An Institutional View on the Curry-Howard-Tait-Isomorphism

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The Curry-Howard-Tait isomorphism

. . . establishes a correspondence between

- propositions and types
- proofs and terms
- proof reductions and term reductions

Can this isomorphism be presented in an institutional setting, as a relation between institutions?

Categories and logical theories

- propositional logic with conjunction $\Leftrightarrow$ cartesian categories
- propositional logic with conjunction and implication $\Leftrightarrow$ cartesian closed categories
- intuitionistic propositional logic $\Leftrightarrow$ bicartesian closed categories
- classical propositional logic $\Leftrightarrow$ bicartesian closed categories with not not-elimination
- first-order logic $\Leftrightarrow$ hyperdoctrines
- Martin-Löf type theory $\Leftrightarrow$ locally cartesian closed categories

Categorical constructions and logical connectives

$\top$	terminal object
$\bot$	initial object
$\wedge$	product
$\vee$	coproduct
$\Rightarrow$	exponential (right adjoint to product)
$\forall$	right adjoint to substitution
$\exists$	left adjoint to substitution
classicality	$c\colon (a \Rightarrow \bot) \Rightarrow \bot \longrightarrow a$

Relativistic institutions

Let $U_X\colon X \longrightarrow \mathbb{S}et$ and $U_Y\colon Y \longrightarrow \mathbb{S}et$ be concrete categories.
An X/Y-institution consists of

- a category $\mathbb{S}ign$ of signatures,
- a sentence/proof functor $\mathsf{Sen}\colon \mathbb{S}ign \longrightarrow X$,
- a model functor $\mathsf{Mod}\colon \mathbb{S}ign^{op} \longrightarrow Y$, and
- a satisfaction relation $\models_\Sigma \subseteq U_X(\mathsf{Sen}(\Sigma)) \times U_Y(\mathsf{Mod}(\Sigma))$ for each $\Sigma \in |\mathbb{S}ign|$,

such that for each $\sigma\colon \Sigma_1 \longrightarrow \Sigma_2 \in \mathbb{S}ign$, $\varphi \in U_X(\mathsf{Sen}(\Sigma_1))$, $M \in U_Y(\mathsf{Mod}(\Sigma_2))$,

$$M \models_{\Sigma_2} U_X(\mathsf{Sen}(\sigma))(\varphi) \text{ iff } U_Y(\mathsf{Mod}(\sigma))(M) \models_{\Sigma_1} \varphi$$

Examples of relativistic institutions

- set/cat: the usual institutions
- set/set: institutions without model morphisms
- cat/cat: institutions with proof categories over individual sentences
- preordcat/cat: institutions with preorder-enriched proof categories over individual sentences $\Rightarrow$ used here
- powercat/cat: institutions with proof categories over sets of sentences

Powercat/cat institutions

$\mathcal{P}: \mathbb{S}et \longrightarrow \mathbb{C}at$ be the functor taking each set to its powerset, ordered by inclusion, construed as a thin (preorder-enriched) category.

Let $\mathcal{P}^{op} = (\_)^{op} \circ \mathcal{P}$ be the functor that orders by the superset relation instead.

We introduce a category $\mathbb{P}owerCat$ as follows:

- Objects (S, P): S is a set (of sentences), and P is a (preorder-enriched) category (of proofs) with $\mathcal{P}^{op}(S)$ a broad product-preserving subcategory of P. Preservation of products implies that proofs of $\Gamma \to \Psi \in P$ are in one-one-correspondence with families of proofs $(\Gamma \to \psi)_{\psi \in \Psi}$, and that there are monotonicity proofs $\Gamma \to \Psi$ whenever $\Psi \subseteq \Gamma$.
- Morphisms $(f, g) \colon (S_1, P_1) \longrightarrow (S_2, P_2)$ consist of a function $f \colon S_1 \longrightarrow S_2$ (sentence translation) and an preorder-enriched functor $g \colon P_1 \longrightarrow P_2$ (proof translation),

such that

$$\begin{array}{ccc}
\mathcal{P}^{op}(S_1) & \subseteq & P_1 \\
\downarrow \mathcal{P}^{op}(f) & & \downarrow g \\
\mathcal{P}^{op}(S_2) & \subseteq & P_2
\end{array}$$

commutes.

