Eigenvalues of normalized Laplacian matrices of fractal trees and dendrimers: Analytical results and applications

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The eigenvalues of the normalized Laplacian matrix of a network plays an important role in its structural and dynamical aspects associated with the network. In this paper, we study the spectra and their applications of normalized Laplacian matrices of a family of fractal trees and dendrimers modeled by Cayley trees, both of which are built in an iterative way. For the fractal trees, we apply the spectral decimation approach to determine analytically all the eigenvalues and their corresponding multiplicities, with the eigenvalues provided by a recursive relation governing the eigenvalues of networks at two successive generations. For Cayley trees, we show that all their eigenvalues can be obtained by computing the roots of several small-degree polynomials defined recursively. By using the relation between normalized Laplacian spectra and eigentime identity, we derive the explicit solution to the eigentime identity for random walks on the two treelike networks, the leading scalings of which follow quite different behaviors. In addition, we corroborate the obtained eigenvalues and their degeneracies through the link between them and the number of spanning trees.

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I. INTRODUCTION

A central issue in the study of complex systems is to understand the topological structure and to further unveil how various structural properties affect the dynamical processes occurring on diverse systems1–3. It has been established that numerous structural and dynamical properties of a networked system are encoded in eigenvalues and eigenvectors of its standard Laplacian matrix. Frequently cited examples include resistance distance4, relaxation dynamic in the framework of generalized Gaussian structure5,6, fluorescence depolarization by quasiresonant energy transfer7–9, continuous-time quantum walks10–12, and so on. Therefore, it is of extreme importance to study the spectra of standard Laplacian matrices of complex systems. Thus far, the eigenvalues of standard Laplacian matrices for some particular graphs have been found exactly, including regular hypercubic lattices1,13, dual Sierpinski gaskets14–16, Vicsek fractals17,18, Vicsek fractals replicated in the shape of dual Sierpinski gaskets19, dendrimers20, and their dual graphs21,22, as well as some small-world networks with a degree distribution of exponential form23–25. These works presented novel approaches and paradigms for the computation about spectra of standard Laplacian matrices.

In addition to the spectrum of standard Laplacian matrix, the eigenvalues and eigenvectors of normalized Laplacian matrix of a network also contain much important information about its architecture and dynamical processes. In the structural aspect, for example, the product of all nonzero eigenvalues of a connected network determines the number of spanning trees in the network26; the nonzero eigenvalues and their orthonormalized eigenvectors can be used to express the resistor resistance between any pair of nodes27. With respect to dynamical processes, many interesting quantities of random walks are related to the eigenvalues and eigenvectors of normalized Laplacian matrix, including mean first-passage time28–30, mixing time28,29, and Kemeny constant31 or eigentime identity29 that can be used as a measure of efficiency of navigation on the network32.

In view of the relevance, it is equally important to compute and analyze the spectra of normalized Laplacian matrices. However, since normalized Laplacian matrix and standard Laplacian matrix of a network behave quite differently33, the spectra of one matrix can be determined does not means that the spectra of the other matrix can also be evaluated. For instance, the eigenvalues of Laplacian matrix of Vicsek fractals can be determined analytically34–36, but it is difficult (maybe impossible) to derive their spectra of normalized Laplacian matrix in a similar way. Thus, the spectra of standard Laplacian matrix and normalized Laplacian matrix must be treated individually. Relative to standard Laplacian matrices, the spectra of normalized Laplacian matrices have received little attention37–40.

In this paper, we present a theoretical study of the normalized Laplacian matrices for a family of proposed fractal trees and Cayley trees20,41 as a classic model of dendrimers. Both networks are constructed in an iterative way, which makes it possible to analytically study the spectra of their normalized Laplacian matrices. For the fractal trees, by making use of the spectrum decimation approach42,43, we deduce a recursion relation for the eigenvalues at every two successive iterations, and derive

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the multiplicity of each eigenvalue. For Cayley trees, the problem of computing eigenvalue spectra is reduced to determining the roots of some small-degree polynomials that are defined recursively.

As an application, we further derive closed-form expressions of eigentime identity for random walks in both networks being studied, by using the obtained recursive relations for eigenvalues and polynomials. The results show that the dominating terms of the Kemeny constants for the two networks display strongly different behaviors, indicating the effects of topologies on the navigation efficiency. Furthermore, we test the validity of the eigenvalue computations using the connection between the number of spanning trees and the product of nonzero eigenvalues of normalized Laplacian matrix of a network.

II. NETWORK CONSTRUCTIONS AND PROPERTIES

In this section, we introduce the constructions and some relevant properties of a family of fractal trees and Cayley trees. Both networks are constructed in an iterative manner. The special constructions allow us to treat analytically their properties and dynamical processes taking place on them.

A. Fractal trees

We first introduce the construction and properties of a family of fractal trees. Let $F_n$ ($n \geq 0$) denote the proposed fractal trees after $n$ iterations. For $n = 0$, $F_0$ is an edge connecting two nodes. For $n \geq 1$, $F_n$ is obtained from $F_{n-1}$ by performing two operations on each existing edge in $F_{n-1}$ as shown in Fig. 1. The first operation is to replace the edge by a path of 3 links long, with the two endpoints of the path being the same endpoints of the original edge. The second operation is to create $m$ (a positive integer) new nodes for each of two middle nodes in the path, and attach them to the middle node. Figure 2 illustrates a network $F_3$ for the case of $m = 2$.

By construction, at each generation $n_i$ ($n_i \geq 1$), the number of newly introduced nodes is $2(m + 1)$ times the number of edges at generation $n_i - 1$. Let $N_n$ and $E_n$ denote, respectively, the number of nodes and the number of edges in $F_n$. Then, $N_n$ and $E_n$ obey the following relations:

\[ N_n = N_{n-1} + 2(m + 1)E_{n-1} \]  \hspace{1cm} (1)

and

\[ E_n = N_n - 1. \]  \hspace{1cm} (2)

Considering $N_0 = 2$ and $E_0 = 1$, Equations (1) and (2) are solved to obtain:

\[ N_n = (2m + 3)^n + 1 \]  \hspace{1cm} (3)

and

\[ E_n = (2m + 3)^n. \]  \hspace{1cm} (4)

Equation (3) shows that after the evolution of one generation, the number of nodes increases by a factor $f_N = 2m + 3$. In addition, it is easy to check that after each iteration the diameter increases by a factor of $f_D = 3$. Thus, the fractal dimension of the trees is

\[ f_B = \ln f_N / \ln f_D = \ln(2m + 3) / \ln 3. \]

Furthermore, the fractal trees are "large-world" with their diameter growing in a power of the network size as $(N_n)^{\ln 3 / \ln(2m+3)}$.\n
\begin{figure}
\centering
\includegraphics[width=0.8\textwidth]{fractal_trees}
\caption{(Color online) Construction method of the fractal trees. The next generation of the fractal trees can be obtained through replacing each edge of the present generation by the clusters on the right-hand side of the arrow.}
\end{figure}
B. Cayley trees

Let \( C_{b,n} \) (\( b \geq 3, n \geq 0 \)) represent the Cayley trees after \( n \) iterations (generations), which can be built in the following iterative way\(^{20,41}\). Initially (\( n = 0 \)), \( C_{b,0} \) consists of only a central node. To form \( C_{b,n+1} \), we create \( b \) nodes and attach them to the central node. For any \( n > 1 \), \( C_{b,n} \) is obtained from \( C_{b,n-1} \) by performing the following operation. For each boundary node of \( C_{b,n-1} \), \( b-1 \) nodes are generated and attached to the boundary node. Figure 3 illustrates a special Cayley tree, \( C_{3,6} \).

