What is a Logic?

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October 16, 2004

TODO: Joseph: Our cardinality spectra generalize traditional cardinality properties from functions of signature cardinality, to functors of signature isomorphism type.

TODO: Replace diagram by equations? Vertical spaces as small as possible; combine paragraphs when possible; reduce model theory details; be cruel with references, use smaller font, less space between refs; better explanations of key material for main point.

TODO: Razvan: mention new results for classical logics obtained by institution theory: Craig interpolation of some fragments of classical logic (see Studia Logica paper).

1 Introduction

Logic is often informally described as the study of *sound reasoning*. As such, it plays a crucial role in several areas of mathematics (especially foundations) and of computer science (especially formal methods), as well as in other fields, such as analytic philosophy and formal linguistics. In an enormous development beginning in the late 19th century, it has been found that a wide variety of different principles are needed for sound reasoning in different domains, and "a logic" has come to mean a set of principles for some form of sound reasoning. But in a subject the essence of which is formalization, it is embarrassing that there is no widely acceptable formal definition of "a logic". It is clear that two key problems here are to define what it means for two presentations of a logic to be equivalent, and to provide effective means to demonstrate equivalence and inequivalence.

This essay addresses these problems using the notion of "institution", which arose within computer science in response to the recent population explosion among logics in use there, with the ambition of doing as much as possible at a level of abstraction independent of commitment to any particular logic [16, 19]. It addresses the *soundness* aspect of sound reasoning by axiomatizing the notion of satisfaction, and it addresses the *reasoning* aspect by calling upon categorical logic, in consonance with its general use of categorical language. In fact, a great deal can be done just with satisfaction (e.g. giving general foundations for modularization of specifications and programs), and much of the institutional literature considers sentences without proofs and models without (homo)morphisms. Richer variants of the institution notion consider entailment relations on sentences and/or morphisms of models, so that they form categories; using proof terms as sentence morphisms provides a richer variant which fully supports proof theory. We call these the set/set, set/cat, cat/set, and cat/cat variants (where the first term refers to sentences, and the second to models); the table in Theorem 6.21 summarizes many of their properties. See [19] for more about the variant notions of institution, and [16, 30, 10, 11, 13, 28] for some non-trivial results in abstract model theory done institutionally.

This essay adds to the literature on institutions a notion of equivalence, such that a logic is an equivalence class of institutions. To support this thesis, we consider a number of logical properties that are, and that are not, preserved under equivalence, and apply them to a number of examples. Perhaps the most interesting invariants are versions of the Lindenbaum algebra; some others concern cardinality of models. We also develop a normal form for institutions under our notion of equivalence, by extending the categorical notion of "skeleton."

We extend the Lindenbaum algebra construction to a *Lindenbaum category* construction, defined on any institution with proofs, by identifying not only equivalent sentences, but also equivalent proofs. We show that this construction is an invariant, i.e., preserved up to isomorphism by our equivalence on institutions. This construction extends the usual approach of categorical logic by having sets of sentences as objects, rather than just single sentences, and thus allows treating a much larger class of logics in a uniform way.

A perhaps unfamiliar feature of institutions is that satisfaction is not dyadic, but rather triadic, a relation among sentence, model, and "signature", where signatures form a category the objects of which are thought of as vocabularies over which the sentences are constructed. In concrete cases, these may be propositional variables, relation symbols, function symbols, and so on. Since these form a category, it is natural that the constructions of sentences (or formulae) and models appear as functors on this category, and it is also natural to have an axiom expressing the invariance of "truth" (i.e., satisfaction) under change of notation. See Definition 2.1 below. When the vocabulary is fixed, the category of signatures is the one morphism category $\mathbb{1}$. (Another device can be used to

eliminate models, giving pure proof theory as a special case, if desired.) If $\sigma\colon \Sigma \longrightarrow \Sigma'$ is an inclusion of signatures, then its application to models (via the model functor) is "reduct." The institutional triadic satisfaction can be motivated philosophically by arguments like those given by Peirce for his "interpretants," which allow for context dependency of denotation in his semiotics, as opposed to Tarski's dyadic satisfaction. We also use this feature to resolve a problem about cardinality raised in [2]; see Example 2.2.

2 Institutions and Logics

We assume the reader is familiar with basic notions from category theory; e.g., see [1, 23] for introductions to this subject. By way of notation, $|\mathbb{C}|$ denotes the class of objects of a category $\mathbb{C}$, and composition is denoted by "$\circ$". The basic concept of this paper in its set/cat variant is as follows[1]:

Definition 2.1 An *institution* $I = (\mathbb{S}ign^{\mathcal{I}}, \mathsf{Sen}^{\mathcal{I}}, \mathsf{Mod}^{\mathcal{I}}, \models^{\mathcal{I}})$ consists of

1. a category $\mathbb{S}ign^{\mathcal{I}}$, whose objects are called *signatures*,
2. a functor $\mathsf{Sen}^{\mathcal{I}}\colon \mathbb{S}ign^{\mathcal{I}} \to \mathbb{S}et$, giving for each signature a set whose elements are called *sentences* over that signature,
3. a functor $\mathsf{Mod}^{\mathcal{I}}\colon (\mathbb{S}ign^{\mathcal{I}})^{\mathrm{op}} \to \mathbb{CAT}$ giving for each signature Σ a category whose objects are called Σ-*models*, and whose arrows are called Σ-(*model*) *homomorphisms*,[2] and
4. a relation $\models^{\mathcal{I}}_{\Sigma} \subseteq |\mathsf{Mod}^{\mathcal{I}}(\Sigma)| \times \mathsf{Sen}^{\mathcal{I}}(\Sigma)$ for each $\Sigma \in |\mathbb{S}ign^{\mathcal{I}}|$, called Σ-*satisfaction*,

such that for each morphism $\sigma\colon \Sigma \to \Sigma'$ in $\mathbb{S}ign^{\mathcal{I}}$, the *satisfaction condition*

$$M' \models^{\mathcal{I}}_{\Sigma'} \mathsf{Sen}^{\mathcal{I}}(\sigma)(\varphi) \quad \text{iff} \quad \mathsf{Mod}^{\mathcal{I}}(\sigma)(M') \models^{\mathcal{I}}_{\Sigma} \varphi$$

holds for each $M' \in |\mathsf{Mod}^{\mathcal{I}}(\Sigma')|$ and $\varphi \in \mathsf{Sen}^{\mathcal{I}}(\Sigma)$. We denote the *reduct* function $\mathsf{Mod}^{\mathcal{I}}(\sigma)$ by $\_\restriction_{\sigma}$ and the sentence translation $\mathsf{Sen}^{\mathcal{I}}(\sigma)$ by $\sigma(\_)$. When $M = M'\restriction_{\sigma}$ we say that M' is a σ-*expansion of* M.

A *set/set institution* is an institution where each model category is discrete; this means that the model functor actually becomes a functor $\mathsf{Mod}^{\mathcal{I}}\colon (\mathbb{S}ign^{\mathcal{I}})^{\mathrm{op}} \to \mathbb{C}lass$ into the quasi-category of classes and functions.
General assumption: We asssume that all institutions are such that satisfaction is invariant under model isomorphism, i.e. if Σ-models M, M' are isomorphic, then $M \models_{\Sigma} \varphi$ iff $M' \models_{\Sigma} \varphi$ for all Σ-sentences φ. □

We now consider classical propositional logic, perhaps the simplest non-trivial example (see the extensive discussion in [2]), and also introduce some derived concepts of the theory of institutions:

Example 2.2 Fix a countably infinite[3] set $\mathcal{X}$ of variable symbols, and let $\mathbb{S}ign$ be the category with finite subsets Σ of $\mathcal{X}$ as objects, and with inclusions as morphisms (or all set maps, if prefered, it matters little). Let $\mathsf{Mod}(\Sigma)$ have $[\Sigma \to \{0,1\}]$ (the set of functions from Σ to $\{0,1\}$) as its objects; these models are the row labels of truth tables. Let a (unique) Σ-model homomorphism $h\colon M \longrightarrow M'$ exist iff for all $p \in \Sigma$, $M(p) = 1$ implies $M'(p) = 1$. Let $\mathsf{Mod}(\Sigma' \hookrightarrow \Sigma)$ be the restriction map $[\Sigma \to \{0,1\}] \to [\Sigma' \to \{0,1\}]$. Let $\mathsf{Sen}(\Sigma)$ be the (absolutely) free algebra generated by Σ over the propositional connectives (we soon consider different choices), with $\mathsf{Sen}(\Sigma \hookrightarrow \Sigma')$ the evident inclusion. Finally, let $M \models_{\Sigma} \varphi$ mean that φ evaluates to true (i.e., 1) under the assignment M. It is easy to verify the satisfaction condition, and to see that φ is a tautology iff $M \models_{\Sigma} \varphi$ for all $M \in \mathsf{Mod}(\Sigma)$. Let **CPL** denote this institution of propositional logic, with the connectives conjunction, disjunction, negation, implication, true and false. Let $\mathbf{CPL}^{\neg,\wedge,false}$ denote propositional logic with negation, conjunction and false, and $\mathbf{CPL}^{\neg,\vee,true}$ with propositional logic negation, disjunction and true[4].