From cat/cat institutions to powercat/cat institutions

$F\colon \mathbb{C}artesianCat \longrightarrow \mathbb{P}owerCat$ maps C to $F(C)$:

Objects: sets of objects in C

Morphisms: $p\colon \Gamma \longrightarrow \Delta$ are families $(p_\varphi\colon \psi_1^\varphi \wedge \ldots \wedge \psi_{n\varphi}^\varphi \longrightarrow \varphi)_{\varphi\in\Delta}$ with $\psi_i^\varphi \in \Gamma$

Identities, composition and functoriality straightforward (however, be careful with coherence!)

Here, we work with preorderedCartesianCat/cat institutions. In other contexts, other types of X/Y institutions may be needed!

Categorical Logics

. . . can be formalized as essentially algebraic theories (i.e. condtional equational partial algebraic theories).

Let $TCat$ be the two-sorted specification of small categories, with sorts $object$ and $morphism$, extended by the specification of an operation $\top : object$ axiomatized to be a terminal object.

A propositional categorical logic L is an extension of $TCat$ with new operations and (oriented) conditional equations. The category of categorical logics has such theories L as objects and theory extension as morphisms. It is denoted by $\mathbb{C}atLog$.

Examples

- propositional logic with conjunction $\Leftrightarrow$ cartesian categories
- propositional logic with conjunction and implication $\Leftrightarrow$ cartesian closed categories
- intuitionistic propositional logic $\Leftrightarrow$ bicartesian closed categories
- classical propositional logic $\Leftrightarrow$ bicartesian closed categories with not not-elimination

Institutional Curry-Howard-Tait Construction

Given a categorical logic L, construct $I(L)$:

- C be the category of L-algebras (=categories),
- $T_L(X)$ be the (absolutely free) term algebra over X,
- $\mathbb{S}ign = \mathbb{S}et$
- $\mathsf{Sen}(\Sigma) = T_L(\Sigma)_{object}$,
- $|\mathsf{Mod}(\Sigma)| = \{m\colon \Sigma \longrightarrow |A|, \text{ where } A \in C\}$,
- $m\colon \Sigma \longrightarrow |A| \models_\Sigma \varphi$ iff $m^\#(\varphi)$ has a global element in A (i.e. there is some morphism $\top \rightarrow m^\#(\varphi)$),
- $\mathsf{Pr}(\Sigma)$ has objects $\mathsf{Sen}(\Sigma)$ and morphisms $p\colon \phi \longrightarrow \psi$ for $L \vdash p\colon \phi \longrightarrow \psi$.

- A model morphism $(F,\mu)\colon(m\colon\Sigma\longrightarrow|A|)\longrightarrow(m'\colon\Sigma\longrightarrow|B|)$ consists of a functor $F\colon A\longrightarrow B\in C$ and a natural transformation $\mu\colon F\circ m\longrightarrow m'$.
- Model reducts are given by composition: $\mathsf{Mod}(\sigma\colon\Sigma_1\longrightarrow\Sigma_2)(m\colon\Sigma_2\longrightarrow|A|)=m\circ\sigma$,
- this also holds for reducts of model morphisms,
- proof reductions are given by term rewriting.

Quotienting out the pre-order

Given a preorder-enriched category C, let $\tilde{C}$ be its quotient by the equivalences generated by the pre-orders on hom-sets.

Given a preordcat/cat institution I, let $\tilde{I}$ be the $\widetilde{\text{cat/cat}}$ institution obtained by replacing each $\mathsf{Pr}(\Sigma)$ with $\widetilde{\mathsf{Pr}(\Sigma)}$.

Theorem. Proof categories in $\widetilde{I(L)}$ are L-algebras.

Corollary. If L has products, then the deduction theorem holds for "proofs with extra assumptions" in $I(L)$:

$$\frac{L \cup \{x\colon \top \longrightarrow \varphi\} \vdash p(x)\colon \psi \longrightarrow \chi}{L \vdash \kappa x \,.\, p(x)\colon \varphi \wedge \psi \longrightarrow \chi}$$

Satisfaction Condition

Theorem. $I(L)$ enjoys the satisfaction condition.