Let \( N_i(n) \) denote the number of nodes in \( C_{b,n} \), which are given birth to at iteration \( i \). It is easy to check that

\[
N_i(n) = \begin{cases} 
1, & i = 0, \\
b(b-1)^{i-1}, & i > 0.
\end{cases}
\]  

Thus, the network size of \( C_{b,n} \) is

\[
N_n = \sum_{i=0}^{n} N_i(n) = \frac{b(b-1)^n - 2}{b-2}.
\]

Different from the fractal trees introduced above, Cayley trees are nonfractal, irrespective of their self-similar architectures. That is, the fractal dimension of Cayley trees is infinite.

After introducing the constructions and features of the fractal trees and Cayley trees, we next study the eigenvalue spectrum of their normalized Laplacian matrices.

III. NORMALIZED LAPLACIAN SPECTRUM OF THE FRACTAL TREES

We now address the spectra of normalized Laplacian matrix of the fractal trees, for which we will give a recursive solution to eigenvalues and determine the degeneracy of each eigenvalue. Moreover, we will use the obtained eigenvalues to compute the eigentime identity of random walks, as well as the number of spanning trees.

A. Eigenvalue spectrum

As is known to us all, the structure of \( F_n \) is encoded in its adjacency matrix \( A_n \), the entry \( A_n(i,j) \) of which is 1 (or 0) if nodes \( i \) and \( j \) are (not) adjacent in \( F_n \).

Then, the normalized Laplacian matrix of \( F_n \), denoted by \( P_n \), is defined as \( P_n = I_n - D_n^{-\frac{1}{2}} A_n D_n^{-\frac{1}{2}} \), where \( I_n \) is the identity matrix of the same order as \( A_n \) and \( D_n \) is the diagonal degree matrix of \( F_n \) with its diagonal entry being the degree, \( d_i(n) \), of node \( i \) in \( F_n \). Notice that \( D_n^{-\frac{1}{2}} A_n D_n^{-\frac{1}{2}} \) is similar to the Markov matrix \( M_n = D_n^{-1} A_n \) of \( F_n \), which can be seen from the equivalent relation \( D_n^{\frac{1}{2}} A_n D_n^{\frac{1}{2}} = D_n^{-1} M_n D_n^{-\frac{1}{2}} \). Hence, both \( D_n^{-\frac{1}{2}} A_n D_n^{-\frac{1}{2}} \) and \( M_n \) have the same set of eigenvalues, although the former is symmetric, while the latter is often asymmetric. Note that if \( \sigma_i(n) \) is an eigenvalue of \( M_n \), then \( 1 - \lambda_i(n) \) is an eigenvalue of the normalized Laplacian matrix of \( P_n \). This one-to-one corresponding relation shows that if one can compute the eigenvalues of one matrix, the eigenvalues of another matrix can be easily found.

Let’s examine the eigenvalue spectrum of \( P_n \). It is easy to verify that the \((i,j)\) entry of \( P_n \) is \( P_n(i,j) = I_n(i,j) - \frac{A_n(i,j)}{\sqrt{d_i(n) d_j(n)}} \), where \( I_n(i,j) \) is the \((i,j)\) entry of \( I_n \). Next we use the decimation technique\(^{42,43}\) to find the eigenvalues of \( P_n \). The decimation method is general and has been applied to calculate the spectra of standard Laplacian matrices of Vicsek fractals\(^{34-36}\) and their extensions\(^{39}\), as well as dual Sierpinski gaskets\(^{14-16}\).

In order to find the recursive solutions to the normalized Laplacian spectra of the fractal trees, we now address the eigenvalue problem for matrix \( P_{n+1} \). Let \( \alpha \) denote the set of nodes belonging to \( F_n \), and \( \beta \) the set of nodes created at \((n+1)\)th iteration. Evidently, \( P_{n+1} \) has a block form

\[
P_{n+1} = \begin{bmatrix}
P_{\alpha,\alpha} & P_{\alpha,\beta} \\
P_{\beta,\alpha} & P_{\beta,\beta}
\end{bmatrix} = \begin{bmatrix}
I_n & P_{\alpha,\beta} \\
P_{\beta,\alpha} & P_{\beta,\beta}
\end{bmatrix},
\]

where we have used the fact \( P_{\alpha,\alpha} = I_n \), which is obvious since there are no edges between any pair of original nodes.

Suppose \( \lambda_i(n+1) \) is an eigenvalue of \( P_{n+1} \), and \( u = (u_\alpha, u_\beta)^\top \) is its corresponding eigenvector, where the superscript \( \top \) stands for transpose and \( u_\alpha \) and \( u_\beta \) correspond to nodes in \( \alpha \) and \( \beta \), respectively. Then, one can write the eigenvalue equation for matrix \( P_{n+1} \) in the following block form:

\[
\begin{bmatrix}
I_n & P_{\alpha,\beta} \\
P_{\beta,\alpha} & P_{\beta,\beta}
\end{bmatrix} \begin{bmatrix}
u_\alpha \\
u_\beta
\end{bmatrix} = \lambda_i(n+1) \begin{bmatrix}
u_\alpha \\
u_\beta
\end{bmatrix},
\]
Equation (8) can be recast into two equations:
\[ I_n u_\alpha + P_{\alpha,\beta} u_\beta = \lambda_i (n+1) u_\alpha, \quad (9) \]
\[ P_{\beta,\alpha} u_\alpha + P_{\beta,\beta} u_\beta = \lambda_i (n+1) u_\beta. \quad (10) \]
Equation (10) implies
\[ u_\beta = [\lambda_i (n+1) - P_{\beta,\beta}]^{-1} P_{\beta,\alpha} u_\alpha, \quad (11) \]
provided that matrix \( \lambda_i (n+1) - P_{\beta,\beta} \) is reversible. Inserting Eq. (11) into Eq. (9) yields
\[ \{ I_n + P_{\alpha,\beta} [\lambda_i (n+1) - P_{\beta,\beta}]^{-1} P_{\beta,\alpha} \} u_\alpha = \lambda_i (n+1) u_\alpha. \quad (12) \]