This arrangement puts truth tables on the side of semantics, and formulas on the side of syntax, each where it belongs, instead of trying to treat them the same way. It also solves the problem raised in [2] that the cardinality of $\mathcal{L}(\Sigma)$ varies with that of Σ, where $\mathcal{L}(\Sigma)$ is the quotient of $\mathsf{Sen}(\Sigma)$ by the semantic equivalence $\models|_{\Sigma}$, defined by $\varphi \models|_{\Sigma} \varphi'$ iff ($M \models_{\Sigma} \varphi$ iff $M \models_{\Sigma} \varphi'$, for all $M \in \mathsf{Mod}(\Sigma)$); it is the Lindenbaum algebra, in this case, the free Boolean algebra over Σ, and its cardinality is 2^{2^n} where n is the cardinality of Σ. Hence this cardinality cannot be considered an invariant of **CPL** without the parameterization by Σ (see also Definition 5.3 below). □

[1] A more concrete definition is given in [20], which avoids category theory by spelling out the conditions for functorality, and assuming a set theoretic construction for signatures. Though less general, this definition is sufficient for everything in this paper; however, it would greatly complicate our exposition. Our use of category theory is modest, oriented towards easy proofs of very general results, which is precisely what is needed for the goals of this essay.

[2] $\mathbb{CAT}$ is the category of all categories. Strictly speaking, this is only a quasi-category living in a higher set-theoretic universe.

[3] The definition also works for finite or uncountable $\mathcal{X}$.

[4] We include a truth constant because otherwise the empty signature would have no sentences at all.

The moral is that everything should be parameterized by signature. Although the construction of the underlying set of the Lindenbaum algebra above works for any institution, its algebraic structure depends on how sentences were defined. However, Section 4 shows how to obtain at least part of this structure for any institution.

Example 2.3 The institution **FOLR** of first-order logic with unsorted relations. A **FOLR** signature Σ is a a family $(\Sigma_n)_{n\in\mathbb{N}}$ of sets of relation symbols of arity n, and a **FOLR** signature morphism $\sigma\colon\Sigma\longrightarrow\Sigma'$ is a family $\sigma_n\colon\Sigma_n\longrightarrow\Sigma'_n$ of arity-preserving functions on relation symbols. An **FOLR** Σ-sentence is of course a closed first-order formula using relation symbols in Σ, and sentence translation is just relation symbol substitution. A **FOLR** Σ-model consists of a set M called the carrier and a subset $R_M\subseteq M^n$ for each $R\in\Sigma_n$. Model translation is reduct with relation translation. A Σ-model morphism $h\colon M\longrightarrow M'$ is a function between the carriers such that $h(R_M)\subseteq R_{M'}$ for all R in Σ. Satisfaction is as usual. The institution **FOL** adds function symbols to **FOLR** in the usual way, and **MSFOL** is its many sorted variant. □

Example 2.4 The institution **Eq** of many sorted equational logic. Signatures are many sorted algebraic signatures consisting of a set of sorts and a set of function symbols (where each function symbol has a string of argument sorts and a result sort). Signature morphisms map sorts and function symbols in a compatible way. Models are just many sorted algebras, i.e. each sort is interpreted as a carrier set, and each function symbol is interpreted as a function between the carrier sets specified by the argument and result sorts. Reducts are constructed as sketched above. Sentences are universally quantified equations between many sorted terms, and sentence translation means replacement of the translated symbols (assuming that variables of distinct sorts never coincide in an equation). Finally, satisfaction is the usual satisfaction of an equation in an algebra. □

Example 2.5 **K** is propositional modal logic, **CPL** with $\Box$ and $\Diamond$ as additional logical connectives. Models are Kripke structures, and satisfaction is defined using possible-world semantics in the usual way. **IPL** is intuitionistic propositional logic, differing from **CPL** in having Kripke structures as models, and possible-world satisfaction. The proof theory of **IPL** (which is considered superior to the model theory by intuitionists) will be discussed in Section 6 below. □

Both intuitionistic and modal logic also come with first-order variants, and both also have variants with constant and with varying domains. Other modal logics restrict **K** by further axioms, such as **S4** or **S5**. All these can be formalized as institutions. So can be substructural logics like linear logic; here one has to take judgements of the form $\varphi_1\ldots\varphi_n\vdash\psi$ as sentences. Also, higher-order [6], polymorphic [29], temporal [15], process [15], behavioural [3], coalgebraic [8] and object-oriented [17] logics have been shown to be institutions. Many familiar basic concepts can be defined over any institution:

Definition 2.6 Given a set of Σ-sentences Γ and a Σ-sentence φ, then φ is a *logical consequence* of Γ, written $\Gamma\models_\Sigma\varphi$, iff for all Σ-models M, we have $M\models_\Sigma\Gamma$ implies $M\models_\Sigma\varphi$, where $M\models_\Sigma\Gamma$ (*semantic entailment*) means $M\models_\Sigma\psi$ for each $\psi\in\Gamma$. Two sentences are *semantically equivalent*, written $\varphi_1\models\mid\varphi_2$, if they are satisfied by the same models. Two models are *elementary equivalent*, written $M_1\equiv M_2$, if they satisfy the same sentences. An institution is *compact* iff $\Gamma\models_\Sigma\varphi$ implies $\Gamma'\models_\Sigma\varphi$ for some finite subset Γ' of Γ. A *theory* is a pair (Σ,Γ) where Γ is a set of Σ-sentences, and is *consistent* iff it has at least one model. □

3 Institution (co)morphisms

Relationships between institutions are captured mathematically by 'institution morphisms', of which there are several variants. We here choose the notion of institution comorphism [19], although the definition of institution equivalence below is largely independent of this choice.

Definition 3.1 Given institutions $\mathcal{I}$ and $\mathcal{J}$, an *institution comorphism* $\rho=(\Phi,\alpha,\beta)\colon\mathcal{I}\longrightarrow\mathcal{J}$ consists of

- a functor $\Phi\colon\mathbb{S}ign^{\mathcal{I}}\longrightarrow\mathbb{S}ign^{\mathcal{J}}$,
- a natural transformation $\alpha\colon\mathsf{Sen}^{\mathcal{I}}\longrightarrow\mathsf{Sen}^{\mathcal{J}}\circ\Phi$,
- a natural transformation $\beta\colon\mathsf{Mod}^{\mathcal{J}}\circ\Phi^{op}\longrightarrow\mathsf{Mod}^{\mathcal{I}}$

such that the following *satisfaction condition* is satisfied for all $\Sigma\in|\mathbb{S}ign^{\mathcal{I}}|$, $M'\in|\mathsf{Mod}^{\mathcal{J}}(\Phi(\Sigma))|$ and $\varphi\in\mathsf{Sen}^{\mathcal{I}}(\Sigma)$:

$$M'\models^{\mathcal{J}}_{\Phi(\Sigma)}\alpha_\Sigma(\varphi)\quad\text{iff}\quad\beta_\Sigma(M')\models^{\mathcal{I}}_{\Sigma}\varphi.$$

With the natural compositions and identities, this gives a category $\mathbb{C}oIns$ of institutions and institution comorphisms. □

A *set/set institution comorphism* is like a set/cat comorphism, expect that β_Σ is just a function on the objects of model categories; the model morphisms are ignored. Hence, the notion of set/set institution comorphism also makes sense for set/set institutions.

Fact 3.2 An institution comorphism is an isomorphism in $\mathbb{C}oIns$ iff all its components are isomorphisms.

However, institution isomorphism is too strong to capture the notion of "a logic," since it can fail to identify logics that differ only in irrelevant details:

Example 3.3 Let $\mathbf{CPL}'$ be $\mathbf{CPL}$ with arbitrary finite sets as signatures. Then $\mathbf{CPL}'$ has a proper class of signatures, while $\mathbf{CPL}$ only has countably many. Hence, $\mathbf{CPL}$ be $\mathbf{CPL}'$ cannot be isomorphic. □

However, $\mathbf{CPL}$ be $\mathbf{CPL}'$ are essentially the same logic. We now give a notion of institution *equivalence* that is weaker than that of institution isomorphism, very much in the spirit of equivalences of categories. The latter weakens isomorphism of categories: two categories are equivalent iff they have isomorphic *skeletons*, where a skeleton is a full subcategory selecting exactly one representative of each isomorphism class of objects.

Definition 3.4 A (set/cat) institution comorphism (Φ, α, β) is a (set/cat) *institution equivalence* iff

- Φ is an equivalences of categories,
- α_Σ has an inverse up to semantic equivalence α'_Σ, (i.e., $\alpha_\Sigma(\alpha'_\Sigma(\varphi)) \models|_\Sigma \varphi$ and $\alpha'_\Sigma(\alpha_\Sigma(\psi)) \models|_{\Phi\Sigma} \psi$) which is natural in Σ, and
- β_Σ is an equivalence of categories, such that its inverse up to isomorphism and the corresponding isomorphism natural transformations are natural in Σ.

$\mathcal{I}$ is *equivalent* to $\mathcal{J}$ if there is an institution equivalence from $\mathcal{I}$ to $\mathcal{J}$. □

This notion arises inevitably as equivalence in the appropriate 2-category of institutions [9]. The requirement for a set/set institution comorphism to be a *set/set equivalence* is weaker: each β_Σ is only required to have an inverse up to elementary equivalence β'_Σ.

