note that this is the *best response myopic strategy* for the coloring game.

Convergence of the Coloring Game The coloring game is said to *converge* if, for every vertex v in G, the color chosen by v is different from the color chosen by all its neighbors. We note that when the coloring game has converged, none of the players have any incentive to change their strategy.

Participants in the Coloring Game We say that a vertex v in graph G is a participant in the coloring game on G if v plays according to the greedy strategy, otherwise we say that v is a *non-participant*. An instance of a coloring game on a graph G which has non-participant nodes, is said to converge, if, for every participant vertex v in G, the color chosen by v is different from the color chosen by all its neighbors.

The main result of this paper can be summarized by the following theorem.

Theorem 1. *Let G be any graph on n vertices, and let Δ be the maximum degree of any vertex in G. If the number of colors available to each vertex is at least $\Delta + 2$, and if each player plays the greedy strategy, then, for any starting assignment of colors, the coloring game on G converges after at most $O(\log(\frac{n}{\delta}))$ rounds with probability at least $1 - \delta$.*

In addition, we show that if there exists a set S of non-participant vertices, such that any vertex $v \in S$ has a fixed color throughout the game, then, the convergence time of the game is still at most $O(\log(\frac{n-|S|}{\delta}))$ round.

Corollary 1 *Let G be any graph on n vertices, Δ be the maximum degree of any vertex in G, and S be a set of non-participant vertices. If the vertices in S do not change their color throughout the game, and the number of colors available to each vertex is at least $\Delta + 2$, and if each player plays the greedy strategy, then, for any starting assignment of colors, the coloring game on G converges after at most $O(\log(\frac{n-|S|}{\delta}))$ rounds with probability at least $1 - \delta$.*

We show that when the number of colors is $\Delta + 1$, there is a graph G and a starting assignment of colors such that the greedy strategy does not converge.

Theorem 2. *There exists a graph G and a starting assignment of colors $\mathcal{C}$ such that if the number of colors is $\Delta + 1$, and if each player plays the greedy strategy, the coloring game on G never converges.*

2 Several Lemmas

We use $c_t(u)$ to denote the color played by player u at round t, and $\mathcal{N}(u)$ to denote the set of neighbors of u in G. We say that player u at round t has a *conflict* if

$$\exists v \in \mathcal{N}(u), \ \ c_t(u) = c_t(v)$$

At any time t, we use *number of conflicts* to mean the number of vertices v which have a conflict. We observe that if a vertex u has no conflict at time t, then $c_{t+1}(u) = c_t(u)$. We use k to denote the number of colors available and Δ to denote the maximum degree of the graph G over which the game is played.

We use the following lemma which is a minor variant of Markov's inequality.

Lemma 1. *Let X be a random variable such that $0 \leq X \leq M$. Then, for any a,*

$$\Pr[X < a] \leq \frac{M - \boldsymbol{E}[X]}{M - a}.$$

We observe that if a vertex v has no conflict at time t, then at round $t+1$, v does not change its color; moreover, no neighbor of v picks color $c_t(v)$, and after round $t+1$, v still has no conflict. Thus we have the following lemma.

Lemma 2. *If a vertex v has no conflict at time t, then it has no conflict at any subsequent time.*

The main idea behind the proof of Theorem 1 is to show that if we consider any two successive rounds of the coloring game, then, the conflict at each vertex is resolved with constant probability. We note that considering the game over two successive rounds is essential; in a single round, it is possible that a player has as little as $(1 - \frac{1}{k-\Delta})^{\Delta}$ chance of getting its conflict resolved.

The main step in the proof of Theorem 1 is the following lemma.

Lemma 3. *Consider an instance of the coloring game played on a graph G, with maximum degree Δ, in which each node has k color choices, where $k \geq \Delta + 2$. If, after round t, some vertex v in G has a conflict, then, there exists some constant c such that*

$$\Pr[v \textit{ has no conflict after round } t+2] \geq c$$

Proof. For vertex v, Let M denote the neighbors of v which do not have a conflict after round t. We define

$$F = \cup_{u \in M} \{c_t(u)\}$$

and let $f = |F|$. Then, the number of choices available to v in rounds $t+1$ as well as $t+2$ is at most $k - f$. However, since t is any arbitrary round, the number of colors available to v after round t can possibly be much less.

