Solutions for Quiz #6 - May 13, 1998

1. Suppose the set $A = \{3, 4\}$. Find the power set of the power set of $A$, $\mathcal{P}(\mathcal{P}(A))$.

\[
\mathcal{P}(A) = \{\emptyset, \{3, 4\}\}
\]
\[
\mathcal{P}(\mathcal{P}(A)) = \{\emptyset, \emptyset, \{\{3, 4\}\}, \{\emptyset, \{3, 4\}\}\}
\]

2. For each of a. and b. prove the statement if it is true and find a counterexample if it is false. Illustrate each statement by drawing a Venn diagram. Assume all sets are subsets of a universal set $U$.

a. For all sets $A$, $B$ and $C$, if $B \cap A \subseteq C$, then $(C - B) \cap (C - A) = \emptyset$.

Illustration:

Counterexample:
Let $A = B = \emptyset$, and $C = \{1\}$. Then $C - B = C - A = C = \{1\}$, and hence

\[(C - B) \cap (C - A) = \{1\} \cap \{1\} = \{1\} \neq \emptyset.\]

b. For all sets $A$, $B$ and $C$, $(C - (A \cup B)) \cap (C^c \cup A) = \emptyset$.

Illustration:

Proof:
Suppose not, i.e. suppose there exists $x \in (C - (A \cup B)) \cap (C^c \cup A)$. This implies $x \in (C - (A \cup B))$ and $x \in (C^c \cup A)$.

From $x \in (C - (A \cup B))$ we get that $x \in C$ (1) and $x \notin A$ (2) and $x \notin B$.

The second expression $x \in (C^c \cup A)$ gives that $x \in C^c$ or $x \notin A$.

Case 1: Let $x \in C^c$.

But by definition of the complement, $x \in C^c$ implies $x \notin C$, which contradicts (1).
Case 2: Let \( x \in A \).
This directly contradicts (2).
So in either case, we get a contradiction. Hence, the supposition is false, and
\[
(C - (A \cup B)) \cap (C^c \cup A) = \emptyset.
\]

3. Suppose that \( A, B, \) and \( C \) are sets and \( B \cap C = \emptyset \). Simplify the following expression as much as possible using theorems 5.2.1-5.2.3 and 5.3.3. Be sure to use each property exactly as stated in the theorem.

\[
[(A - B) \cup (A - C)] \cap (B - B^c) = [(A \cap B^c) \cup (A \cap C^c)] \cap (B \cap (B^c)^c)
\]
by the alternate representation
for set difference law
\[
= [(A \cap B^c) \cup (A \cap C^c)] \cap (B \cap B)
\]
by the double negative law
\[
= [(A \cap B^c) \cup (A \cap C^c)] \cap B
\]
by the idempotent law
\[
= [A \cap (B^c \cup C^c)] \cap B
\]
by the distributive law
\[
= [A \cap (B \cap C)^c] \cap B
\]
by De Morgan’s law
\[
= [A \cap \emptyset] \cap B
\]
by hypothesis
\[
= [A \cap U] \cap B
\]
by complement of \( \emptyset \) law
\[
= A \cap B
\]
by intersection with \( U \) law

4. a. Assuming that the following sentence is a statement, prove that there exists a biggest prime number \( p \).

If this sentence is true, then there exists a biggest prime number \( p \).

Assuming that the given sentence is a statement, it is either true or false.
By definition of truth values for an if-then statement, the only way the statement can be false is for its hypothesis to be true and its conclusion false.
But if the hypothesis is true, then the statement is true and therefore it is not false. So it is impossible for the sentence to be false, and hence it is true.
Consequently, what its hypothesis asserts is true, and so (again by definition of truth values for if-then statements) its conclusion must also be true.
Therefore, there exists a biggest prime number \( p \).

b. What can you deduce from part a. about the status of “This sentence is true”? Why?

We can deduce that “This sentence is true” is not a statement. For if it were a statement, then since “there exists a biggest prime number \( p \)” is also a statement, the sentence “If this sentence is true, then there exists a biggest prime number \( p \)” would also be a statement.
It would then follow by part a. that there exists a biggest prime number \( p \), which we know to be false.
So “This sentence is true” is not a statement.