Proposition 3.5 Both set/cat and set/set equivalence of institutions are equivalence relations. Set/cat equivalence implies set/set equivalence.

Example 3.6 $\mathbf{CPL}$ and $\mathbf{CPL}'$ are set/cat equivalent, as are $\mathbf{CPL}^{\neg,\vee,true}$ and $\mathbf{CPL}^{\neg,\wedge,false}$. Signatures and models are translated identically, while sentences are translated using de Morgan's laws. Indeed, $\mathbf{CPL}^{\neg,\vee,true}$ and $\mathbf{CPL}^{\neg,\wedge,false}$ are isomorphic, but the isomorphism is far more complicated than the equivalence. □

A category is *skeletal* iff all its isomorphisms are identities. A *skeleton* $\mathbb{S}$ of a category $\mathbb{C}$ is an isomorphism-dense skeletal subcategory. In this case, the inclusion $\mathbb{S} \hookrightarrow \mathbb{C}$ has a left inverse (i.e. a retraction) $\mathbb{C} \to \mathbb{S}$ mapping each object to the unique representative of its isomorphism class (see [23]).

Definition 3.7 Given a set/cat institution $\mathcal{I}$, an institution $\mathcal{J}$ is a set/cat *skeleton* of $\mathcal{I}$, if

- $\mathbb{S}ign^{\mathcal{J}}$ is a skeleton of $\mathbb{S}ign^{\mathcal{I}}$,
- $\mathsf{Sen}^{\mathcal{J}}(\Sigma) \cong \mathsf{Sen}^{\mathcal{I}}(\Sigma)/\models|$ for $\Sigma \in |\mathbb{S}ign^{\mathcal{J}}|$ (the isomorphism being natural in Σ), and $\mathsf{Sen}^{\mathcal{J}}(\sigma)$ is the induced mapping between the equivalence classes,
- $\mathsf{Mod}^{\mathcal{J}}(\Sigma)$ is a skeleton of $\mathsf{Mod}^{\mathcal{I}}(\Sigma)$, and $\mathsf{Mod}^{\mathcal{J}}(\sigma)$ is both the restriction and corestriction of $\mathsf{Mod}^{\mathcal{I}}(\sigma)$,
- $M \models^{\mathcal{J}}_\Sigma [\varphi]$ iff $M \models^{\mathcal{I}}_\Sigma \varphi$.

Set/set skeletons are defined similarly, the difference being that $\mathsf{Mod}^{\mathcal{J}}(\Sigma)$ is $\mathsf{Mod}^{\mathcal{I}}(\Sigma)/\equiv$. □

Theorem 3.8 Assuming the axiom of choice, every institution has a skeleton. Every institution is equivalent to any of its skeletons. Any two skeletons of an institution are isomorphic. Institutions are equivalent iff they have isomorphic skeletons. □

We have now reached a central point, where we can claim

The identity of a logic is the isomorphism type of its skeleton.

This isomorphism type even gives a normal form for equivalent logics. It follows that a *property of a logic* must be a property of institutions that is invariant under equivalence, and the following sections explore a number of such properties.

4 Model-Theoretic Invariants of Institutions

This section discusses some model-theoretic invariants of institutions; Theorem 6.21 provides a summary.

Every institution has a Galois connection between its sets Γ of Σ-sentences and its classes $\mathcal{M}$ of Σ-models, defined by $\Gamma^\bullet = \{M \in \mathsf{Mod}(\Sigma) \mid M \models_\Sigma \Gamma\}$ and $\mathcal{M}^\bullet = \{\varphi \in \mathsf{Sen}(\Sigma) \mid \mathcal{M} \models_\Sigma \varphi\}$. A Σ-theory Γ is *closed* if $(\Gamma^\bullet)^\bullet = \Gamma$.[5] Closed Σ-theories are closed under arbitrary intersections; hence they form a complete lattice. This leads to a functor $\mathcal{C}^\models\colon \mathbb{S}ign \longrightarrow \mathbb{C}Lat$. Although $\mathcal{C}^\models$ is esssentially preserved under equivalence, the closure operator $(\_^\bullet)^\bullet$ on theories is not. This means it makes too fine-grained distinctions; for example, in **FOL**, $(true^\bullet)^\bullet$ is infinite, while in a skeleton of **FOL**, $([true]^\bullet)^\bullet$ is just the singleton $\{[true]\}$. As already noted in [26], the closure operator simultaneously is too coarse for determining the identity of a logic: while e.g. proof-theoretic falsum in a sound and complete logic (see Section 6) is preserved by homeomorphisms of closure operators in the sense of [26], external semantic falsum is not. Because the theory closure operator is not preserved under equivalence, we do not study it further, but instead use the closed theory lattice functor $\mathcal{C}^\models$ and the Lindenbaum functor $\mathcal{L}$. (We note in passing that this Galois connection generalizes some results considered important in the study of ontologies in the computer science sense.)

The category of theories of an institution is often more useful than its lattice of theories, where a theory morphism $(\Sigma, \Gamma) \longrightarrow (\Sigma', \Gamma')$ is a signature morphism $f\colon \Sigma \longrightarrow \Sigma$ such that $f(\Gamma) \models_{\Sigma'} \Gamma'$. Let $\mathbb{T}h(\mathcal{I})$ denote this category (it should be skeletized to become an invariant). The following result is basic for combining theories, and has important applications to both specification and programming languages [16]:

Theorem 4.1 The category of theories of an institution has whatever colimits its signatures have. □

Definition 4.2 An institution has *external semantic conjunction* [31] if for any pair of sentences φ_1, φ_2 over the same signature, there is a sentence ψ such that ψ holds in a model iff both φ_1 and φ_2 hold in it. ψ will also be denoted $\varphi_1 \circledwedge \varphi_2$, a meta-notation which may not agree with the syntax for sentences in the institution. Similary, one can define what it means for an institution to have external semantic disjunction, negation, implication, equivalence, true, false, and we will use similar circle notations for these. An institution is *truth functionally complete*, if any Boolean combination of sentences is equivalent to a single sentence. □

Example 4.3 **FOL** is truth functionally complete, while **Eq** has no external semantic connectives. □

Definition 4.4 The Lindenbaum construction of Example 2.2 works for any institution $\mathcal{I}$. Let $\Xi^\mathcal{I}$ be the single sorted algebraic signature having that subset of the operations $\{\circledwedge, \circledvee, \circleddash, \circledminus, \circledminus, \circledt, \circledf\}$ (with standard arities) that are external semantic for $\mathcal{I}$; $\Xi^\mathcal{I}$ may include connectives not provided by the institution $\mathcal{I}$, or provided by $\mathcal{I}$ with a different syntax. We later prove that $\Xi^\mathcal{I}$ is invariant under equivalence. For any signature in $\mathcal{I}$, let $\mathcal{L}(\Sigma)$ have as carrier set the quotient $\mathsf{Sen}(\Sigma)/\!\models\!|$, as in Example 2.2. Every external semantic operation of $\mathcal{I}$ has a corresponding operation $\circledwedge$ on $\mathcal{L}(\Sigma)$, so $\mathcal{L}(\Sigma)$ can be given a $\Xi^\mathcal{I}$-algebra structure. Any subsignature of $\Xi^\mathcal{I}$ can also be used (indicated with superscript notation as in Example 2.2), in which case crypto-isomorphisms[6] can provide Lindenbaum algebra equivalence. Moreover, $\mathcal{L}$ is a functor $\mathbb{S}ign \longrightarrow \mathsf{Alg}(\Xi^\mathcal{I})$ because $\models\!|$ is preserved by translation along signature morphisms[7]. If $\mathcal{I}$ is truth functionally complete, then $\mathcal{L}(\Sigma)$ is a Boolean algebra. A proof-theoretic variant of $\mathcal{L}(\Sigma)$ is considered in Section 6 below. □

Definition 4.5 An institution has *external semantic universal $\mathcal{D}$-quantification* [32] for a class $\mathcal{D}$ of signature morphisms, iff for each $\sigma\colon \Sigma \to \Sigma'$ in $\mathcal{D}$ and each Σ'-sentence, there exists a Σ-sentence $\forall\sigma.\varphi$ such that $M \models_\Sigma \forall\sigma.\varphi$ if and only if $M' \models \varphi$ for each σ-expansion M' of M. External semantic existential quantification is defined similarly. □

This definition accomodates quantification over any entities which are part of the relevant concept of signature. For conventional model theory, this includes second order quantification by taking $\mathcal{D}$ to be all extensions of signatures by operation and relation symbols. First order quantification is modelled with $\mathcal{D}$ the *representable signature morphisms* [10, 12] defined below, building on the observation that an assignment for a set of (first order) variables corresponds precisely to a model homomorphism from the free (term) model over that set of variables:

Definition 4.6 A signature morphism $\chi\colon \Sigma \to \Sigma'$ is *representable* iff there are a Σ-model M_χ called the *representation of* χ and a category isomorphism i_χ such that the following commutes:

[5] These closed theories can serve as models in institutions lacking (non-trivial) models.