Let \( T_n = [\lambda_i (n+1) - P_{\beta,\beta}]^{-1} \), the \((i,j)\)th element of which is represented by \( T_n (i,j) \), and let \( Q_n = I_n + P_{\alpha,\beta} [\lambda_i (n+1) - P_{\beta,\beta}]^{-1} P_{\beta,\alpha} = I_n + P_{\alpha,\beta} T_n P_{\beta,\alpha} \). By construction, each edge in \( F_n \) will generate \( 2(m+1) \) new nodes belonging to \( F_{n+1} \), among which two nodes (denoted as \( k \) and \( l \)) have a degree of \( m+2 \), the other \( 2n \) nodes have a single degree. In the Appendix, we prove that
\[ Q_n = \frac{T_n (l,k)}{m+2} P_n + \left( 1 + \frac{T_n (l,l) + T_n (l,k)}{m+2} \right) I_n, \quad (13) \]
which related \( Q_n \) to \( P_n \), and thus enables one to represent the eigenvalues of matrix \( P_{n+1} \) in terms of those of \( P_n \).

Substituting Eq. (13) into Eq. (12) yields
\[ \left[ \left( 1 + \frac{T_n (l,l) + T_n (l,k)}{m+2} \right) I_n - \frac{T_n (l,k)}{m+2} P_n \right] u_\alpha = \lambda_i (n+1) u_\alpha. \quad (14) \]
Define \( x = 1 + \frac{T_n (l,l) + T_n (l,k)}{m+2} \) and \( y = \frac{T_n (l,k)}{m+2} \). Equation (14) can be rewritten as
\[ P_n u_\alpha = \frac{\lambda_i (n+1) - x}{y} u_\alpha. \quad (15) \]
Thus, if \( \lambda_i (n) \) is an eigenvalue of \( P_n \) with associated eigenvector \( u_\alpha \), then
\[ \lambda_i (n) = \frac{\lambda_i (n+1) - x}{y}. \quad (16) \]

Plugging the results for \( T_n (l,l) \) and \( T_n (l,k) \) obtained in the Appendix into Eq. (16), we obtain
\[
(m + 2) [\lambda_i (n+1)]^4 - 4(m + 2) [\lambda_i (n+1)]^3 + (4m^2 + 19m + 21) [\lambda_i (n+1)]^2 - 6m + 9 + \lambda_i (n) [\lambda_i (n+1)] + \lambda_i (n) = 0. \quad (17)
\]
Equation (17) relates \( \lambda_i (n+1) \) to \( \lambda_i (n) \), with each \( \lambda_i (n) \) producing four eigenvalues of \( P_{n+1} \).

B. Multiplicities of eigenvalues

In order determine the multiplicity of each eigenvalue, we first calculate numerically the eigenvalues for some networks of small sizes. The eigenvalues of \( F_0 \) are \( 0 \) and \( 2 \); the eigenvalues of \( F_1 \) are \( 0, 2, \frac{1}{m+2}, \frac{2m+3}{m+2} \) and \( 1 \) eigenvalue 1 has a degeneracy of \( 2m \), while the other four eigenvalues have a single degeneracy. For \( n \geq 2 \), the eigenvalue spectrum of \( F_n \) exhibits the following properties: (i) Eigenvalues 0, 2, \( \frac{1}{m+2} \) and \( \frac{2m+3}{m+2} \) are present at any iteration, each having a single degeneracy. (ii) All eigenvalues present at a given generation \( n_i \) will always exist at its succeeding generation \( n_i + 1 \), and all new eigenvalues at generation \( n_i + 1 \) are just those generated via Eq. (17) by substituting \( \lambda_i (n_i) \) \( \lambda_i (n_i) \neq 0, 2 \) for \( \lambda_i (n_i) \), where \( \lambda_i (n_i) \) are those eigenvalues that are newly added to generation \( n_i \); moreover each new eigenvalue at generation \( n_i + 1 \) keeps the degeneracy of its father.

Using property (ii) of eigenvalues, we now determine the degeneracy of eigenvalue 1 and the multiplicities of its offspring. Let \( D_n^{\text{mul}}(\lambda) \) represent the multiplicity of eigenvalue \( \lambda \) of matrix \( P_n \). And let \( r(M) \) denote the rank of matrix \( M \). Then, the multiplicity of eigenvalue 1 of \( P_n \) is
\[ D_n^{\text{mul}}(\lambda = 1) = N_n - r(P_n - I_n). \quad (18) \]

Thus, to determine \( D_n^{\text{mul}}(\lambda = 1) \), we can alternatively compute \( r(P_n - I_n) \). By using some elementary matrix operations, we can eliminate all nonzero elements at \( P_{n,\beta} \) leading to \( r(P_n - I_n) = r(P_{\beta,\beta} - I_{\beta,\beta}) \), where \( I_{\beta,\beta} \) is the identity matrix with the same order as \( P_{\beta,\beta} \). By construction, it is easy to see that \( r(P_{\beta,\beta} - I_{\beta,\beta}) \) is block diagonal with the rank of each of \( (2m+3)^{n-1} \) blocks being 4, then \( r(P_{\beta,\beta} - I_{\beta,\beta}) = 4(2m+3)^{n-1} - 1 \) and
\[ D_n^{\text{mul}}(\lambda = 1) = (2m - 1)(2m + 3)^{n-1} + 1. \quad (19) \]

Since each resultant eigenvalue of \( P_n \) inherits the multiplicity of its father, the degeneracy of every first-generation offspring of eigenvalue 1 is \( (2m - 1)(2m + 3)^{n-1} + 1 \), the degeneracy of every second-generation offspring of eigenvalue 1 is \( (2m - 1)(2m + 3)^{n-3} + 1 \), and so on. Then, the total number of eigenvalue 1 and its descendants in \( P_n \) \( (n \geq 1) \) is given by
\[ N_n(\lambda_1^{\text{seed}}) = \sum_{i=1}^{n} [(2m - 1)(2m + 3)^{i-1} + 1] \times 4^n - i \]
\[ = (2m + 3)^n - \frac{2^{2n+1} + 1}{3}. \quad (20) \]

Let \( N_n(\lambda_2^{\text{seed}}) \) denote the number of eigenvalue \( \frac{1}{m+2} \) and its descendants of \( P_n \) \( (n \geq 0) \), and \( N_n(\lambda_3^{\text{seed}}) \) the number of eigenvalue \( \frac{2m+3}{m+2} \) and its descendants of \( P_n \). Then, it is obvious that
\[ N_n(\lambda_2^{\text{seed}}) = N_n(\lambda_3^{\text{seed}}) = \sum_{i=1}^{n} 4^n - i = 4^n - \frac{1}{3}. \quad (21) \]
In addition to eigenvalues 1, $\frac{1}{m+2}, \frac{2m+3}{m+2}$ and their offsprings, $P_n$ ($n \geq 1$) has two more eigenvalues 0 and 2. Summing up the number of eigenvalues, we obtain

$$N_n(\lambda_1) + N_n(\lambda_2) + N_n(\lambda_3) + 2 = N_n,$$  \hspace{1cm} (22)

which implies that we have found all eigenvalues of $P_n$.