The proof of this lemma proceeds in two steps. First, we show in Lemma 4 that after round $t+1$, with constant probability, the vertex v has at least $\frac{k-f}{6}$ color choices. Next, we show in Lemma 5 that, given that v has $\frac{k-f}{6}$ color choices after round $t+1$, with constant probability, v has no conflict after round $t+2$. Combining these two facts, we get a proof of the lemma.

After round t, for any color $i \in [k]$, we define the random variable Y_i as follows:

$$Y_i = \begin{cases} 1 & \text{if } c_{t+1}(u) \neq i, \text{ for all } u \in \mathcal{N}(v) \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Let $Y = \sum_{i \in [k]} Y_i$; thus Y is the number of colors available to vertex v after round $t+1$.

For any $u \in \mathcal{N}(v) \setminus M$, we let $\chi(u)$ be the set of colors available to vertex u after round t. Further, let $p_u = \frac{1}{|\chi(u)|}$. For $u \in M$, we define $\chi(u) = \{c_t(u)\}$, and $p_u = 1$. Note that if $u \in \mathcal{N}(v) \setminus M$, p_u is the probability with which vertex u picks a fixed color available to it during round $t+1$. Also, as $k \geq \Delta + 2$, each vertex $u \in \mathcal{N}(v) \setminus M$ has at least two color choices available, and therefore $|\chi(u)| \geq 2$. We are interested in the probability that $Y \geq \frac{k-f}{6}$. We show the following lemma.

Lemma 4.

$$\Pr(Y \geq \frac{k-f}{6}) \geq \frac{1}{25}$$

Proof. From the definition of Y_i, we can write that:

$$\Pr(Y_i = 1) = \prod_{\{u \in \mathcal{N}(v) | i \in \chi(u)\}} (1 - p_u)$$

The expectation of Y can therefore be estimated as:

$$\begin{aligned}\mathbf{E}(Y) &= \sum_{i \in [k]} \prod_{\{u \in \mathcal{N}(v) | i \in \chi(u)\}} (1 - p_u) \\ &\geq \sum_{i \in [k] \setminus F} \prod_{\{u \in \mathcal{N}(v) \setminus M | i \in \chi(u)\}} (1 - p_u)\end{aligned}$$

The second step follows by the definition of F and M. As, for each $u \in \mathcal{N}(v) \setminus M$, $|\chi(u)| \geq 2$, we can use the inequality $1 - x \geq e^{-\frac{3}{2}x}$ for $0 \leq x \leq \frac{1}{2}$ to write:

$$\mathbf{E}(Y) \geq \sum_{i \in [k] \setminus F} e^{-\frac{3}{2} \sum_{\{u \in \mathcal{N}(v) \setminus M | i \in \chi(u)\}} p_u}$$

Using the convexity of the exponential function,

$$\mathbf{E}[Y] \geq (k - f) e^{-\frac{3}{2} \frac{1}{k-f} \sum_{i \in [k] \setminus F} \sum_{\{u \in \mathcal{N}(v) \setminus M | i \in \chi(u)\}} p_u}$$

Observe that

$$\sum_{i \in [k] \setminus F} \sum_{\{u \in \mathcal{N}(v) \setminus M | i \in \chi(u)\}} p_u \leq \sum_{u \in \mathcal{N}(v) \setminus M} p_u \chi(u)$$

and therefore, we can write that:

$$\begin{aligned}\mathbf{E}[Y] &\geq (k - f) e^{-\frac{3}{2} \frac{1}{k-f} \sum_{u \in \mathcal{N}(v) \setminus M} p_u \chi(u)} \\ &\geq (k - f) e^{-\frac{3}{2}} \geq \frac{k - f}{5}\end{aligned}$$

The final step follows from the fact that $|\{u \in \mathcal{N}(v) \setminus M\}| \leq k - f - 2$. To complete the lemma, we now use Lemma 1, which is a variant of Markov's inequality. □

Now we analyze the dynamics in round $t + 2$ given that $Y > \frac{k-f}{6}$.

For any color $i \in [k]$, we define variables $\tilde{Y}_i$ as follows:

$$\tilde{Y}_i = \begin{cases} 1 & \text{if } c_{t+2}(u) \neq i \text{ for all } u \in \mathcal{N}(v) \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

Let $\tilde{Y} = \sum_{i \in [k]} \tilde{Y}_i$. We are interested in the probability of the event that $\tilde{Y} \geq \frac{k-f}{7e^9}$.