[6] A cryptomorphism is a homomorphism between algebras of different signatures linked by a signature morphism; the homomorphism goes from the source algebra into the reduct of the target algebra.

[7] $\mathcal{L}$ is also functorial in the institution.

$$\begin{tikzcd} \mathsf{Mod}(\Sigma') \arrow[r, "i_\chi"] \arrow[dr, "\mathsf{Mod}(\chi)"'] & (M_\chi/\mathsf{Mod}(\Sigma)) \arrow[d, "U"] \\ & \mathsf{Mod}(\Sigma) \end{tikzcd}$$

where $(M_\chi/\mathsf{Mod}(\Sigma))$ is a comma category and U is the forgetful functor. □

It seems highly likely that if external semantic universal quantification over representable quantifiers is included in the signature $\Xi^{\mathbf{FOL}}$, then our Lindenbaum algebra functor yields cylindric algebras, though not all details have been checked as of this writing.

The following is important for studying invariance properties of instiutions under equivalence:

Lemma 4.7 If $(\Phi, \alpha, \beta): \mathcal{I} \longrightarrow \mathcal{J}$ is a set/cat or set/set institution equivalence, then then for any signature Σ in $\mathcal{I}$, any $\Gamma \cup \{\varphi\} \subseteq \mathsf{Sen}(\Sigma)$,

$$\Gamma \models^{\mathcal{I}}_{\Sigma} \varphi \text{ iff } \alpha_\Sigma(\Gamma) \models^{\mathcal{J}}_{\Phi(\Sigma)} \alpha_\Sigma(\varphi)$$

and for any $M_1, M_2 \in Mod(\Phi(\Sigma))$,

$$M_1 \equiv M_2 \text{ iff } \beta_\Sigma(M_1) \equiv \beta_\Sigma(M_2).$$

Theorem 4.8 Let $(\Phi, \alpha, \beta): \mathcal{I} \longrightarrow \mathcal{J}$ be an institution equivalence. If $\mathcal{I}$ has universal (or existential) representable quantification, then $\mathcal{J}$ also has universal (or existential) representable quantification.

Example 4.9 Horn Clause logic is not equivalent to **FOL**, because it does not have negation (nor implication, nor disjunction). Horn Clause logic with predicates and without predicates are not equivalent: in the latter logic, model categories of theories have (regular epi,mono)-factorizations, which is not true for the former logic. □

Example 4.10 Propositional logic **CPL** and propositional modal logic **K** are not equivalent: the former has external semantic disjunction, the latter has not. Indeed, the Lindenbaum algebra signature for **CPL** is $\{\textcircled{\wedge}, \textcircled{\vee}, \textcircled{\neg}, \textcircled{\ominus}, \textcircled{\ominus}, \textcircled{t}, \textcircled{f}\}$, while that for **K** is $\{\textcircled{\wedge}, \textcircled{\ominus}, \textcircled{t}, \textcircled{f}\}$. Likewise, first-order logic and first-order modal logic are not equivalent. These assertions also hold if "modal" is replaced with "intuitionistic". □

Definition 4.11 For any classes $\mathcal{L}$ and $\mathcal{R}$ of signature morphisms in an institution $\mathcal{I}$, the institution has the *external semantic* $\langle \mathcal{L}, \mathcal{R} \rangle$*-Craig interpolation property* [31], if for any pushout in $\mathbb{S}ign$ such that $\sigma_1 \in \mathcal{L}$ and $\sigma_2 \in \mathcal{R}$

$$\begin{tikzcd} \Sigma \arrow[r, "\sigma_1"] \arrow[d, "\sigma_2"] & \Sigma_1 \arrow[d, "\theta_2"] \\ \Sigma_2 \arrow[r, "\theta_1"] & \Sigma' \end{tikzcd}$$

any set of Σ_1-sentences φ_1 and any set of Σ_2-sentences φ_2 with $\theta_2(\varphi_1) \models \theta_1(\varphi_2)$, there exists a set of Σ-sentences φ (called the *interpolant*) such that $\varphi_1 \models \sigma_1(\varphi)$ and $\sigma_2(\varphi) \models \varphi_2$. □

This generalizes the conventional formulation of interpolation from intersection-union squares of signatures to arbitrary classes of pushout squares. While **FOL** has interpolation for all pushout squares [14], many sorted first order logic has it only for those where one component is injective on sorts [7, 5, 21], and **Eq** and Horn clause logic only have it for pushout squares where $\mathcal{R}$ consists of injective morphisms [27, 13]. Using sets of sentences rather than single sentences accomodates interpolation results for equational logic [27] as well as for other institutions having Birkhoff-style axiomatizability properties [13]. Robinson consistency can be defined in a similar style [31], and shown equivalent to Craig interpolation under reasonable assumptions.

Definition 4.12 An institution $\mathcal{I}$ is *(semi-)exact*, if Mod maps finite colimits (pushouts) to limits. An institution $\mathcal{I}$ admits *elementary amalgamation* [13] if given a pushout as in Definition 4.11 above, any pair $(M_1, M_2) \in \mathsf{Mod}(\Sigma_1) \times \mathsf{Mod}(\Sigma_2)$ that is *compatible up to elementary equivalence* in the sense that M_1 and M_2 reduce to elementary equivalent Σ-models can be *amalgamated* to a unique Σ'-model M (i.e., there exists a unique $M \in \mathsf{Mod}(\Sigma')$ that reduces to M_1 and M_2, respectively). □

Semi-exactness is important, because many model theoretic results depend on it. For institutions with (external semantic) quantification, even the satisfaction condition requires this property. Semi-exactness is important for instantiations of parameterized specifications. It means that given a pushout as in Definition 4.11 above, any pair $(M_1, M_2) \in \mathsf{Mod}(\Sigma_1) \times \mathsf{Mod}(\Sigma_2)$ that is *compatible* in the sense that M_1 and M_2 reduce to the same Σ-model

can be *amalgamated* to a unique Σ'-model M (i.e., there exists a unique $M \in \mathsf{Mod}(\Sigma')$ that reduces to M_1 and M_2, respectively), and similarly for model homomorphisms.

It is also known how to define ultraproducts, ultrafactors, Łoś sentences and Łoś institutions [10], elementary diagrams of models [11], and Beth definability[8], all in an institution independent way, such that the expected theorems hold under reasonable assumtions. All this is very much in the spirit of "abstract model theory," in the sense advocated by Jon Barwise, but it goes much further, including even some new results for known logics, such as many sorted first order logic [21].

5 Concrete Institutions and Cardinality Properties

Cardinality properties associate cardinalities to objects in a category. The easiest way to do so is via *concrete categories*, which are equipped with a faithful functor to $\mathbb{S}et$ (called the *forgetful functor*).

Since in general we deal with many sorted logics, we slightly generalise this concept to a many sorted setting, and work with the categories of many sorted sets $\mathbb{S}et^S$, where the sets S range over sets of sorts in institution signatures. The following notion adds to institutions the concept of a carrier set for models.

Definition 5.1 A *concrete institutions* is an institution $\mathcal{I}$ together with a functor $sorts^{\mathcal{I}}\colon \mathbb{S}ign^{\mathcal{I}} \to \mathbb{S}et$ and a natural transformation $|\_|^{\mathcal{I}}\colon \mathsf{Mod}^{\mathcal{I}} \to \mathbb{S}et^{(sorts^{\mathcal{I}})^{op}(\_)}$ between functors from $\mathbb{S}ign^{op}$ to $\mathbb{CAT}$ such that for each $\Sigma \in |\mathbb{S}ign^{\mathcal{I}}|$, the functor $|\_|^{\mathcal{I}}_{\Sigma}\colon \mathsf{Mod}^{\mathcal{I}}(\Sigma) \to \mathbb{S}et^{sorts^{\mathcal{I}}(\Sigma)}$ is faithful (that is, for each $\Sigma \in |\mathbb{S}ign^{\mathcal{I}}|$, $\mathsf{Mod}^{\mathcal{I}}(\Sigma)$ is a concrete category, with forgetful functors $|\_|^{\mathcal{I}}_{\Sigma}\colon \mathsf{Mod}^{\mathcal{I}}(\Sigma) \to \mathbb{S}et^{sorts^{\mathcal{I}}(\Sigma)}$ natural in Σ). Here, $\mathbb{S}et^{(sorts^{\mathcal{I}})^{op}(\_)}$ stands for the functor that maps each signature $\Sigma \in |\mathbb{S}ign^{\mathcal{I}}|$ to the category of $sorts^{\mathcal{I}}(\Sigma)$-sorted sets.

Institution comorphisms can be extended to concrete institutions in a natural way: given concrete institutions $\mathcal{I}$ and $\mathcal{J}$, a *concrete comorphism* from $\mathcal{I}$ to $\mathcal{J}$ extends an institution comorphism $(\Phi, \alpha, \beta)\colon \mathcal{I} \longrightarrow \mathcal{J}$ by a natural transformation $\delta\colon sorts^{\mathcal{I}} \to sorts^{\mathcal{J}} \circ \Phi$ and a family of functions $\mu_{\Sigma}\colon |\mathsf{Mod}^{\mathcal{I}}(\Sigma)|^{\mathcal{I}}_{\Sigma} \to (|\mathsf{Mod}^{\mathcal{J}}(\Phi(\Sigma))|^{\mathcal{J}}_{\Phi(\Sigma)})\restriction_{\delta_{\Sigma}}$ required to be natural in $\mathcal{I}$-signatures Σ.