C. Applications of normalized Laplacian eigenvalues

We next use the above obtained eigenvalues to determine some relevant quantities for the family of fractal trees $F_n$, including the eigentime identity of random walks and the number of spanning trees. Since for any tree, the number of its spanning trees is 1, so our aim for computing spanning trees of $F_n$ is to verify that our computation of eigenvalues of the normalized Laplacian matrix is right.

D. Eigentime identity

Since the normalized Laplacian matrix of a network is derived from its transition matrix that describes isotropic discrete-time random walks on the network, many important quantities about unbiased random walks can be extracted from the eigenvalues of normalized Laplacian matrix. For instance, the sum of reciprocals of each nonzero eigenvalue of normalized Laplacian matrix for a network determines the eigentime identity of random walks on the network, which is a global characteristic of the network, and reflects the architecture of the whole network.

Let $H_{ij}(n)$ be the mean-first passage time from node $i$ to node $j$ in $F_n$, which is the expected time for a particle starting off from node $i$ to arrive at node $j$ for the first time. The stationary distribution for random walks on $F_n$ is $\pi = (\pi_1, \pi_2, \ldots, \pi_N)^T$, where $\pi_i = d_i(n)/(2E_n)$, obeying relations $\sum_{i=1}^{N} \pi_i = 1$ and $\pi^T M_n = \pi^T$. Let $H_n$ represent the eigentime identity for random walks on $F_n$, which is defined as the expected time for a walker going from a node $i$ to another node $j$, chosen randomly from all nodes accordingly to the stationary distribution. That is,

$$H_n = \sum_{j=1}^{N_n} \pi_j H_{ij}(n).$$  \hspace{1cm} (23)

According to previous results, $H_n$ can be expressed in terms of the nonzero eigenvalues of $P_n$ as

$$H_n = \sum_{i=2}^{N_n} \frac{1}{\lambda_i(n)},$$  \hspace{1cm} (24)

where we have assumed that $\lambda_1(n) = 0$. We next explicitly evaluate the sum on the right-hand side of Eq. (24). Let $\Omega_n$ denote the set of all the $N_n - 1$ nonzero eigenvalues of $P_n$, $\Omega_n = \{\lambda_2(n), \lambda_3(n), \ldots, \lambda_{N_n}(n)\}$, where we neglect the distinctness of the elements. We can further classify $\Omega_n$ into three nonoverlapping subsets $\Omega_n^{(1)}$, $\Omega_n^{(2)}$, and $\Omega_n^{(3)}$, satisfying $\Omega_n = \Omega_n^{(1)} \cup \Omega_n^{(2)} \cup \Omega_n^{(3)}$, where $\Omega_n^{(1)}$ only contains eigenvalue 2, $\Omega_n^{(2)}$ consists of eigenvalue 1 and all its offsprings, and $\Omega_n^{(3)}$ includes $\{\frac{1}{m+2}, \frac{2m+3}{m+2}\}$, as well as all their descendants. Then, $H_n$ can be recast as

$$H_n = \sum_{\lambda_i(n) \in \Omega_n^{(1)}} \frac{1}{\lambda_i(n)} + \sum_{\lambda_i(n) \in \Omega_n^{(2)}} \frac{1}{\lambda_i(n)} + \sum_{\lambda_i(n) \in \Omega_n^{(3)}} \frac{1}{\lambda_i(n)}.$$  \hspace{1cm} (25)

It is obvious that $\sum_{\lambda_i(n) \in \Omega_n^{(1)}} \frac{1}{\lambda_i(n)} = \frac{1}{2}$. Hence, we only need to compute the two sum terms $\sum_{\lambda_i(n) \in \Omega_n^{(2)}} \frac{1}{\lambda_i(n)}$ and $\sum_{\lambda_i(n) \in \Omega_n^{(3)}} \frac{1}{\lambda_i(n)}$.

Equation (17) shows that each eigenvalue $\lambda_i(n-1)$ in $\Omega_n^{(2)}$ (or $\Omega_n^{(3)}$) derives four eigenvalues, $\lambda_i(n), \lambda_i(n+1), \lambda_i, \lambda_i(n+4)$, which belong to $\Omega_n^{(2)}$ (or $\Omega_n^{(3)}$). According to Vieta’s formulas, we have

$$\frac{1}{\lambda_{i,1}(n)} + \frac{1}{\lambda_{i,2}(n)} + \frac{1}{\lambda_{i,3}(n)} + \frac{1}{\lambda_{i,4}(n)} = 1 + \frac{6m+9}{\lambda_i(n-1)}$$  \hspace{1cm} (26)

and

$$\lambda_{i,1}(n) \times \lambda_{i,2}(n) \times \lambda_{i,3}(n) \times \lambda_{i,4}(n) = \frac{\lambda_i(n-1)}{(m+2)^2}.$$  \hspace{1cm} (27)

Therefore, for $n \geq 2$, we obtain

$$\sum_{\lambda_i(n) \in \Omega_n^{(2)}} \frac{1}{\lambda_i(n)}$$

$$= (6m+9) \sum_{\lambda_i(n-1) \in \Omega_n^{(2)-1}} \frac{1}{\lambda_i(n-1)}$$

$$+ N_{n-1}(\lambda_1^{seed}) + (2m-1)(2m+3)^{n-1} + 1$$  \hspace{1cm} (28)

and

$$\sum_{\lambda_i(n) \in \Omega_n^{(3)}} \frac{1}{\lambda_i(n)}$$

$$= N_{n-1}(\lambda_2^{seed}) + N_{n-1}(\lambda_3^{seed}) + (m+2) + \frac{m+2}{2m+3}$$

$$+ (6m+9) \sum_{\lambda_i(n-1) \in \Omega_n^{(3)-1}} \frac{1}{\lambda_i(n-1)}.$$  \hspace{1cm} (29)