Proof. simple universal algebra: $(m \circ \sigma)^{\#} = m^{\#} \circ \mathsf{Sen}(\sigma)$.

$$
\begin{array}{lll}
 & & m|_{\sigma} \models \varphi \\
 & \text{iff} & m \circ \sigma \models \varphi \\
\text{Hence,} & \text{iff} & (m \circ \sigma)^{\#}(\varphi) \text{ has a global element} \\
 & \text{iff} & m^{\#} \circ \mathsf{Sen}(\sigma)(\varphi) \text{ has a global element} \\
 & \text{iff} & m \models \sigma(\varphi).
\end{array}
$$

Soundness

Theorem. $I(L)$ is a sound institution.

Proof.

Assume $\varphi \vdash \psi$.

Also assume $m \models_\Sigma \varphi$.

This is: $L \vdash p\colon \varphi \longrightarrow \psi$ and $x\colon T \longrightarrow m^{\#}(\varphi)$.

These imply $p \circ x\colon T \longrightarrow m^{\#}(\psi)$, i.e. $m \models_\Sigma \psi$.

Altogether, $\varphi \models \psi$.

Completeness

Theorem. If L has products (i.e. conjunction), $I(L)$ is a complete institution.

Proof.

If $\varphi \models_\Sigma \psi$, this holds also for the free L-algebra $\eta\colon \Sigma \longrightarrow F$ over Σ and $x\colon \top \longrightarrow \varphi$.

Because $\eta \models_\Sigma \varphi$, also $\eta \models_\Sigma \psi$, i.e. there is $p(x) : \top \rightarrow \eta^{\#}(\psi)$.

Since in the free algebra, a ground atomic sentence holds exactly iff it is provable, $L \cup \{x\colon \top \longrightarrow \varphi\} \vdash p(x)\colon \top \longrightarrow \psi$.

By the deduction theorem, $L \vdash \kappa x\,.\ p(x)\colon \varphi \wedge \top \longrightarrow \psi$, therefore Ł $\vdash \kappa x\,.\ p(x) \circ \pi_2\colon \varphi \longrightarrow \psi$. Hence $\varphi \vdash \psi$.

The Curry-Howard-Tait isomorphism

There is (e.g.) an institution morphism from $Prop$ to $I(biCCCnotnot)$:

- identity on signatures; trivial isomorphism on sentences
- a Boolean-valued valuation of propositional variables in particular is a valuation into the $biCCCnotnot$-category, i.e. Boolean algebra, $\{false, true\}$.
- a $biCCCnotnot$-proof is mapped to a Gentzen-style proof
- $biCCCnotnot$-reductions $\longrightarrow$ cut elimination?

biCCCnotnot $=$ bicartesian closed categories with notnot-elemination.

The L construction is functorial

A theory extension $L_1 \subseteq L_2$ easily leads to an institution comorphism $I(L_1) \to I(L_2)$.

Conclusion and Future Work

- canonical way of obtaining institutions with proofs
- usual collapsing problems (i.e. classical biCCCs are Boolean algebras) are avoided through the preorder structure
- generic deduction, soundness and completeness theorem
- extension to propositional model logic?
- extension to FOL, HOL requires different treatment of signatures. Extract signature category from the index category of a hyperdoctrine?

Hyperdoctrines and cat/- institutions

A hyperdoctrine is an indexed category $P\colon C^{op} \longrightarrow \mathbb{C}at$ s.t.

- each $P(A)$ is cartesian closed
- for each $f \in C$,
  - $P(f)$ preservers exponentials
  - $P(f)$ has a right adjoint $\forall_f$
  - $P(f)$ has a left adjoint $\exists_f$
  - P satisfies the Beck condition

This is pretty close to a cat/- institution having proof-theoretic $\top$, $\wedge$, $\Rightarrow$, $\forall$, $\exists$: take P to be the sentence/proof functor $\mathsf{Pr}\colon \mathbb{S}ign \longrightarrow \mathbb{C}at$ and C the subcategory of $\mathbb{S}ign^{op}$ consisting of the representable morphisms.