A concrete institution comorphism is a *concrete equivalence* if the underlying institution comorphism is an equivalence and for each $\Sigma \in |\mathbb{S}ign^{\mathcal{I}}|$, both δ_{Σ} and μ_{Σ} are bijective. □

Further generalization equipes signatures with the notion of the set of their symbols:

Definition 5.2 A *concrete institution with symbols* is a concrete institution $\mathcal{I}$ together with a faithful functor $Symb^{\mathcal{I}}\colon \mathbb{S}ign^{\mathcal{I}} \to \mathbb{S}et$ that naturally extends $sorts^{\mathcal{I}}$, that is, such that for each $\Sigma \in |\mathbb{S}ign^{\mathcal{I}}|$, $sorts^{I}(\Sigma) \subseteq Symb^{\mathcal{I}}(\Sigma)$, and for each σ in $\mathbb{S}ign^{\mathcal{I}}$, $Symb^{\mathcal{I}}(\sigma)$ extends $sorts^{\mathcal{I}}(\sigma)$.

Then, given concrete institutions with symbols $\mathcal{I}$ and $\mathcal{J}$, a *concrete comorphism with symbols* from $\mathcal{I}$ to $\mathcal{J}$ extends an institution comorphism $(\Phi, \alpha, \beta)\colon \mathcal{I} \longrightarrow \mathcal{J}$ by a natural transformation $\delta\colon Symb^{\mathcal{I}} \to Symb^{\mathcal{J}} \circ \Phi$ and a family of functions $\mu_{\Sigma}\colon |\mathsf{Mod}^{\mathcal{I}}(\Sigma)|^{\mathcal{I}}_{\Sigma} \to (|\mathsf{Mod}^{\mathcal{J}}(\Phi(\Sigma))|^{\mathcal{J}}_{\Phi(\Sigma)})\restriction_{\delta'_{\Sigma}}$ required to be natural in $\mathcal{I}$-signatures Σ, where δ'_{Σ} is the restriction of δ_{Σ} to $sorts^{\mathcal{I}}(\Sigma)$. Notice that then $(\Phi, \alpha, \beta, \delta', \mu)$ is a concrete comorphism.

Finally, such a concrete comorphism with symbols is a *concrete equivalence with symbols* if the underlying institution comorphism is an equivalence and for each $\Sigma \in |\mathbb{S}ign^{\mathcal{I}}|$, both δ_{Σ} and μ_{Σ} are bijective. □

Note that practically all institutions in use come in as concrete institutions with symbols, with the extra structure defined in the obvious way.

Via the faithful functors to $\mathbb{S}et$, it makes sense to speak about cardinalities of signatures and models in a concrete institutions with symbols.

Definition 5.3 The *Lindenbaum cardinality spectrum* of a concrete institution with symbols maps each cardinal number κ to the maximum number of non-equivalent sentences that a signature of cardinality κ can have.

The *model cardinality spectrum* of a concrete institution with symbols maps each pair of cardinal numbers (κ, λ) to the maximum number of non-isomorphic models of cardinality λ that a signature of cardinality κ can have. **TODO: perhaps use theories here? Look at Shelah's work on classification theory** □

Definition 5.4 A concrete institution with symbols has the *finite model property* if each satisfiable theory has a finite model.

A concrete institution with symbols *admits free models* if all the forgetful functors for model categories have left adjoints. □

[8] As of this writing, the formal proof of the expected results on Beth definability are still in progress, though we ae confident of success.

Theorem 5.5 Sentence and model cardinality spectra, the finite model property and admission of free models are preserved under concrete equivalence.

By restricting signature morphisms, it is often possible to view these cardinality constraints to a subcategory as functors.

6 Proof-theoretic Properties

Proof-theoretic institutions add proof-theoretic structure to the sentences. Usually, categorical logic works with categories of sentences, where morphisms are (equivalence classes of) proof terms [22]. However, this captures provability between single sentences only, while logic traditionally studies provability from a *set* of sentences. Here, we reconcile both approaches by considering categories of sets of sentences.

Definition 6.1 A *cat/cat institution* is like a set/cat institution, except that now $\mathsf{Sen}\colon \mathbb{S}ign \longrightarrow \mathsf{Sen}$ comes with an additional categorical structure on sets of sentences, which is carried by a functor $\mathsf{Pr}\colon \mathbb{S}ign \longrightarrow \mathbb{C}at$ such that $(\_)^{op} \circ \mathcal{P} \circ \mathsf{Sen}$ is a subfunctor of Pr, and the inclusion $\mathcal{P}(\mathsf{Sen}(\Sigma))^{op} \hookrightarrow \mathsf{Pr}(\Sigma)$ is broad and preserves prodcuts. Here $\mathcal{P}\colon \mathbb{S}et \longrightarrow \mathbb{C}at$ is the functor taking each set to its powerset, ordered by inclusion, construed as a thin category, Preservation of products implies that proofs of $\Gamma \to \Psi$ are in one-one-correspondence with families of proofs $(\Gamma \to \psi)_{\psi \in \Psi}$, and that there are monotonicity proofs $\Gamma \to \Psi$ whenever $\Psi \subseteq \Gamma$.[9]

A *cat/cat institution comorphism* between cat/cat institutions I and J consists of a set/cat institution comorphism $(\Phi, \alpha, \beta)\colon I \longrightarrow J$ and a natural transformation $\gamma\colon \mathsf{Pr}^I \longrightarrow \mathsf{Pr}^J \circ \Phi$ such that translation of sentence sets is compatible with translation of single sentences: $|\gamma_\Sigma| = \mathcal{P}(\alpha_\Sigma)$.

A cat/cat institution comorphism $(\Phi, \alpha, \beta, \gamma)$ is a *cat/cat equivalence*, if (Φ, α, β) is a set/cat equivalence and each proof translation functor γ_Σ is an equivalence of categories, with the witnesses for the equivalence being natural in Σ.

As before, all this also extends to the case of omission of model morphisms, i.e. the cat/set case. Henceforth, the term *proof-theoretic institution* will refer to both the cat/cat and the cat/set case. □

Given an arbitrary but fixed proof-theoretic institution, we can define an entailment relation $\vdash_\Sigma$ between sets of Σ-sentences as follows: $\Gamma \vdash_\Sigma \Psi$ if there exists a morphism $\Gamma \to \Psi$ in $\mathsf{Pr}(\Sigma)$. A proof-theoretic institution is *sound* if $\Gamma \vdash_\Sigma \Psi$ implies $\Gamma \models_\Sigma \Psi$; it is *complete*, if the converse implication holds.

Proposition 6.2 $\vdash$ satisfies the properties of an *entailment system* [24], i.e. it is reflexive, transitive, monotonic and stable under translation along signature morphisms. In fact, entailment systems are in one-one correspondence with proof-theoretic institutions having trivial model theory (i.e. $\mathsf{Mod}(\Sigma) = \emptyset$) and having thin[10] categories of proofs.

The requirement for sentence translation in proof-theoretic institution equivalences is very close to the notion of translational equivalence introduced in [25]. A set/set institution equivalence basically requires that the back-and-forth translation of sentence is semantically equivalent to the original sentence (i.e. $\alpha'_\Sigma(\alpha_\Sigma(\varphi)) \models\!\mid \varphi$); a similar notion would arise when using $\vdash\!\dashv$. Note, however, that this does not work well for modal logics, since e.g. in **S5**, $\varphi \vdash\!\dashv \Box\varphi$. Therefore, [25] require $\vdash \alpha'_\Sigma(\alpha_\Sigma(\varphi)) \leftrightarrow \varphi$. However, this is based upon the presence of equivalence as a proof-theoretic connective, which is not present in all institutions. Our solution to this problem comes naturally out of the above definition of proof-theoretic (i.e. cat/cat or cat/set) equivalence: $\alpha'_\Sigma(\alpha_\Sigma(\varphi))$ and φ have to be isomorphic in the category of proofs. We thus neither identify φ and $\Box\varphi$ in modal logics, nor rely on the presence of a connective $\leftrightarrow$.

Definition 6.3 A proof-theoretic institution is *finitary*, if $\Gamma \vdash_\Sigma \varphi$ implies $\Gamma' \vdash_\Sigma \varphi$ for some finite $\Gamma' \subseteq \Gamma$.

A proof-theoretic institution has *proof-theoretic conjunction*, if each category $\mathsf{Pr}(\Sigma)$ has products of singletons, which are singletons again. In terms of derivability, this means that for φ_1, φ_2 Σ-sentences, there is a product sentence $\varphi_1 \boxed{\wedge} \varphi_2$, and two "projection" proof terms $\pi_1\colon \varphi_1 \boxed{\wedge} \varphi_2 \longrightarrow \varphi_1$ and $\pi_2\colon \varphi_1 \boxed{\wedge} \varphi_2 \longrightarrow \varphi_2$, such that for any ψ with $\psi \vdash_\Sigma \varphi_1$ and $\psi \vdash_\Sigma \varphi_2$, then $\psi \vdash_\Sigma \varphi_1 \boxed{\wedge} \varphi_2$.