Then, we have the following recursive relation for $H_n$:

$$H_n = \frac{1}{2}$$

$$= (6m+9) \left( H_{n-1} - \frac{1}{2} \right) + N_{n-1}(\lambda_1^{seed}) + N_{n-1}(\lambda_2^{seed})$$

$$+ N_{n-1}(\lambda_3^{seed}) + (2m-1)(2m+3)^{n-1} + 1$$

$$+ (m+2) + \frac{m+2}{2m+3}.$$  \hspace{1cm} (30)
Using Eq. (22), Eq. (30) is reduced to
\[ H_n = (6m + 9)H_{n-1} + \frac{2m(2m + 3)^n - 4m^2 - 9m - 4}{2m + 3}. \] (31)

Considering \( H_1 = 3m + \frac{5}{2} + \frac{m+2}{2m+3} \), Eq. (31) is solved to yield
\[ H_n = \frac{8(m + 1)^2(6m + 9)^n - 6m^2(2m + 3)^n}{12m^2 + 34m + 24} \]
\[ - \frac{8m(2m + 3)^n - 4m^2 - 9m - 4}{12m^2 + 34m + 24} , \] (32)
which can be expressed as a function of network size \( N_n \) as
\[ H_n = \frac{4(m + 1)^2}{6m^2 + 17m + 12}(N_n - 1)^{1 + \ln 3/\ln(2m+3)} \]
\[ - \frac{6m^2 + 8m}{12m^2 + 34m + 24}N_n \]
\[ + \frac{10m^2 + 17m + 4}{12m^2 + 34m + 24} , \] (33)
Thus, for very large networks, i.e., \( N_n \to \infty \),
\[ H_n \sim (N_n)^{1 + \ln 3/\ln(2m+3)} \]
\[ = (N_n)^{1+1/f_n} , \] (34)
which grows superlinearly with the network size \( N_n \), consistent with the general result in\(^{49}\).

**E. Spanning trees**

In addition to eigentime identity, the eigenvalues of normalized Laplacian matrix of a connected network also determine the number of its spanning trees. From the well established results\(^{26,27}\), the number of spanning trees \( N_{st}(F_n) \) for \( F_n \) is
\[ N_{st}(F_n) = \frac{\prod_{i=1}^{N_n} d_i(n) \prod_{i=2}^{N_n} \lambda_i(n)}{\sum_{i=1}^{N_n} d_i(n)} . \] (35)

Let \( \Phi_n, \Psi_n \) and \( \Lambda_n \) represent \( \sum_{i=1}^{N_n} d_i(n), \prod_{i=1}^{N_n} d_i(n) \) and \( \prod_{i=2}^{N_n} \lambda_i(n) \), respectively. Then, the following two recursive relations hold:
\[ N_{st}(F_n) = \frac{\Psi_n \times \Lambda_n}{\Phi_n} \] (36)
and
\[ N_{st}(F_{n-1}) = \frac{\Psi_{n-1} \times \Lambda_{n-1}}{\Phi_{n-1}} , \] (37)
from which we can derive the relation between \( N_{st}(F_n) \) and \( N_{st}(F_{n-1}) \).

Obviously,
\[ \Phi_n = \sum_{i=1}^{N_n} d_i(n) = 2E_n = 2(3m + 3)^n , \] (38)
leading to
\[ \Phi_n = (3m + 3)\Phi_{n-1} . \] (39)

On the other hand, by the construction and structural properties of the fractal trees, we have
\[ \Psi_n = (m + 2)^{E_{n-1}}\Psi_{n-1} . \] (40)

Finally,
\[ \Lambda_n = \frac{1}{2} \prod_{\lambda_i(n) \in \Omega_n^{(2)}} \lambda_i(n) \prod_{\lambda_i(n) \in \Omega_n^{(2)}} \lambda_i(n) , \] (41)
where the two product terms on the right-hand side satisfy
\[ \prod_{\lambda_i(n) \in \Omega_n^{(2)}} \lambda_i(n) = \left( \frac{1}{m + 2} \right)^{2N_{n-1}(\lambda_3^{med})} \prod_{\lambda_i(n-1) \in \Omega_{n-1}^{(2)}} \lambda_i(n-1) \] (42)
and
\[ \prod_{\lambda_i(n) \in \Omega_n^{(2)}} \lambda_i(n) = \left( \frac{1}{m + 2} \right)^{2[N_{n-1}(\lambda_2^{med})+N_{n-1}(\lambda_1^{med})]} \] \times
\[ \prod_{\lambda_i(n-1) \in \Omega_{n-1}^{(2)}} \lambda_i(n-1) . \] (43)

Thus,
\[ \Lambda_n = \left( \frac{1}{m + 2} \right)^{2[N_{n-1}(\lambda_2^{med})+N_{n-1}(\lambda_1^{med})]} \times
\[ \frac{2m + 3}{(m + 2)^2} \times \Lambda_{n-1} \] (44)

Equations (39), (40), and (44) indicate that
\[ N_{st}(F_n) = N_{st}(F_{n-1}) = 1 , \] (45)
which implies that our computation for eigenvalues of the normalized Laplacian matrix \( P_n \) is correct.

**IV. NORMALIZED LAPLACIAN SPECTRUM FOR CAYLEY TREES**

This section is devoted to the eigenvalue problem and their applications of normalized Laplacian matrices for Cayley trees. Here, we focus on a particular Cayley tree corresponding to \( b = 3 \), since for other values of \( b \) the computation and result is similar but the formulas are very lengthy.
A. Characteristic polynomial and eigenvalues

Since the above method for computing the spectrum of the normalized Laplacian matrix for the fractal trees is not applicable to the Cayley trees, we use the elementary matrix operations to reduce the related matrix to lower triangle matrix.

For the convenience description, we use the same notations as those of the fractal trees. Let $A_n$ and $D_n$ be the adjacency matrix and diagonal degree matrix of $C_{3,n}$. Then, its normalized Laplacian matrix is $P_n = I_n - D_n^{-1}A_n D_n^{-1}$, which has the same set of eigenvalues as matrix $I_n - D_n^{-1}A_n$. Below we concentrate on matrix $I_n - D_n^{-1}A_n$ and still denote it as $P_n$ in the case without confusion.

Note that the numbers of nodes and edges in $C_{3,n}$ are $N_n = 3 \times 2^n - 2$ and $E_n = 3 \times 2^n - 3$, respectively. The $N_n = 3 \times 2^n - 2$ nodes can be divided into $n + 1$ levels: the 0th level contains only one node (i.e., the central node) labeled by 1; the $i$th $(1 \leq i \leq n)$ level has $N_i(n) = N_i - N_{i-1}$ nodes, which are labeled sequentially by $N_{i-1} + 1, N_{i-1} + 2, \cdots, N_i$. Here $N_i$ is defined by Eq. (6).