Lemma 5.

$$\Pr(\tilde{Y} \geq \frac{k - f}{7e^9} | Y \geq \frac{k - f}{6}) \geq \frac{1}{42e^9}$$

Proof. We define $\tilde{M}$ to be the set of vertices in $\mathcal{N}(v)$ which have no conflicts after round $t + 1$, and $\tilde{F}$ as the set $\cup_{u \in \tilde{M}} \{c_{t+1}(u)\}$. Note, by Lemma 2, that $\tilde{M} \supseteq M$ and $|\tilde{F}| \geq |F|$. Further, we let $\tilde{f} = |\tilde{F}|$. Also we define $H = [k] \setminus \cup_{u \in \mathcal{N}(v)} \{c_{t+1}(u)\}$. Thus H is the number of colors available to v in round $t + 2$. Obviously, $|H| \leq k - \tilde{f}$; given that $Y \geq \frac{k-f}{6}$, $|H| \geq \frac{k-f}{6} \geq \frac{k-\tilde{f}}{6}$.

For any $u \in \mathcal{N}(v) \setminus \tilde{M}$, we define $\tilde{\chi}(u)$ as the set of colors available to vertex u after round $t + 1$, and $\tilde{p}_u$ as $\frac{1}{|\tilde{\chi}(u)|}$ respectively. For $u \in \tilde{M}$, we define $\tilde{\chi}(u) = \{c_{t+1}(u)\}$ and $\tilde{p}_u = 1$. We can write:

$$\Pr(\tilde{Y}_i = 1) = \prod_{\{u \in \mathcal{N}(v) | i \in \tilde{\chi}(u)\}} (1 - \tilde{p}_u)$$

Similar to the proof of Lemma 4, we can write:

$$\begin{aligned} \mathbf{E}[\tilde{Y}] &= \sum_{i \in [k]} \prod_{\{u \in \mathcal{N}(v) | i \in \tilde{\chi}(u)\}} (1 - \tilde{p}_u) \\ &\geq \sum_{i \in H} \prod_{\{u \in \mathcal{N}(v) \setminus \tilde{M} | i \in \tilde{\chi}(u)\}} (1 - \tilde{p}_u) \end{aligned}$$

As, for each $u \in \mathcal{N}(v) \setminus \tilde{M}$, $|\tilde{\chi}(u)| \geq 2$, we can write:

$$\mathbf{E}(Y) \geq \sum_{i \in H} e^{-\frac{3}{2} \sum_{\{u \in \mathcal{N}(v) \setminus \tilde{M} | i \in \tilde{\chi}(u)\}} \tilde{p}_u}$$

Using the convexity of the exponential function,

$$\mathbf{E}[\tilde{Y}] \geq \frac{1}{6}(k - \tilde{f}) e^{-\frac{3}{2} \frac{6}{k - \tilde{f}} \sum_{i \in H} \sum_{\{u \in \mathcal{N}(v) \setminus \tilde{M} | i \in \tilde{\chi}(u)\}} \tilde{p}_u}$$

Observe that

$$\sum_{i \in H} \sum_{\{u \in \mathcal{N}(v) \setminus \tilde{M} | i \in \tilde{\chi}(u)\}} \tilde{p}_u \leq \sum_{u \in \mathcal{N}(v) \setminus \tilde{M}} \tilde{p}_u \tilde{\chi}(u)$$

and therefore, we can write that:

$$\begin{aligned} \mathbf{E}[\tilde{Y}] &\geq \frac{1}{6}(k - \tilde{f}) e^{-\frac{3}{2} \frac{1}{k - \tilde{f}} \sum_{u \in \mathcal{N}(v) \setminus \tilde{M}} \tilde{p}_u \tilde{\chi}(u)} \\ &\geq \frac{1}{6}(k - \tilde{f}) e^{-9} \geq \frac{k - f}{6e^9} \end{aligned}$$

The lemma now follows by an application of Lemma 1. □

Now, given that the event $\tilde{Y} \geq \frac{k - \tilde{f}}{7e^9}$ occurs, there is a set $\mathcal{C}$ of $\frac{k - \tilde{f}}{7e^9}$ colors, such that the conflict of v is resolved if it picks a color from $\mathcal{C}$ in round $t + 2$. Since v picks one out of at most $k - \tilde{f}$ colors, given the event $\tilde{Y} \geq \frac{k - \tilde{f}}{7e^9}$, v has no conflict after round $t + 2$ with probability at least $\frac{1}{42e^9}$. Combining this with the probability of occurrence of the event $Y > \frac{k - f}{6}$, Lemma 3 follows for $c = \frac{1}{1050e^9}$. □

3 Proofs of the main theorems

First we prove Theorem 1.