Similarly, a proof theoretic institution has *proof-theoretic disjunction (true, false)*, if each proof category has coproducts of singletons that are singeltons (a singleton terminal object, a singleton initial object).

A proof theoretic institution has *proof-theoretic implication*, if each functor $\_ \cup \{\varphi\}\colon \mathsf{Pr}(\Sigma) \longrightarrow \mathsf{Pr}(\Sigma)$ has a right adjoint, denoted by $\varphi \boxed{\to} \_$, such that $\varphi \boxed{\to} \_$ actually restricts to a functor on $\mathsf{Sen}(\Sigma)$. In case that $\boxed{f}$ is

[9] Instead of having two functors Pr and Sen, it is also possible to use one functor into a comma category.

[10] A category is *thin*, if between two given objects, there is at most one morphism, i.e. the category is a pre-order.

present, it has *proof-theoretic negation*, if each sentence ψ has a negation $\boxed{\neg}\psi$ such that $Hom(\Gamma \cup \{\psi\}, \boxed{f})$ is in natural bijective correspondence to $Hom(\Gamma, \{\boxed{\neg}\psi\})$.

A proof-theoretic institution is *propositional*, if it has proof-theoretic conjunction, disjunction, implication, negation, true and false. □

Definition 6.4 A proof-theoretic institution with proof-theoretic negation has $\neg\neg$-*elimination*, if for each Σ-sentence φ, $\boxed{\neg}\boxed{\neg}\varphi \vdash_\Sigma \varphi$ (the converse relation holds by definition). **TODO: Check the latter statement.** □

For example, **CPL** and **FOL** have $\neg\neg$-elimination, while **IPL** has not.

Clearly, any sound and complete proof-theoretic institution with external semantic and proof-theoretic negation has $\neg\neg$-elimination.

Proposition 6.5 A sound and complete proof-theoretic institution is finitary iff it is compact. A proof-theoretic institution having proof-theoretic implication enjoys the deduction theorem and modus ponens for $\vdash_\Sigma$.

Example 6.6 Modal logic **K** does not have proof-theoretic implication, nor negation, and this is a difference to intuitionstic logic **IPL**, showing that the two logics are not equivalent.

While **K** does not have proof-theoretic implication, it still has a form of *local* implication, one which does not satisfy the deduction theorem. This can be axiomatized as follows.

Definition 6.7 A proof-theoretic institution has *Hilbert implication*, if for each siganture Σ, there is a unique binary operator $\boxed{\Rightarrow}$ on Σ-sentences satisfying the Hilbert axioms for implication, i.e.

$$\begin{array}{ll}
(K) & \emptyset \vdash_\Sigma \{\varphi \boxed{\Rightarrow} \psi \boxed{\Rightarrow} \varphi\} \\
(S) & \emptyset \vdash_\Sigma \{(\varphi \boxed{\Rightarrow} \psi \boxed{\Rightarrow} \chi) \boxed{\Rightarrow} (\varphi \boxed{\Rightarrow} \psi) \boxed{\Rightarrow} \varphi \boxed{\Rightarrow} \chi\} \\
(MP) & \{\varphi \boxed{\Rightarrow} \psi, \varphi\} \vdash_\Sigma \{\psi\}
\end{array}$$

□

Definition 6.8 There is a proof-theoretic variant of the Lindenbaum algebra of Def. 4.4. Let $\Psi^{\mathcal{I}}$ be the single-sorted algebraic signature having a subset of the operations $\{\boxed{\wedge}, \boxed{\vee}, \boxed{\neg}, \boxed{\Rightarrow}, \boxed{\leftrightarrow}, \boxed{t}, \boxed{f}\}$ (with standard arities), chosen according to whether $\mathcal{I}$ has proof-theoretic conjunction, disjunction, negation etc., but for implication, Hilbert implication is used. Note that like the signature $\Xi^{\mathcal{I}}$ introduced in Def. 4.4, $\Psi^{\mathcal{I}}$ may include connectives not provided by the instution $\mathcal{I}$, or are provided by $\mathcal{I}$ with a different syntax. By Theorem 6.11, $\Xi^{\mathcal{I}}$ is a subsignature of $\Psi^{\mathcal{I}}$. Consider $\mathcal{L}^{\vdash}(\Sigma) = \mathsf{Sen}(\Sigma)/\cong$, where $\cong$ is isomorphism in $\mathsf{Pr}(\Sigma)$. Since products etc. are unique up to isomorphism, it is straightforward to make this a $\Psi^{\mathcal{I}}$-algebra.

The Lindenbaum algebra is the basis for the *Lindenbaum category* $\mathcal{LC}^{\vdash}(\Sigma)$, which has object set $\mathcal{P}(\mathcal{L}^{\vdash}(\Sigma))$. By chosing a system of canonical representatives for $\mathsf{Sen}(\Sigma)/\cong$, this object set can be embedded into $|\mathsf{Pr}^{(}\Sigma)|$; hence we obtain an induced full subcategory. $\mathcal{LC}^{\vdash}(\Sigma)$ is then the category of fractions [4] obtained by this full subcategory through turning all arrows $\langle h_\varphi \rangle \colon [\Gamma] \longrightarrow [\Gamma']$ with $h_\varphi \colon \Gamma \longrightarrow \varphi$ for $\varphi \in \Gamma$ into isomorphisms. **TODO: Improve formulation**

While the Lindenbaum category construction are functorial, the Lindenbaum algebra construction is generally not.

Also, the closed theory functor $\mathcal{C}^{\models}$ has a proof theoretic counterpart $\mathcal{C}^{\vdash}$ taking theories closed under $\vdash$. □

Definition 6.9 A proof-theoretic institution is *compatible*, if for each circled operator in $\Xi^{\mathcal{I}}$, the corresponding boxed operator in $\Psi^{\mathcal{I}}$ is present, and both agree. It is *bicompatible*, if also the converse holds. □

Proposition 6.10 Assume a compatible proof-theoretic institution with thin proof categories. If deduction is sound and complete, then $\mathcal{L}(\Sigma)$ and $\mathcal{L}^{\vdash}(\Sigma)|_{\Xi^{\mathcal{I}}}$ are isomorphic; just soundness gives a surjective cryptomorphism $\mathcal{L}^{\vdash}(\Sigma)|_{\Xi^{\mathcal{I}}} \to \mathcal{L}(\Sigma)$, and just completeness gives one in the opposite direction.

Theorem 6.11 A sound and complete proof-theoretic institution with thin proof categories is compatible, but not necessarily bicompatible.

Example 6.12 Intuitionistic propositional logic shows that proof-theoretic disjunction does not imply external semantic disjunction (although the proof categories are not thin here, one can factor them out to become thin).

Example 6.13 **S4** has a non-idempotent operator (congruent with $\boxed{\wedge}$ and $\boxed{t}$) on its Lindenbaum algebra, while **S5** does not have one. Hence, **S4** and **S5** are not equivalent.

TODO: Prove this, or perhaps better use the following:

Definition 6.14 A proof-theoretic institution is called *classical modal*, if its Lindenbaum algebras $\mathcal{L}^{\vdash}(\Sigma)$ are Boolean algebras (also having implication) with an operator $\Box$ (congruent with $\boxed{\wedge}$ and $\boxed{t}$). A classical modal proof-theoretic institution is called *normal*, if the operator satisfies the necessitation law: $\varphi \vdash_\Sigma \Box\varphi$. (Note that modus ponens already follows from implication being present in $\Psi^{\mathcal{I}}$.)

It is clear that equivalences between classical modal proof-theoretic institutions need to preserve $\mathcal{L}^{\vdash}$ (but not necessarily the operator). We can hence apply the results of [25]: **TODO: The following is still a bit speculative!**

Theorem 6.15 Let $\mathcal{I}$ and $\mathcal{J}$ be classical modal proof-theoretic institutions, such that $\mathcal{I}$ is a "proper sublogic" (how to formalize this?) of $\mathcal{J}$, both having the finite model property. Then $\mathcal{I}$ and $\mathcal{J}$ are not equivalent.

Definition 6.16 Given a cat/cat institution $\mathcal{I}$, an institution $\mathcal{J}$ is a cat/cat *skeleton* of $\mathcal{I}$, if

- $\mathbb{S}ign^{\mathcal{J}}$ is a skeleton of $\mathbb{S}ign^{\mathcal{I}}$,
- $\mathsf{Sen}^{\mathcal{J}}(\Sigma) = \mathsf{Sen}^{\mathcal{I}}(\Sigma)/\cong$ for $\Sigma \in |\mathbb{S}ign^{\mathcal{J}}|$, and $\mathsf{Sen}^{\mathcal{J}}(\sigma)$ is the induced mapping between the equivalence classes,
- $\mathsf{Pr}^{\mathcal{J}}(\Sigma) = \mathcal{LC}^{\vdash}(\Sigma)$, the Lindenbaum category,
- $\mathsf{Mod}^{\mathcal{J}}(\Sigma)$ is a skeleton of $\mathsf{Mod}^{\mathcal{I}}(\Sigma)$, and $\mathsf{Mod}^{\mathcal{J}}(\sigma)$ is both the restriction and corestriction of $\mathsf{Mod}^{\mathcal{I}}(\sigma)$,
- $M \models^{\mathcal{J}}_{\Sigma} [\varphi]$ iff $M \models^{\mathcal{I}}_{\Sigma} \varphi$.

