1. Let \( A = \{3, 4, 5\} \) and \( B = \{4, 5, 6\} \) and let \( \mathcal{D} \) be the “divides” relation. That is,

\[
\text{for all } (x, y) \in A \times B, \ x \mathcal{D} y \iff x \mid y.
\]

State explicitly which ordered pairs are in \( \mathcal{D} \) and \( \mathcal{D}^{-1} \).

\[
\mathcal{D} = \{(3, 6), (4, 4), (5, 5)\}
\]

\[
\mathcal{D}^{-1} = \{(6, 3), (4, 4), (5, 5)\}
\]

2. Let \( A \) be a set with at least two elements and \( \mathcal{P}(A) \) the power set of \( A \). Define a relation \( \mathcal{R} \) on \( \mathcal{P}(A) \) as follows:

\[
\text{for all } X, Y \in \mathcal{P}(A), \ X \mathcal{R} Y \iff X \subseteq Y \text{ or } Y \subseteq X.
\]

Determine whether the given binary relation \( \mathcal{R} \) is reflexive, symmetric, transitive, or none of these. Justify your answers.

\( \mathcal{R} \) is reflexive:
Suppose \( X \in \mathcal{P}(A) \). By definition of subset, \( X \subseteq X \). By definition of \( \mathcal{R} \), then, \( X \mathcal{R} X \).

\( \mathcal{R} \) is symmetric:
Suppose \( X \) and \( Y \) are sets in \( \mathcal{P}(A) \), and \( X \mathcal{R} Y \). By definition of \( \mathcal{R} \), this means either \( X \subseteq Y \) or \( Y \subseteq X \). In either case, the statement “\( X \subseteq Y \) or \( Y \subseteq X \)” is also true. Hence, \( Y \mathcal{R} X \).

If \( A \) has at least two elements, then \( \mathcal{R} \) is not transitive:
Counterexample: Let \( x \) and \( y \) be in \( A \), \( x \neq y \). Then it is \( \{x\} \mathcal{R} A \) and \( A \mathcal{R} \{y\} \) because \( \{x\} \), \( \{y\} \), and \( A \) are sets in \( \mathcal{P}(A) \), and \( \{x\} \subseteq A \) as well as \( \{y\} \subseteq A \).
But it is neither \( \{x\} \subseteq \{y\} \) nor \( \{y\} \subseteq \{x\} \), so \( \{x\} \) is not related to \( \{y\} \) by \( \mathcal{R} \).
If \( A \) has a single element, then \( \mathcal{R} \) is transitive:
In this case, given any two subsets of \( A \), either one is a subset of the other or the other is a subset of the one. Hence regardless of the choice of \( X \), \( Y \), and \( Z \), it must be the case that \( X \subseteq Z \) or \( Z \subseteq X \), and so \( X \mathcal{R} Z \).

3. Recall the definition on page 357 for the inverse image of a set.

Given two functions \( f : X \to Y \) and \( g : Y \to Z \), prove

\[
\text{For all subsets } E \text{ of } Z, \ (g \circ f)^{-1}(E) = f^{-1}(g^{-1}(E)).
\]
Proof:

1. Show $(g \circ f)^{-1}(E) \subseteq f^{-1}(g^{-1}(E))$.
   Let $x \in (g \circ f)^{-1}(E)$, and let $y := f(x)$ and $z := g(y)$.
   By definition of $(g \circ f)$ and definition of the inverse image, this means
   \[
   (g \circ f)(x) = g(f(x)) = g(y) = z \in E
   \]
   But if $z \in E$ then by definition of the inverse image, $y \in g^{-1}(E)$ (because there exists $z$ in $E$ such that $g(y) = z$).
   But now, since $y \in g^{-1}(E)$, for the same reason as before, it must be that $x \in f^{-1}(g^{-1}(E))$.

2. Show $f^{-1}(g^{-1}(E)) \subseteq (g \circ f)^{-1}(E)$.
   Let $x \in f^{-1}(g^{-1}(E))$.
   By definition of the inverse image, this means that there exists $y \in g^{-1}(E)$, such that $f(x) = y$.
   But as $y \in g^{-1}(E)$, there must exist a $z \in E$ such that $g(y) = z$.

Putting this together yields that there exists $z \in E$ such that
\[
   z = g(y) = g(f(x)) = (g \circ f)(x).
\]
But this means that $x \in (g \circ f)^{-1}(E)$.

Now as $(g \circ f)^{-1}(E) \subseteq f^{-1}(g^{-1}(E))$ and $f^{-1}(g^{-1}(E)) \subseteq (g \circ f)^{-1}(E)$, it is $(g \circ f)^{-1}(E) = f^{-1}(g^{-1}(E))$ as was to be shown.

4. Prove that a union of three countable sets is countable.
   Let $A$, $B$ and $C$ be countable sets.
   To show that $A \cup B \cup C$ is countable, we must find a one-to-one correspondence from $\mathbb{Z}^+$ to $A \cup B \cup C$.
   W.l.o.g. assume that $A$, $B$ and $C$ are mutually disjoint. Otherwise just remove elements which occur more than once from some of the sets, which won’t change that they’re countable. (This assumption assures that the correspondence will be one-to-one.)
   Consider the case if all three sets $A$, $B$ and $C$ are countable infinite. Then there exist one-to-one correspondences $f_A$, $f_B$ and $f_C$ from $\mathbb{Z}^+$ to each of the sets. Define $F : \mathbb{Z}^+ \to A \cup B \cup C$ by
   \[
   F(n) = \begin{cases}
   f_A\left([n + 2]/3\right) & \text{if } n \text{ mod } 3 = 1 \\
   f_B\left([n + 2]/3\right) & \text{if } n \text{ mod } 3 = 2 \\
   f_C\left([n + 2]/3\right) & \text{if } n \text{ mod } 3 = 0
   \end{cases}
   \]
   Then $F$ is one-to-one because $f_A$, $f_B$ and $f_C$ are one-to-one and we assumed that $A$, $B$ and $C$ are mutually disjoint.
   If $x \in A$ then there exists $k$ such that $f_A(k) = x$ (because $f_A$ is onto). But then $x = f_A(k) = F(3(k \mod 1) + 1)$.
   Similarly, for $y \in B$ or $C$ there exists a number $n$ (namely $3(k \mod 1) + 2$ and $3(k \mod 1) + 3$) such that $F(n) = y$. Hence, $F$ is onto.
   In the case that some of the sets $A$, $B$ and $C$ are finite, simply count their elements first and then proceed similarly with the remaining (finite) set(s).