Proof. (Of Theorem 1) For a vertex v, and a time t, we define random variables $X_v(t)$ as follows:

$$X_v(t) = \begin{cases} 1 & \text{if vertex } v \text{ has a conflict after round } t \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

From Lemma 3, for any vertex v, for some constant c,

$$\Pr(X_v(t + 2) = 1 | X_v(t) = 1) \leq 1 - c \quad (4)$$

Using Lemma 2, for any τ,

$$\begin{aligned}\Pr(X_v(2\tau) = 1|X_v(0) = 1) &= \Pr(\bigcap_{i=1}^{\tau} X_v(2i) = 1|X_v(0) = 1) \\ &= \prod_{i=1}^{\tau} \Pr(X_v(2i) = 1|\bigcap_{j=1}^{i-1} X_v(2j) = 1) \\ &\leq (1-c)^{\tau} \leq e^{-c\tau}\end{aligned}$$

Plugging in $\tau = \frac{1}{c}\log(\frac{n}{\delta})$, we get, for any vertex $v \in G$,

$$\Pr(X_v(2\tau) = 0|X_v(0) = 1) \geq 1 - \frac{\delta}{n} \tag{5}$$

The theorem follows by applying an union bound over all vertices in G. □

Proof. (Of Corollary 1) All steps and lemmas in the proof of Theorem 1 remain valid except that we use $n - |S|$ in place of n. □

It remains to prove Theorem 2.

Proof. (Of Theorem 2) Let G be a cycle of length 5, and let $V = \{v_1, \ldots, v_5\}$. Let $k = 3$, and suppose the initial configuration is $(c(v_1), c(v_2), c(v_3), c(v_4), c(v_5)) = (1, 0, 2, 2, 0)$. Then, only v_3 and v_4 have conflicts. If v_3 and v_4 follow the dynamics, then, in the next round, $c(v_3) = c(v_4) = 1$. This again causes them both to have conflicts, so in the following round, $c(v_3) = c(v_4) = 2$, and we have a cycle in the dynamics. □

References

1. Esteban Arcaute amd Ramesh Johari and Shie Mannor. Two stage myopic dynamics in network formation games. In *Workshop on Network and Economics (WINE)*, 2008.
2. Irene Finocchi, Alessandro Panconesi, and Riccardo Silvestri. Experimental analysis of simple, distributed vertex coloring algorithms. In *SODA*, pages 606–615, 2002.
3. M. R Garey and D.S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W.H. Freeman, New York, 1979.
4. Thomas P. Hayes, Juan C. Vera, and Eric Vigoda. Randomly coloring planar graphs with fewer colors than the maximum degree. In *STOC '07: Proceedings of the thirty-ninth annual ACM symposium on Theory of computing*, pages 450–458, New York, NY, USA, 2007. ACM.
5. Tom Hayes and Eric Vigoda. Coupling with the stationary distribution and improved sampling for colorings and independent sets. In *SODA '05: Proceedings of the sixteenth annual ACM-SIAM symposium on Discrete algorithms*, pages 971–979, Philadelphia, PA, USA, 2005. Society for Industrial and Applied Mathematics.
6. Mark Jerrum. A very simple algorithm for estimating the number of k-colorings of a low-degree graph. *Random Struct. Algorithms*, 7(2):157–165, 1995.
7. Michael Kearns, Siddharth Suri, and Nick Montfort. An experimental study of the coloring problem on human subject networks. *Science*, 313(5788):824–827, 2006.
8. Subhash Khot. Improved inaproximability results for maxclique, chromatic number and approximate graph coloring. In *FOCS*, pages 600–609, 2001.
9. Michael Luby. Removing randomness in parallel computation without a processor penalty. In *FOCS*, pages 162–173, 1988.
10. Eric Vigoda. Improved bounds for sampling colorings. In *Proc. of FOCS*, 1999.