$\Box$

Cat/set skeletons are defined similarly; the difference being that $\mathsf{Mod}^{\mathcal{J}}(\Sigma)$ is $\mathsf{Mod}^{\mathcal{I}}(\Sigma)/\equiv$.

Definition 6.17 A cat/cat institution *has proof-theoretic universal (existential) quantification*, if for each representable signature morphism σ, $\mathsf{Pr}(\sigma)$, when restricted to singletons, has a right (left) adjoint, denoted by $\forall\sigma.\_$ ($\exists\sigma.\_$). $\Box$

This means that there is a one-one-correspondence between proofs of $\varphi \vdash_\Sigma \forall\sigma.\psi$ and proofs of $\sigma(\varphi) \vdash_{\Sigma'} \psi$. The counit of the adjunction gives $\sigma(\forall\sigma.\varphi) \vdash_{\Sigma'} \varphi$. The situation for $\exists$ is dual. **TODO: more explanation, mention Lawvere**

One may define a proof-theoretic concept of consistency. A theory (Σ, Γ) is *consistent* when its closure under $\vdash$ is a proper subset of $\mathsf{Sen}(\Gamma)$.

Definition 6.18 The notions of Craig interpolation and Robinson consistency come also in proof-theoretic versions by replacing $\models_\Sigma$ with $\vdash_\Sigma$ and the semantic concept of consistency by its proof-theoretic correspondent. The proof-theoretic version of explicit definability (and hence, the Beth definability property) relies on the institution having universal quantification, equivalence, and (infinitary) conjunction. $\Box$

Given a set/cat institution $\mathcal{I}$, we can obtain a (sound and complete) cat/cat institution $\mathcal{I}^{\models}$ by letting $\mathsf{Pr}(\Sigma)$ be the pre-order defined by $\Gamma \leq \Psi$ if $\Gamma \models_\Sigma \Psi$, considered as a category. Actually, this construction also works for set/set, cat/set and cat/cat institutions. Some of the proof-theoretic notions are useful when interpreted in $\mathcal{I}^{\models}$:

Definition 6.19 An institution $\mathcal{I}$ has *internal semantic conjunction*, if $\mathcal{I}^{\models}$ has proof-theoretic conjunction; similarly for the other connectives. $\Box$

Example 6.20 Intuitionistic logic **IPL** has internal, but not external semantic implication. Higher-order intuitionistic logic interpreted in a fixed topos has proof-theoretic and Hilbert implication, but neither external nor internal semantic implication. Modal logic **S5** has just Hilbert implication.

Theorem 6.21 The following properties are invariant under set/set, set/cat, cat/set and cat/cat equivalence, resp.[11] and cat/cat case and the corresponding notions, Sect. 6.) Properties in *italics* rely on concrete institutions, see Sect. 5.

[11] Functors such as the Lindenbaum algebra functor are preserved in the sense that $\mathcal{L}_{\mathcal{I}}$ is naturally isomorphic to $\mathcal{L}_{\mathcal{J}} \circ \Phi$.

set/set	set/cat
compactness, elementary amalgamation, external semantic Craig interpolation, Robinson consistency and Beth definability, having external semantic conjunction, disjunction, negation, true, false, being truth functionally complete, Lindenbaum signature $\Xi^{\mathcal{I}}$, Lindenbaum algebra functor $\mathcal{L}$, closed theory lattice functor $\mathcal{C}^{\models}$, cocompleteness of the signature category, *Lindenbaum cardinality spectrum, finite model property.*	all of set/set, having external semantic universal or existential quantification, exactness, semi-exactness, liberality, (co)completeness of model categories, existence of initial or terminal models, existence of prodcts or reduced products, preservation for fomulae along products or reduced products, being a Łoś-institution, elementary diagrams, *model cardinality spectrum, admission of free models.*
cat/set	**cat/cat**
all of set/set, soundness, completeness, finitariness, having proof-theoretic conjunction, disjunction, implication, negation, true, false, being propositional, Hilbert implication, $\neg\neg$-elimination, proof-theoretic Craig interpolation, Beth definability and Robinson consistency, Lindenbaum signature $\Psi^{\mathcal{I}}$, Lindenbaum algebra construction $\mathcal{L}^{\vdash}$, Lindenbaum category, closed theory lattice functor $\mathcal{C}^{\vdash}$.	all of set/set, set/cat and cat/set, having proof-theoretic universal or existential quantification, compatibility, balancedness.

TODO: Relate the table to its applications. Check that there enough applications, i.e. separation of institutions using the invariant properties.

There are some further proof-theoretic properties that we have not treated, like (strong) normal forms for proofs (this would require $Sen(\Sigma)$ to become 2-category of sentences, proof terms and proof term reductions). A related topic is cut elimination, which would require an even finer structure on $\mathsf{Pr}(\Sigma)$, with proof rules of particular format. We hope this essay provides a good starting point for such investigations. We do not consider proof-theoretic ordinals here either, because they are a measure for the proof-theoretic strength of a *theory* in a logic, not a measure for the logic itself.

TODO: Till: Say something about numberings of sentences and recursiveness of entailment.

7 Conclusions

TODO: needs rewriting We believe that this essay has established three main points: (1) The notion of "a logic" should depend on the purpose at hand – no single definition will suffice for all purposes, and in particular, proof theory and model theory sometimes treat essentially the same issue in different ways. **TODO: Joseph: write something about the usefulness of the institutional framework.** (2) Every plausible notion of equivalence of logics can be formalized using institutions and various equivalence relations on them. (3) Inequivalence of logics can be established using various constructions on institutions that are invariant under the appropriate equivalence, such as Lindenbaum algebra and cardinality spectra.

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The problem "What is a Logic?" is important philosophically, but for most practical purposes, such as proving general theorems about large classes of logics, it seems sufficient to simply work with institutions.

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A Parts to be thrown out of the paper, but to be kept for a later journal version

A.1 General stuff

Proposition A.1 The condition $\alpha'_\Sigma(\alpha_\Sigma(\varphi)) \models|_\Sigma \varphi$ for a cat/cat equivalence between sound proof-theoretic institutions already follows from the conditions for γ.

Proof: We have $\{\alpha'_\Sigma(\alpha_\Sigma(\varphi))\} = \gamma'_\Sigma(\gamma_\Sigma(\{\varphi\})) \cong \{\varphi\}$, hence by soundness $\alpha'_\Sigma(\alpha_\Sigma(\varphi)) \models|_\Sigma \varphi$. □

Model-theoretic consistency implies satisfiablity. Equational logic with $\forall x.x = x$ provides a counterexample to the converse implication. However, in presence of false, the converse implication holds.

Furthermore, soundness $\Rightarrow$ (Model-theoretic consistency implies proof-theoretic consistency) The converse implication holds if the institution has proof-theoretic negation and false, and model-theoretic false.

completeness $\Rightarrow$ (proof-theoretic consistency implies Model-theoretic consistency) The converse implication holds if the institution has model-theoretic negation and proof.theoretic or Hilbert implication.

Recall: Model-theoretic consistency = closure of the theory is not equal to all setencens. Proof-theoretic consistency = proof-th. closure of the theory is not equal to all setencens. Satisfiability = having a model

A.2 Invariance Proofs

Theorem A.2 Compactness, exactness, semi-exactness, liberality, having conjunction, disjunction, negation, true, false, being truth-functionally complete, (co)completeness of model categories, existence of initial or terminal models, being a Łoś-institution, the signature $\Xi^{\mathcal{I}}$ of the Lindenbaum algebra and the Lindenbaum algebra functor are invariant under equivalence.

Proof: Let $(\Phi, \alpha, \beta): \mathcal{I} \longrightarrow \mathcal{J}$ be an institution equivalence. We will either prove preservation of the property from $\mathcal{I}$ to $\mathcal{J}$ or vice versa. By symmetry, this makes no difference.

Compactness Suppose that $\mathcal{I}$ is compact, and let $\Gamma \models^{\mathcal{J}}_{\Phi\Sigma} \varphi$. By Lemma. 4.7, $\alpha_\Sigma(\Gamma) \models^{\mathcal{I}}_{\Sigma} \alpha_\Sigma(\varphi)$. By compactness of $\mathcal{I}$, $\Gamma' \models^{\mathcal{I}}_{\Sigma} \alpha_\Sigma(\varphi)$ for some finite $\Gamma' \subseteq \alpha_\Sigma(\Gamma)$. Then there is some finite $\Gamma'' \subseteq \Gamma$ with $\alpha_\Sigma(\Gamma'') = \Gamma'$, and by Lemma 4.7 again, $\Gamma'' \models \varphi$. Hence, $\mathcal{J}$ is compact as well.