We now address the eigenvalue problem of $P_n$. By definition, all eigenvalues of $P_n$ are actually the roots of characteristic equation $\det (P_n - \lambda I_n) = 0$. Let $X_n = P_n - \lambda I_n$. Then $X_n$ is a determinant of order 3 $\times$ 2 $-2$. Next we apply the row operations of determinants to transform $X_n$ into a lower triangle determinant.

Let $R_k$ represent the $k$th row of $X_n$ and its variants after being performed row operations. In order to have a lower triangle matrix, the row operations are performed as follows. First, we keep rows $R_k (N_{n-1}+1 \leq k \leq N_n)$ unchanged, and define their diagonal entries as $f_1(\lambda) = 1 - \lambda$ as in the original matrix. Then for each $1 \leq i \leq n$, we repeat the following two operations: (i) For each $k (N_{n-i-1}+1 \leq k \leq N_{n-i})$, we multiply $R_k$ by $f_i(\lambda)$; (ii) For each $k (N_{n-i-1}+1 \leq k \leq N_{n-i})$, we add the sum of $R_{2k+1}$ and $R_{2k+2}$ times $\frac{1}{2}$ to $R_k$. Note that we have assumed $N_{-1} = 0$. After performing the two operations, the diagonal entry of $R_k (N_{n-i-1}+1 \leq k \leq N_{n-i})$ becomes $f_i(\lambda)$, while the other entries of $R_k$ on the right-hand side of the diagonal entry $f_i(\lambda)$ are zeros. Finally, we add $R_2$ times $\frac{1}{2}$ to $R_1$.

The above row operations reduce matrix $X_n$ to lower triangle matrix $Y_n$, the diagonal elements of which are as follows: $f_{n+1}(\lambda) - \frac{1}{2} f_{n-1}(\lambda)$ for the first row, $f_i(\lambda) (1 \leq i \leq n)$ for those rows starting from $N_{n-i+1}$ to $N_{n-i}$.

It follows above row operations that the functions $f_i(\lambda)$ obey the following recursive relations:

\[
 f_i(\lambda) = \begin{cases} 
 1 - \lambda, & i = 1, \\
 \lambda^2 - 2\lambda + \frac{1}{3}, & i = 2, \\
 (1 - \lambda)f_{i-1}(\lambda) - \frac{2}{3} f_{i-2}(\lambda), & 3 \leq i \leq n+1.
 \end{cases}
 \]  

(46)

According to the properties of determinants, we have $\det X_n = \det Y_n / D(\lambda)$, where $D(\lambda)$ is the overall factor. From the above procedure, we can obtain

\[
 D(\lambda) = f_n(\lambda) \prod_{i=1}^{n-1} (f_i(\lambda))^{3 \times 2^{n-i-1}}.
 \]  

(47)

Thus, we have

\[
 \det X_n = \frac{[f_n(\lambda)]^3 [f_{n+1}(\lambda) - \frac{1}{2} f_{n-1}(\lambda)] \prod_{i=1}^{n-1} [f_i(\lambda)]^{3 \times 2^{n-i}}}{D_n(\lambda)}
 = \frac{[f_n(\lambda)]^2 [f_{n+1}(\lambda) - \frac{1}{2} f_{n-1}(\lambda)] \prod_{i=1}^{n-1} [f_i(\lambda)]^{3 \times 2^{n-i-1}}}{D_n(\lambda)}.
 \]  

(48)

Since the eigenvalues of $P_n$ are the roots of $\det X_n = 0$ and the right-hand side of Eq. (48) is factorized, the problem of computing eigenvalues of $P_n$ becomes to find the roots of functions $f_n(\lambda)$ and $f_{n+1}(\lambda) - \frac{1}{2} f_{n-1}(\lambda)$. From Eq. (46), it is obvious that $f_i(\lambda)$ is a polynomial of $\lambda$ with degree $i$. Every $f_i(\lambda)$ provides $i$ different roots, each of which is an eigenvalue of $P_n$ having a multiplicity $3 \times 2^{n-i-1}$, with the exception of $f_n(\lambda)$, whose roots are 2-fold degenerate. In addition, $f_{n+1}(\lambda) - \frac{1}{2} f_{n-1}(\lambda)$ generates $n + 1$ eigenvalues. Thus the total number of eigenvalues is

\[
 2n + (n+1) + \sum_{i=1}^{n-1} (3i \times 2^{n-i-1}) = 3 \times 2^n - 2 = N_n,
 \]  

(49)

implying that all eigenvalues of $P_n$ are produced by $f_i(\lambda) = 0 (i = 1, 2, \ldots, n)$ and $f_{n+1}(\lambda) - \frac{1}{2} f_{n-1}(\lambda) = 0$.

B. Applications of normalized Laplacian eigenvalues

Although we failed to obtain the explicit recursive expressions for the eigenvalues of $P_n$, below we will show that the above obtained characteristic polynomial for $P_n$ contains enough information of the eigenvalues such that we can use it to determine the eigenvalue identity and the number of spanning trees in Cayley trees.

C. Eigentime identity

For each $f_i(\lambda)$ $(1 \leq i \leq n)$, it is an $i$-degree polynomial, and we can rewrite it as:

\[
 f_i(\lambda) = \alpha_i + \beta_i \lambda + \gamma_i \lambda^2 + \cdots.
 \]  

(50)

Thus, equation $f_i(\lambda) = 0$ has $i$ nonzero roots, labeled by $\lambda^{(i)}_1, \lambda^{(i)}_2, \cdots, \lambda^{(i)}_i$. According to Vieta’s formulas, we have

\[
 \sum_{j=1}^{i} \frac{1}{\lambda^{(i)}_j} = -\frac{\beta_i}{\alpha_i},
 \]  

(51)
and
\[ \prod_{j=1}^{i} \lambda_j^{(i)} = \alpha_i , \quad (52) \]

where we have used the fact that the coefficient of the term \( \lambda^j \) in \( f_1(\lambda) \) equals \((-1)^j\).

