Universal Portfolios
with and without transaction costs [BK97]

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Outline

1. Constant Rebalanced Portfolios

2. Universal Algorithm
   - Universal guarantees
   - Simple algorithm Split
   - Universal algorithm

3. Simple Analysis

4. Transaction Costs

5. Predicting From Expert Advice

6. Implementation
Constant Rebalanced Portfolios (CRPs)

Definition

Portfolio with same distribution of wealth each day, e.g.

Day | Stock #1 | Stock #2 | Holdings of CRP(1/2, 1/2) | = | New Holdings
---|---|---|---|---|---
0 | $1.00 | $1.00 | $1.00 | = | $0.50 + $0.50
1 | $1.00 | $2.00 | $1.50 | = | $0.50 + $1.00
2 | $1.00 | $1.00 | $1.12 | = | $0.75 + $0.37
3 | $1.00 | $2.00 | $1.68 | = | $0.56 + $1.12
4 | $1.00 | $1.00 | $1.26 | = | $0.84 + $0.42

... 

2n | $1.00 | $1.00 | $1 | = | $0.63 + $0.63

Remark

In hindsight, we see that a \((1/2, 1/2)\)-CRP is optimal among all CRPs.
**Constant Rebalanced Portfolios (CRPs)**

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Portfolio with same distribution of wealth each day, e.g.

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<tr>
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<td>$1.00 = $0.50 + $0.50</td>
</tr>
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*In hindsight, we see that a $(\frac{1}{2}, \frac{1}{2})$-CRP is optimal among all CRPs.*
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2. *Universal* Algorithm
   - *Universal* guarantees
   - Simple algorithm Split
   - *Universal* algorithm

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Universal guarantees

- Let $n$ be the number of days, $m$ be the number of stocks
Universal guarantees

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$$\frac{\text{wealth of Universal}}{\text{wealth of best CRP}} \geq \frac{1}{(n + 1)^{m-1}}$$
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- avg. per-day ratio $\geq \left[\frac{1}{(n+1)^{m-1}}\right]^{1/n} \to 1$
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- With fixed % commission $c < 1$,

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- With fixed % commission $c < 1$,

$$\frac{\text{wealth of Universal}}{\text{wealth of best CRP}} \geq \frac{1}{(n(1 + c) + 1)^{m-1}}$$

- $EG(\eta)$ algorithm
Warm-up algorithm *Split*

- Initially invest an equal amount in each stock
- Let it sit. (no trades)
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Warm-up algorithm *Split*

- Initially invest an equal amount in each stock
- Let it sit. (no trades)
- \( \text{wealth of SPLIT} = \text{avg. of stocks} \)
  \[
  \frac{\text{wealth of SPLIT}}{\text{wealth of best stock}} \geq \frac{1}{m}
  \]
- \( \text{avg. per-day ratio} \geq \frac{1}{m^{1/n}} \to 1 \text{ as } n \to \infty \)
Universal algorithm

- Split money evenly among all CRPs
- Let it sit (i.e. Do not transfer between CRPs)
Universal algorithm

- Split money evenly among all CRPs
- Let it sit (i.e. Do not transfer between CRPs)
- 4 CRPs

| CRP(\(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\)) | CRP(0,0,1) | CRP(0,1,0) | CRP(1,0,0) |
Universal algorithm

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4 CRPs

| CRP(1/3, 1/3, 1/3) | CRP(0,0,1) | CRP(0,1,0) | CRP(1,0,0) |

100 CRPs

| CRP(1/3, 1/3, 1/3) | ... | CRP(1/7, 2/7, 4/7) | ... | CRP(0,0,1) |
Universal algorithm

- Split money evenly among all CRPs
- Let it sit (i.e. Do not transfer between CRPs)
- 4 CRPs
  \[
  \begin{array}{|c|c|c|}
  \hline
  \text{CRP}\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) & \text{CRP}(0,0,1) & \text{CRP}(0,1,0) \\
  \hline
  \end{array}
  \]
- 100 CRPs
  \[
  \begin{array}{|c|c|c|}
  \hline
  \text{CRP}\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) & \ldots & \text{CRP}\left(\frac{1}{7}, \frac{2}{7}, \frac{4}{7}\right) \\
  \hline
  \end{array}
  \]
- Limit is *Universal* algorithm.
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Proof idea

- Universal achieves avg. wealth of all CRP’s.
- “Near” CRPs do nearly as well.
- Lots of CRP’s are “near” the optimal CRP. (figure?)
Proof.

1. $x$ is “near” $y$ if $x = \frac{n}{n+1} y + \frac{1}{n+1} z$
where $x, y, z \in \beta = \{\text{set of CRPs}\}$ and $n = \# \text{ of days}$
Proof.

1. $x$ is “near” $y$ if $x = \frac{n}{n+1} y + \frac{1}{n+1} z$
   where $x, y, z \in \beta = \{\text{set of CRPs}\}$ and $n = \# \text{ of days}$

2. CRP$_x$ day’s gain $\geq \frac{n}{n+1}$ CRP$_y$ day’s gain

   \[
   \frac{\text{Wealth of CRP}_x}{\text{Wealth of CRP}_y} \geq (\frac{n}{n+1})^n \geq 1/e
   \]
Proof.

1. $x$ is “near” $y$ if $x = \frac{n}{n+1}y + \frac{1}{n+1}z$
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3. Prob\{a random $x$ is “near” $y$\} is

   \[
   \frac{\text{Vol}\{ \frac{n}{n+1}y + \frac{1}{n+1}z | z \in \beta \}}{\text{Vol} \beta} = \frac{\text{Vol}\{ \frac{1}{n+1}z | z \in \beta \}}{\text{Vol} \beta} = \left( \frac{1}{n+1} \right)^{m-1}
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Proof.

1. $x$ is “near” $y$ if $x = \frac{n}{n+1} y + \frac{1}{n+1} z$
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2. $CRP_x$ day’s gain $\geq \frac{n}{n+1}$ $CRP_y$ day’s gain

   \[
   \frac{\text{Wealth of } CRP_x}{\text{Wealth of } CRP_y} \geq \left(\frac{n}{n+1}\right)^n \geq \frac{1}{e}
   \]

3. $\text{Prob}\{\text{a random } x \text{ is “near” } y\}$ is

   \[
   \frac{\text{Vol}\left\{ \frac{n}{n+1} y + \frac{1}{n+1} z | z \in \beta \right\}}{\text{Vol}\beta} = \frac{\text{Vol}\left\{ \frac{1}{n+1} z | z \in \beta \right\}}{\text{Vol}\beta} = \left(\frac{1}{n+1}\right)^{m-1}
   \]

4. \[
   \frac{\text{Wealth of Universal}}{\text{Wealth of best CRP}} \geq \left(\frac{1}{n+1}\right)^{m-1} \frac{1}{e}
   \]
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Transaction Costs

- Fixed % commission charged on purchases, paid for by sales
- CRPs pay commission as well

\[ x = (1 - t)y + tz \]  \hspace{1cm} (1)
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