Exactness, semi-exactness Suppose that $\mathcal{J}$ is exact, i.e. $\mathsf{Mod}^{\mathcal{J}}$ preserves limits. Since Φ preserves colimits and $\mathsf{Mod}^{\mathcal{I}}$ is naturally isomorphic to $\mathsf{Mod}^{\mathcal{J}} \circ \Phi^{op}$, $\mathcal{I}$ is exact as well. Preservation of semi-exactness is shown similarly.

Elementary amalgamation In the notation of Def. 4.12, let $M_1 \in \mathsf{Mod}(\Sigma_1)$ and $M_2 \in \mathsf{Mod}(\Sigma_2)$ reduce to elementary equivalent Σ-models. By isomorphism-density of β_{Σ_i}, there is an β_{Σ_i}-preimage M'_i of M_i $(i = 1, 2)$. By naturality of β_{Σ_i}, closure of satisfaction under model isomorphism and Lemma 4.7, M'_1 and M'_2 have elementary equivalent $\Phi\Sigma$-reducts. Hence, assuming that there is an amalgamation M' of M'_1 and M'_2 in $\mathcal{J}$, $\beta_{Sigma_R}(M')$ is the amalgamation of M_1 and M_2 in $\mathcal{I}$.

Liberality Let $\sigma\colon\langle\Sigma_1,\Gamma_1\rangle\longrightarrow\langle\Sigma_2,\Gamma_2\rangle$ be a theory morphism in $\mathcal{I}$. By the satisfaction condition, for $i=1,2$, β_{Σ_i} restricts to $\beta_{\langle\Sigma_i,\Gamma_i\rangle}\colon\mathsf{Mod}^{\mathcal{J}}(\langle Sig(\Phi\Sigma_i), Ax(\Phi\Sigma_i)\cup\alpha_{\Sigma_i}(\Gamma_i)\rangle)\longrightarrow\mathsf{Mod}^{\mathcal{I}}(\langle\Sigma_i,\Gamma_i\rangle)$. Since β_{Σ_i} is a categorical equivalence, liberality of $\Phi(\Sigma)$ leads to liberality of σ.

Having conjunction, disjunction, negation, implication, true, false; truth-functional completeness Since model translation is a natural equivalence, by the satisfaction condition, if ψ is a conjunction of φ_1 and φ_2, then $\alpha_\Sigma(\psi)$ is a conjunction of $\alpha_\Sigma(\varphi_1)$ and $\alpha_\Sigma(\varphi_2)$. We are done because α_Σ is surjective up to semantical equivalence. The proof for the other logical connectives and truth-functional completeness is similar.

(Co)completeness of model categories Obvious.

Existence of initial or terminal models Obvious.

Distributivity To be done!

Craig interpolation In the notation of Def. 4.11, let $\theta_2(\varphi_1)\models\theta_1(\varphi_2)$. By Lemma 4.7 and naturality of α, $\Phi\theta_2(\alpha(\varphi_1))\models\Phi\theta_1(\alpha(\varphi_2))$. Since we assume that $\mathcal{J}$ has Craig interpolation, there is an interpolant ψ with $\alpha(\varphi_1)\models\Phi\sigma_1(\psi)$ and $\Phi\sigma_2(\psi)\models\alpha(\varphi_2)$. Since α_Σ is surjective up to semantical equivalence, there is some Σ-sentence φ with $\alpha_\Sigma(\varphi)$ semantically equivalent to ψ. Again by Lemma 4.7, this is the desired interpolant, showing that $\mathcal{I}$ has Craig interpolation as well.

Łoś institution Since model translation is a categorical equivalence, reduced products are preserved. Preservation of satisfaction is just the satisfaction condition.

Beth definability, Robinson consistency To be done!

Lindenbaum algebra

Some of the properties need more than equivalence for their preservation:

Theorem A.3 Cocompleteness of the signature category and elementary diagrams are preserved under equivalence.

Let $(\Phi,\alpha,\beta)\colon\mathcal{I}\longrightarrow\mathcal{J}$ be an institution equivalence. We will either prove preservation of the property from $\mathcal{I}$ to $\mathcal{J}$ or vice versa. By symmetry, this makes no difference.
PROOF:

Cocompleteness of the signature category Obvious.

Elementary diagrams TODO: Razvan: To be done

Theorem A.4 Soundness, completeness, having proof-theoretic conjunction, disjunction, implication, negation, true and false, $\neg\neg$-elimination, proof-theoretic Craig interpolation, Beth definability, Robinson consistency and the Lindenbaum signature $\Psi^{\mathcal{I}}$ and the Lindenbaum algebra functor $\mathcal{L}^{\vdash}$ are invariant under cat/set equivalence.

A.3 Weak equivalence

Equivalence of institutions sometimes may be still too fine-grained:

Example A.5 $\mathbf{FOL}^{=}$ and $\mathbf{FOLR}^{=}$ are not equivalent.

PROOF: Take in $\mathbf{FOL}^{=}$ the signature Σ consisting of one sort s and one constant $c:s$. Then $\mathsf{Mod}(\Sigma)$ has a zero object (the one-point model). But in $\mathbf{FOLR}^{=}$, for any signature, the one-point model with full relations is terminal but not initial, hence a zero object cannot exist. Since zero objects are preserved by equivalences of categories, $\mathbf{FOL}^{=}$ and $\mathbf{FOLR}^{=}$ cannot be equivalent. □

Still, $\mathbf{FOL}^{=}$ and $\mathbf{FOLR}^{=}$ are very close to being equivalent. While the identity of a logic should certainly determined using the notion of equivalence above, there is a notion of weak equivalence that still preserves most of the relevant logical structure, ignoring it only at some points that are often minor to the consideration. In particular, weak equivalences may confuse signatures and theories. We first need to introduce the category of theories.

In an arbitrary institution $\mathcal{I}$, a *theory* is a pair $T=\langle\Sigma,\Gamma\rangle$, where $\Sigma\in|\mathbb{S}ign|$ and $\Gamma\subseteq\mathsf{Sen}(\Sigma)$ (we set $Sig(T)=\Sigma$ and $Ax(T)=\Gamma$). *Theory morphisms* $\sigma\colon\langle\Sigma,\Gamma\rangle\longrightarrow\langle\Sigma',\Gamma'\rangle$ are those signature morphisms $\sigma\colon\Sigma\longrightarrow\Sigma'$ for which $\Gamma'\models_{\Sigma'}\sigma(\Gamma)$, that is, axioms are mapped to logical consequences. By inheriting composition and

identities from $\mathbb{S}ign$, we obtain a category **Th** of theories. A theory morphism $\sigma: T_1 \longrightarrow T_2$ is a *definitional extension* iff $\mathsf{Mod}(\sigma): \mathsf{Mod}(T_2) \longrightarrow \mathsf{Mod}(T_1)$ maps models bijectively.

It is easy to extend the Sen and Mod components of an institution to start from **Th** by putting $\mathsf{Sen}(\langle \Sigma, \Gamma \rangle) = \mathsf{Sen}(\Sigma)$ and letting $\mathsf{Mod}(\langle \Sigma, \Gamma \rangle)$ be the full subcategory of $\mathsf{Mod}(\Sigma)$ induced by the class of those models M satisfying Γ. In this way, we get the *institution of theories* $\mathcal{I}^{th} = (\mathbf{Th}, \mathsf{Sen}, \mathsf{Mod}, \models)$ over $\mathcal{I}$.

Definition A.6 A *weak equivalence of institutions* $\mathcal{I}$ *and* $\mathcal{J}$ is an institution comorphism $\rho = (\Phi, \alpha, \beta): \mathcal{I} \longrightarrow \mathcal{J}^{th}$ such that

- Φ preserves colimits,
- for any $\mathcal{I}$-signature Σ, α is surjective up to semantical equivalence, which means that that for each Σ-sentence φ, there is a $\Phi(\Sigma)$-sentence ψ with $\alpha_\Sigma(\psi)$ semantically equivalent to φ, and
- for any $\mathcal{I}$-signature Σ, β_Σ is an equivalence of categories.

Here, J^{th} is the institution of theories over $\mathcal{J}$. Together with Kleisli composition, this gives a category of weak equivalences. □

Two institutions are *weakly equivalent*, if there are weak equivalences between them in both directions. Obviously, this is an equivalence relation.

Proposition A.7 Equivalence of institutions implies weak equivalence.

Proof: By symmetry (Prop. 3.5), an institution equivalence is always accompanied by an institution equivalence in the reverse direction. With the results on ajointness of morphisms and comorphisms [19], one obtains the desired two comorphisms.

Example A.8 $\mathbf{FOL}^=$ and $\mathbf{FOLR}^=$ are weakly equivalent.

We will discuss more examples at the end of the next section.

TODO: Till: The Lindenbaum algebra functor is preserved by equivalences; for preservation under weak equivalence, it needs to be formulated for theories instead of signatures. Work out how this interacts with separation of modal logics.

A.4 (In)Equivalences of particular institutions

Order-sorted algebra [18] with subsorts interpreted as inclusions and order-sorted algebra with subsorts interpreted as injections are equivalent. (In order to ensure that satisfaction is invariant under model isomorphism, we assume that all signatures are locally filtered.)

Open question: Are many-sorted algebra and order-sorted algebra weakly equivalent? Note that the standard translation from OSA to MSA does not preserve colimits of signatures.