From the recursion relation in Eq. (46), we obtain
\[ \alpha_{i+1} = \alpha_i - \frac{2}{9} \alpha_{i-1} , \quad (53) \]
\[ \beta_{i+1} = \beta_i - \frac{2}{9} \beta_{i-1} , \quad (54) \]
and
\[ \gamma_{i+1} = \gamma_i - \frac{2}{9} \gamma_{i-1} . \quad (55) \]

Considering the initial conditions \( \alpha_0 = 1, \beta_0 = -1, \gamma_0 = 0 \), \( \alpha_1 = \frac{1}{4}, \beta_2 = -2 \) and \( \gamma_2 = 1 \), Eqs. (53-55) are solved to yield
\[ \alpha_i = 3^{-(i-1)} , \quad (56) \]
\[ \beta_i = -3^{-(i-1)} (2^{i+2} - 3i - 4) , \quad (57) \]
and
\[ \gamma_i = \frac{1}{2} 3^{-(i-2)} (2^{i+3} i - 9 \times 2^{i+2} + 3i^2 + 17i + 36) . \quad (58) \]

Therefore, for any \( 1 \leq i \leq n \),
\[ \sum_{j=1}^{i} \frac{1}{\lambda_j^{(i)}} = - \frac{\beta_i}{\alpha_i} = 2^{i+2} - 3i - 4 , \quad (59) \]
and
\[ \prod_{j=1}^{i} \lambda_j^{(i)} = 3^{-(i-1)} , \quad (60) \]

On the other hand, we can see that the only zero eigenvalue of \( P_n \) is generated by \( f_{n+1}(\lambda) - \frac{1}{9} f_{n-1}(\lambda) = 0 \), since
\[ f_{n+1}(\lambda) - \frac{1}{9} f_{n-1}(\lambda) = \lambda \left[ (\beta_{n+1} - \frac{1}{9} \beta_{n-1}) + (\gamma_{n+1} - \frac{1}{9} \gamma_{n-1}) \lambda + \cdots \right] , \quad (61) \]
which implies that the \( n \) nonzero roots, \( \lambda_1^{(n+1)}, \lambda_2^{(n+1)}, \ldots, \lambda_n^{(n+1)} \), satisfy
\[ \sum_{j=1}^{n} \frac{1}{\lambda_j^{(n+1)}} = - \frac{\gamma_{n+1} - \frac{1}{9} \gamma_{n-1}}{\beta_{n+1} - \frac{1}{9} \beta_{n-1}} . \quad (62) \]

Let \( \lambda_j(n) \) (\( j = 1, 2, \ldots, n \)) denote the \( n \) eigenvalues of the normalized Laplacian matrix \( P_n \) of the Cayley tree \( C_{3,n} \), where we assume that \( \lambda_1(n) = 0 \). Then, the eigenvalue identity for random walks on \( C_{3,n} \) is given by
\[ H_n = \sum_{j=2}^{N_n} \frac{1}{\lambda_j(n)} = - \left[ \sum_{i=1}^{n-1} 3 \times 2^{n-i-1} \times \frac{\beta_i}{\alpha_i} \right] - 2 \times \frac{\beta_n}{\alpha_n} - \frac{\gamma_{n+1} - \frac{1}{9} \gamma_{n-1}}{\beta_{n+1} - \frac{1}{9} \beta_{n-1}} \]
\[ = 3n \times 4^{n+1} - 13 \times 2^{2n+1} + 35 \times 2^n - 9 \]
\[ \sim 3n \times 4^{n+1} - 13 \times 2^{2n+1} + 35 \times 2^n - 9 \], \quad (64)
which can be expressed in terms of network size \( N_n \) as
\[ H_n = \frac{2(N_n)^2 \ln(N_n + 2)}{(N_n - 1) \ln 2} - \frac{(N_n)^2(13 \ln 2 + 6 \ln 3)}{3(N_n - 1) \ln 2} \]
\[ + \frac{8N_n \ln(N_n + 2)}{(N_n - 1) \ln 2} + \frac{N_n(248 \ln 3)}{6(N_n - 1) \ln 2} \]
\[ + \frac{8 \ln(N_n + 2)}{(N_n - 1) \ln 2} + \frac{25 \ln 2 - 48 \ln 3}{6(N_n - 1) \ln 2} . \quad (65) \]
For very large networks (i.e., \( N_n \to \infty \)), the leading term of \( H_n \) obey
\[ H_n \sim N_n \ln N_n , \quad (66) \]
which is in sharp contrast to its counterpart of the fractal trees \( F_n \) obtained in Sec. III D. Note that for the case \( b > 3 \), the dominating term of \( H_n \) follows the same scaling as that corresponding to the case \( b = 3 \).

D. Spanning trees

From above results, we can also obtain the product of nonzero eigenvalues for \( P_n \):
\[ \prod_{j=2}^{N_n} \lambda_j(n) = \prod_{i=1}^{n-1} (\alpha_i)^3 \times (2^{2n-i-1}) \times (\alpha_n)^2 \times \left( \frac{1}{9} \beta_{n-1} - \beta_{n+1} \right) \]
\[ = 2(2^n - 1) 3^{-\frac{3}{2}(2^n-2)} , \quad (67) \]
In addition, we can derive
\[ \prod_{i=1}^{N_n} d_i(n) = 3^{N_{n-1}} = 3^{3 \times 2^{n-1} - 2} , \quad (68) \]
and
\[ \sum_{i=1}^{N_n} d_i(n) = 3N_{n-1} + (N_n - N_{n-1}) = 6 \times 2^n - 6. \quad (69) \]

Then, the number of spanning trees in \( C_{3,n} \) is
\[ N_{st}(C_{3,n}) = \prod_{i=1}^{N_n} d_i(n) \prod_{i=2}^{N_n} \lambda_i(n) \sum_{i=1}^{N_n} d_i(n) = 1, \quad (70) \]

which means that our computation about the characteristic polynomial and eigenvalues for related matrix of \( C_{3,n} \) is valid.

V. CONCLUSIONS

The eigenvalue spectrum of the normalized Laplacian matrix of a network is relevant in the topological aspects and random-walk dynamics that is closely related to a large variety of other dynamical processes of the network. In this paper, we have studied the eigenvalue problem of the normalized Laplacian matrices of a class of fractal trees and Cayley trees, with the latter being a typical model of dendrimers. Both networks under study are constructed in an iterative way, which allows to analytically treat the eigenvalues and their degeneracies.

For the fractal trees, by applying the decimation technique, we have provided an exact recursive relation governing the eigenvalues of the networks at two consecutive generations. Then, we have also derived the multiplicity of each eigenvalue. For the Cayley trees, through the elementary matrix operations we have reduced the related matrix to lower triangle matrix and reduced the eigenvalue problem to computing the roots of some polynomials of very small degrees, with the polynomials being defined recursively.

On the basis of these obtained recursion relations of eigenvalues for fractal trees and of related polynomials for Cayley trees, we have further evaluated the eigen-time identity for random walks on the two networks, the leading scalings of which display disparate behaviors. In addition, we have verified our computation and results for the eigenvalues of the normalized Laplacian matrices for the two different types of networks by enumerating the number of spanning trees in them.

The fractal trees being studied have a similar topology to the Vicsek fractals modeling regular hyperbranched polymers, both fractals are thus suggested to exhibit similar dynamical behaviors, e.g., superlinear growth of eigen-time identity. Since eigen-time identity is an important quantity rooted in the inherent network topology, the distinct scalings of eigen-time identity enable to distinguish two classic macromolecules—dendrimers and hyperbranched polymers. On the other hand, although we have limited our analysis to unweighted treelike networks, the approaches and procedures applied here could be extended to the case of weighted and directed networks, even those with loops.

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Appendix: Proof of equation (13)

The \( N_{n-1} - N_n \) nodes in set \( \beta \) can be divided into \( E_n \) sections. All nodes in each section associates an edge in \( E_n \) that generates these nodes. Concretely, each section contains \( 2(m+1) \) nodes, among which two nodes have a degree of \( m + 2 \), the other \( 2m \) nodes have a single degree. The two nodes with degree \( m + 2 \) are connected to each other by an edge; moreover, each multiple-degree node is linked to \( m \) single-degree nodes in the same section. Notice that there is no edge between any pair of nodes belonging to different sections. Hence, \( P_{3,\beta} \) is a block diagonal matrix including \( E_n \) blocks, with each block having the form

\[
\begin{bmatrix}
1 & -\frac{1}{\sqrt{d}} & \ldots & -\frac{1}{\sqrt{d}} & -\frac{1}{3} & 0 & \ldots & 0 \\
-\frac{1}{\sqrt{d}} & 1 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-\frac{1}{\sqrt{d}} & 0 & \ldots & 1 & 0 & 0 & \ldots & 0 \\
-\frac{1}{\sqrt{d}} & 0 & \ldots & 0 & 1 & -\frac{1}{\sqrt{d}} & \ldots & -\frac{1}{\sqrt{d}} \\
0 & 0 & \ldots & 0 & -\frac{1}{\sqrt{d}} & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 1
\end{bmatrix}
\]

\[(A.1)\]

where \( d = m + 2 \) represents the degree of the two non-single-degree nodes, denoted by \( k \) and \( l \), respectively. According to above arguments, we can conclude that \( T_n = [\lambda_i(n+1) - P_{3,\beta}]^{-1} \) is also block diagonal, with each of \( E_n \) blocks being identical. It is not difficult to derive that their entries satisfy relations \( T_n(k, k) = T_n(l, l) \) and \( T_n(l, k) = T_n(k, l) \), the exact expressions of which are...
\[ T_n(k, k) = \frac{(m + 2)^2[\lambda_i(n + 1)]^3 - 3(m + 2)^2[\lambda_i(n + 1)]^2 + (2m + 10m + 12)\lambda_i(n + 1) - 2m - 4}{(m + 2)^2[\lambda_i(n + 1)]^4 - 4(m + 2)^2[\lambda_i(n + 1)]^3 + (4m^2 + 20m + 23)\lambda_i(n + 1)^2 - (8m + 14)\lambda_i(n + 1) + 3} \] (A.2)

\[ T_n(l, k) = \frac{-(m + 2)[\lambda_i(n + 1)]^2 + (2m + 4)\lambda_i(n + 1) - m - 2}{(m + 2)^2[\lambda_i(n + 1)]^4 - 4(m + 2)^2[\lambda_i(n + 1)]^3 + (4m^2 + 20m + 23)\lambda_i(n + 1)^2 - (8m + 14)\lambda_i(n + 1) + 3} \] (A.3)

which are useful to derive Eq. (17) in the main text.

We proceed to determine the entries \( Q_n(i, j) \) of \( Q_n = I_n + P_{\beta, \beta}[\lambda_i(n + 1) - P_{\beta, \beta}]^{-1}P_{\beta, \alpha} \). By definition,

\[ Q_n(i, j) = \delta_{i,j} + \frac{1}{\sqrt{d_i(n + 1)d_j(n + 1)}} \sum_{u \in \beta, v \in \beta} \frac{T_n(u,v)}{\sqrt{d_u(n + 1)d_v(n + 1)}} \] (A.4)

where \( i \sim u \) means that nodes \( i \) and \( u \) are adjacent in \( F_{n+1} \), and \( \delta_{i,j} \) is defined as: \( \delta_{i,j} = 1 \) if \( i = j \), \( \delta_{i,j} = 0 \) otherwise. Note that for nodes in \( \beta \) set, only those with multiple degrees have a link connected to an old node in set \( \alpha \). Then, \( d_u(n + 1) = d_v(n + 1) = m + 2 \) and Eq. (A.4) becomes

\[ Q_n(i, j) = \delta_{i,j} + \frac{1}{(m + 2)^2d_i(n + 1)d_j(n + 1)} \sum_{u \in \beta, v \in \beta} \frac{T_n(u,v)}{\sqrt{d_u(n + 1)d_v(n + 1)}} \] (A.5)

Equation (A.5) can be simplified by distinguishing two cases: \( i = j \) and \( i \neq j \). For the case \( i = j \), we have

\[ Q_n(i, i) = 1 + \frac{d_i(n + 1)}{(m + 2)d_i(n + 1)} T_n(u, u) = 1 + \frac{T_n(l, l)}{m + 2} \] (A.6)

For the other case \( i \neq j \),

\[ Q_n(i, j) = T_n(u, v) \frac{1}{m + 2} \frac{d_i(n + 1)}{d_i(n + 1)} T_n(u, v) = 1 + \frac{T_n(l, l)}{m + 2} \] (A.7)

Moreover, if nodes \( i \) and \( j \) are not adjacent in \( F_n \), the non-diagonal elements \( Q_n(i, j) \) of \( Q_n \) are zeros, because in this case \( u \) and \( v \) are in different sections, leading to \( T_n(u, v) = 0 \). Otherwise, if nodes \( i \) and \( j \) are directly connected by an edge in \( F_n \), then \( T_n(u, v) = T_n(l, k) \) with the latter given by Eq. (A.3). Notice that for any old node \( i \) in \( \alpha \), we have \( d_i(u) = d_i(n + 1) \). Thus, Eq. (A.7) can be recast as

\[ Q_n(i, j) = T_n(l, k) \frac{A_n(i, j)}{m + 2 \sqrt{d_i(n + 1)d_j(n)}} = T_n(l, k) \frac{A_n(i, j)}{m + 2} P_n(i, j) \] (A.8)

Combining the results provided by Eq. (A.6) and (A.8), we can obtain

\[ Q_n = -\frac{T_n(l, k)}{m + 2} P_n + \left(1 + \frac{T_n(l, l) + T_n(l, k)}{m + 2}\right) I(A.9) \]

This completes the proof of Eq. (13).